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PARTIALLY CLOSED BRAIDS

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1. **Introduction.** The purpose of this paper is to define partially closed braids (§3) and to prove that every partially closed braid has a canonical form easily obtainable (§5). These objects are of interest because they can be used to represent knots tied in a string.

2. **Notation.** Braids have an obvious intuitive meaning to which we shall refer. Braids are also elements of the braid groups of E. Artin [1], defined for each integer n greater than one by the presentation

$$B_n = \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1};\quad \sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{if} \quad |i-j| \geq 2, \quad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\rangle.$$

There is a natural homomorphism π_n mapping B_n onto the symmetric group on $\{1, 2, \ldots, n\}$ in which π_n takes $\sigma_i \in B_n$ to the interchange of i and $i+1$. Each braid in the kernel of π_n, that is one that does not permute its strands, can be put in a canonical form due to Artin [2], called *combed* form, in which the braid is represented as a product of $n-1$ factors. In the first factor, the strands other than the first do not change their positions relative to one another, but the first strand is twisted among them, finishing in its original position. The first factor is said to be *one-pure*. In the second factor, called *two-pure*, the first strand is absolutely uninvolved, the second strand twists about strands 3 to n, and these latter strands do not change their positions relative to one another. The third factor, *three-pure*, does not involve strands one or two, but has the third strand twist about the unmoving remainder. In general, the ith factor is *i-pure*, not involving strands one to $i-1$ and having the ith strand twist about the remainder. Finally, the $(n-1)$th factor has the $(n-1)$th strand twisted around the nth. Figure 1 shows a combed

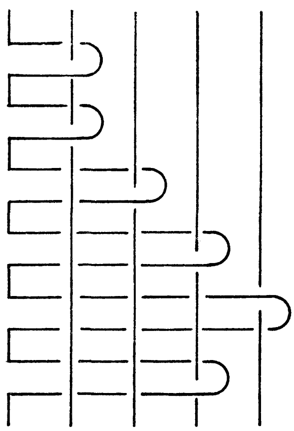

FIGURE 1

braid in which the twists of the first strand around the other strands are separated from one another.

The fact that a non-permuting braid, that is one in the kernel of π_n, can be combed allows every such braid to be represented as a product of twists like those illustrated in Fig. 1, in which a strand, say the ith, passes behind its neighbours to the right, loops once around the $(i+j)$th, and then returns to its original position. And any product of such twists is a non-permuting braid represented by a two-column matrix of integers,

$$\begin{bmatrix} i_1 & j_1 \\ i_2 & j_2 \\ & \cdot \\ & \cdot \\ & \cdot \\ i_k & j_k \end{bmatrix},$$

in which the rows represent successive twists.

(1) Each first-column entry $i_1, i_2, \ldots, i_k$, is positive and represents the number of the left-hand or moving strand;

(2) Each second-column entry $j_1, j_2, \ldots, j_k$, contains two pieces of information:

(a) $|j_l|$ is the number of the right-hand strand around which the (i_l)th strand loops, and

(b) the sign of j_l indicates the sense of the lth twist, the first twist of Fig. 1 being positive.

(3) The identity, in which nothing happens, can adequately be represented by [ ].

Obvious consequences of the conditions (1) and (2) are the following.

(4) In B_n, the first-column entries come from $\{1, 2, \ldots, n-1\}$ and the right-column entries from $\{2, -2, 3, -3, \ldots, n, -n\}$.

(5) $|j_l| > i_l \quad (l=1, 2, \ldots, k)$.

(6) The product of two braids is represented by writing the matrix of the second factor beneath that of the first factor as a continuation.

(7) The inverse of $[i \quad j]$ is $[i \quad -j]$.

$$\begin{matrix}[i & -j] \\ [i & j\ ]\end{matrix} = \begin{bmatrix} i & -j \\ i & j \end{bmatrix} = [\quad].$$

One object of introducing this notation is that it allows combing to be performed quickly by a computer [3]. A braid is combed when its left column is monotone increasing.

3. **Partial closure.** We define an operator which we call rotation and denote R. The effect of R applied to a non-permuting braid is expressed in terms of the matrix notation and using the function

$$\operatorname{sgn}(x) = \begin{cases} +1 & x > 0, \\ -1 & x < 0. \end{cases}$$

R is defined thus.

$$R[\quad] = [\quad];$$

$$R\begin{bmatrix} a & b \\ c & d \\ & \cdot \\ & \cdot \\ & \cdot \\ g & h \\ i & j \end{bmatrix} = \begin{bmatrix} i-1 & \operatorname{sgn}(j) & (|j|-1) \\ a & & b \\ c & & d \\ \\ & \cdots & \\ g & & h \end{bmatrix}.$$

R brings the bottom row to the top, while reducing its entries by one in absolute value. R can be applied only when the bottom left entry is greater than one, in view of (4). Considered topologically, R takes the bottom twist $[i \;\; j](i>1)$ of an open n-strand braid and puts it at the top as though the tops and bottoms of the strands were connected as in Fig 2, where a (topological) partially closed braid is shown. The restriction $i>1$ corresponds to the disconnection of the bottom of the first strand.

We define an algebraic partially closed braid (abbreviated to p.c.b.) to be an equivalence class of braids in a single braid group B_n. Two braids of B_n are considered equivalent, or to represent the same p.c.b., if one can be transformed to

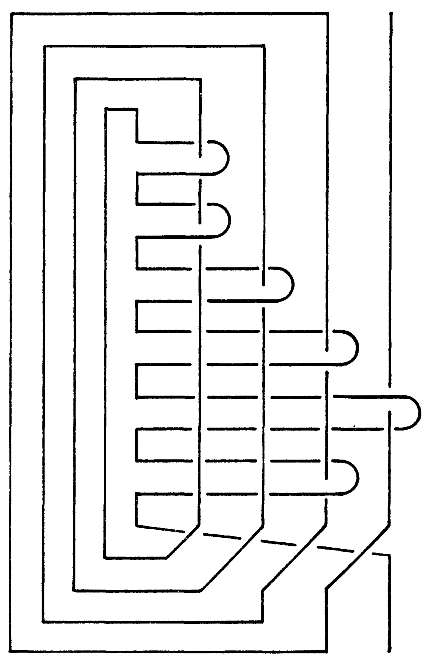

FIGURE 2

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the other by the action of the rotation operator. It is natural to note that R has an inverse R^{-1} defined as follows.

$$R^{-1}[\ \ ]=[\ \ ];$$

$$R^{-1}\begin{bmatrix} a & b \\ c & d \\ & \cdot \\ & \cdot \\ & \cdot \\ i & j \end{bmatrix} = \begin{bmatrix} c & & d \\ & \cdots & \\ i & & j \\ a+1 & \operatorname{sgn}(b) & (|b|+1) \end{bmatrix}.$$

In order that R^{-1} not transform a braid from B_n to B_{n+1}, R^{-1} can be applied only when the top right entry is less than n in absolute value, in view of (4). Topologically and algebraically, the action of R^{-1} is opposite to that of R.

4. **Motivation.** It is because in euclidean space every polygonal arc with inaccessible ends can be represented by partially closed braids that they are here defined in the hope that they may help in the classification of such arcs. Alexander [4] showed that every polygonal system of closed curves can be represented as a braid with ends identified.

LEMMA 1. *In euclidean space, every continuous arc made up of a finite number of line segments and two rays can be continuously deformed into a braid with all ends identified pairwise except one pair, which are produced indefinitely.*

Proof. There is a plane perpendicular to none of the finite number of line segments and rays. Let such a fixed plane π be used as the projection plane in which to discuss the given arc A. The parts of the arc can be so deformed that the perpendicular projection into π has only double-points at worst, not triple-points, and that no vertex is a double-point [5]. Let O be a point in π not on the image I of the deformed arc nor on any ray or segment of I produced. Such choice of O ensures that a radius vector based at O and tracing out the image I will always be turning or reversing direction, never merely lengthening or shortening, and will never shrink to nothing. One of the rays can be moved so that the projection of two rays and a line segment in π but not in I form a whole line. The projection in π must now look something like Fig. 3. Choose one ray, and consider the rotation of the radius vector at O as it traces this ray R_1 in from infinity. This rotation is, say, clockwise. Clockwise is then the basic direction; as the other ray R_2 is traced out to infinity, the direction of rotation is also clockwise. As the image I is traced from R_1 to R_2, the radius vector moves sometimes clockwise, sometimes not. Every line segment in A causing an anticlockwise rotation can be broken into overcrossing and undercrossing subsegments if necessary. Each subsegment can be replaced by (deformed to) two line segments with images in π along which rotation is clockwise. It may be that no subdivision is needed, as in Fig. 4. When the

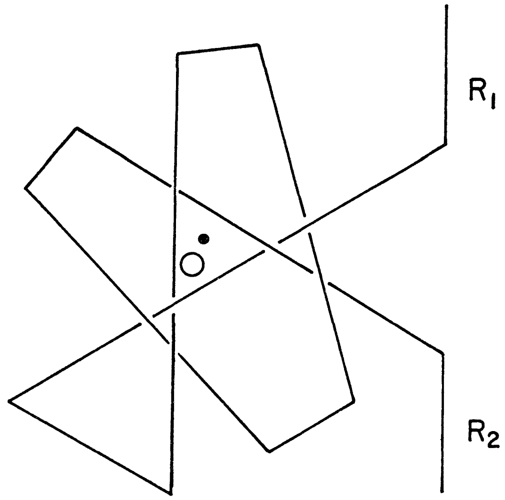

FIGURE 3

arc A is deformed in the manner described, its projection on π is as required. For consider a ray from O passing through no double-point of I and not cutting the line of R_1 and R_2. The points at which this ray cuts I, as in Fig. 5, can be taken as the identified tops and bottoms of a braid's strands as in Fig. 6. This is Alexander's proof [4] modified to take account of altered hypotheses. Figures 3 to 6 show how Reidemeister's knot 7_7 split open at the right can be made into a (topological) partially closed braid.

A braid produced by the construction of lemma 1 is not in the tidy form of Fig. 2. It does, however, have two properties that allow it to be tidied up.

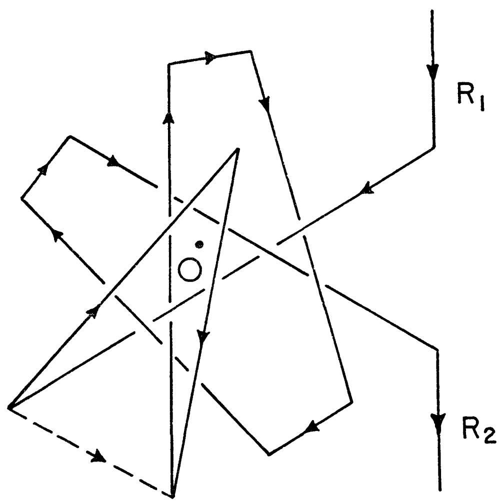

FIGURE 4

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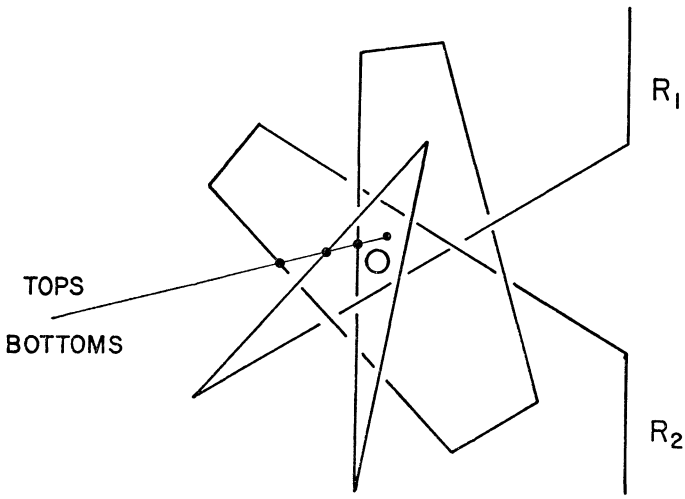

FIGURE 5

It has two loose ends at the far right and if it is in B_n, then its image under π_n is a single cycle involving all the numbers from one to n. It is easy to see that, for a braid mapped to a two-cycle permutation, the strands without the loose ends close to make a separate circle, not a part of a single arc. We must show that a braid with nth top and bottom produced can be put in the form of Fig. 2.

LEMMA 2. *For every braid $b \in B_n$ mapped by π_n to a single n-cycle in the symmetric group, there exists a braid $c \in B_n$ such that*

(1) $$c \text{ contains no } \sigma_{n-1} \text{ or } \sigma_{n-1}^{-1}$$

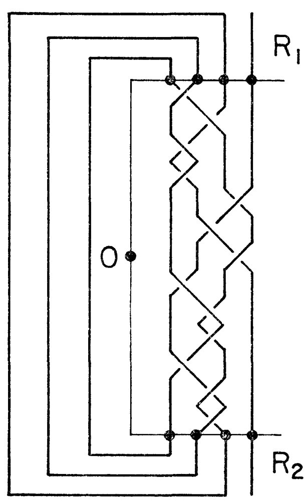

FIGURE 6

and

$$\pi_n(c^{-1}bc) = (n, n-1, n-2, \ldots, 2, 1). \tag{2}$$

Proof. Let $\pi_n(b)$ be written with the n first, $\pi_n(b)=(n, i_2, i_3, \ldots, i_n)$. Either $i_2=n-1$, or $i_2<n-1$. If $i_2=n-1$, let $b_2=b$. If $i_2<n-1$, let

$$b_2 = \sigma_{n-2}^{-1}\sigma_{n-3}^{-1}\cdots\sigma_{i_2+1}^{-1}\,\sigma_{i_2}^{-1}b\sigma_{i_2}\sigma_{i_2+1}\cdots\sigma_{n-2}.$$

In either case $\pi_n(b_2)=(n, n-1, i_3', \ldots, i_n')$. In the same way b_3 can be formed from b_2, without using σ_{n-1} or σ_{n-2}, so that $\pi_n(b_3)=(n, n-1, n-2, i_4'', \ldots, i_n'')$. The process can be continued until b_{n-1} is formed such that

$$\pi_n(b_{n-1}) = (n, n-1, \ldots, 3, 2, i_n^{(n-1)}).$$

But $i_n^{(n-1)}$ can only be 1. The required braid c has not been written down, but its formation has been outlined, assuring that it exists and contains no σ_{n-1}. This completes the proof.

We now know that from each polygonal arc of lemma one we can produce a braid projecting to the particular n-cycle $(n, n-1, \ldots, 1)$. It remains to show that the braid can be expressed as the product of a non-permuting braid and the standard ending $\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{n-1}^{-1}$. If $\pi_n(c^{-1}bc)=(n, n-1, \ldots, 2, 1)$, then

$$\begin{aligned}\pi_n(c^{-1}bc\sigma_{n-1}\sigma_{n-2}\cdots\sigma_2\sigma_1) &= (n, n-1, \ldots, 2, 1)(1, 2, \ldots, n)\\ &= (1)(2)\cdots(n),\end{aligned}$$

but $c^{-1}bc=(c^{-1}bc\sigma_{n-1}\sigma_{n-2}\cdots\sigma_2\sigma_1)(\sigma_1^{-1}\,\sigma_2^{-1}\cdots\sigma_{n-1}^{-1})$, which is a product of the required type. Accordingly, we have proved the theorem following.

THEOREM 1. *Every continuous arc in euclidean space and made up of a finite number of line segments and two rays can be continuously deformed into a (topological) partially closed braid.*

Finally, we remark that there is an obvious one-to-one relationship between each topological p.c.b. of §4 and an algebraic p.c.b. of §3. We consider the algebraic objects further in §5 and return to the topological in §6.

5. **The canonical form.** The canonical form of a p.c.b. is obtained, briefly, by applying the operator R as often as it is possible to do so, a finite number of times. Consider a braid in B_n in its combed form $b_1, b_2, \ldots, b_{n-1}$, where b_j is j-pure. In matrix representation, the braid b_j has, say, r_j rows ($j=1, 2, \ldots, n-1$). Accordingly, R can be applied $\sum_{j=2}^{n-1} r_j$ times to the braid as it stands. Let b_j^{j-1} represent the result of r_j applications of R to b_j; b_j^{j-1} is $(j-1)$-pure. In general, let b_j^k represent the result of making b_j k-pure. If

$$b_j = \begin{bmatrix} j & p \\ j & q \\ \cdots & \\ j & s \end{bmatrix},$$

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then

$$b_j^k = \begin{bmatrix} k & \operatorname{sgn}(p) & (|p|-j+k) \\ k & \operatorname{sgn}(q) & (|q|-j+k) \\ & \cdots & \\ k & \operatorname{sgn}(s) & (|s|-j \quad k) \end{bmatrix} \in B_n.$$

The result of $\sum_{j=2}^{n-1} r_j$ applications of R to $b_1, b_2, \ldots, b_{n-1}$ is $b_2^1 b_3^2 \cdots b_{n-1}^{n-2} b_1$. The braid is no longer in combed form, nor can R be applied further. If the braid is recombed, $b_3^2 b_4^3 \cdots b_{n-1}^{n-2}$ will be moved to the bottom, b_2^1 will be left at the top, and b_1 will be replaced by the more complicated 1-pure braid produced by combing $b_3^2 \cdots b_{n-1}^{n-2}$ past it. Now R can be applied again $\sum_{j=3}^{n-1} r_j$ times, and no more until the braid is recombed. This cycle, rotate and comb, rotate and comb, as often as possible, can be called *purification* and results in a 1-pure braid that can be called the *pure* form of the p.c.b.

The above description of the iterative purification process can serve as a definition of the canonical pure form of a p.c.b., but a more explicit description can be given using the notation introduced above.

THEOREM 2. *The result of purifying the braid* $b_1 b_2 \cdots b_{n-1} \in B_n$ (b_j *j-pure*) *is the* 1*-pure factor of the combed form of*

$$b_{n-1}^1 (b_{n-2}^1 b_{n-1}^2)(b_{n-3}^1 b_{n-2}^2 b_{n-1}^3)(\cdots)(b_3^1 b_4^2 \cdots b_{n-1}^{n-3})(b_2^1 b_3^2 \cdots b_{n-1}^{n-2}) b_1.$$

Proof. It is a characteristic of the combing process that, if $j > i$ and a j-pure factor is combed past an i-pure factor, the j-pure braid emerges unchanged from the process. With that fact in mind, it is easy to see that the expression of the theorem is obtained by placing before b_1 what the first step of purification places above it by several uses of R, by placing before that what passes by b_1 in the combing and is placed above b_2^1 by several uses of R, etc.

The statement of the purified form of $b_1 b_2 \cdots b_{n-1}$ as the 1-pure factor of a specific braid of B_n assures its being uniquely defined for a given p.c.b. That means that we must show that two braids related by an application of R have the same pure form.

LEMMA 3. *If* $a \in B_n$ *and* $R(a) = c$, *then the pure form of a is the pure form of c.*

Proof. It will be convenient to mix the notations for braids, writing the product of a braid $b \in B_n$ and a single twist $[i \quad j]$ either as $[i \quad j]b$ or as $b[i \quad j]$ respectively as b is below or above the twist $[i \quad j]$.

Consider a braid, not necessarily in combed form, and the effect of R on it:

$$a = b[i \quad j] \xrightarrow{R} [i-1 \ \operatorname{sgn}(j)(|j|-1)]b = c.$$

We must show that the pure forms of both these braids are the same. Now in its combed form

$$b = b_1 b_2 \cdots b_{i-1} b_i b_{i+1} \cdots b_{n-1}.$$

And $b[i \quad j]$ in its combed form is $b_1 \cdots b_i d_i b_{i+1} \cdots b_{n-1}$, where d_i is the result, with $[i \quad j]$ in its centre, of combing $b_{i+1} \cdots b_{n-1}$ past $[i \quad j]$. Now we apply the first stage of the purification to both forms a and c;

$$a = b_1 \cdots b_i d_i b_{i+1} \cdots b_{n-1}$$

becomes under the operation of R sufficiently often, but no combing,

$$a' = b_2^1 b_3^2 \cdots b_i^{i-1} d_i^{i-1} b_{i+1}^i \cdots b_{n-1}^{n-2} b_1.$$

And

$$c = [i-1 \quad \mathrm{sgn}(j)(|j|-1)] b_1 b_2 \cdots b_i \cdots b_{n-1}$$

becomes under the operation of R sufficiently often, but no combing,

$$b_2^1 b_3^2 \cdots b_i^{i-1} b_{i+1}^i \cdots b_{n-1}^{n-2} [i-1 \quad \mathrm{sgn}(j)(|j|-1)] b_1.$$

Combing the portion $b_{i+1}^i \cdots b_{n-1}^{n-2} [i-1 \quad \mathrm{sgn}(j)(|j|-1)]$ gives

$$c' = b_2^1 b_3^2 \cdots b_i^{i-1} x b_{i+1}^i \cdots b_{n-1}^{n-2} b_1$$

where x is the result, with $[i-1 \quad \mathrm{sgn}(j)(|j|-1)]$ in its centre, of combing $b_{i+1}^i \cdots b_{n-1}^{n-2}$ past $[i-1 \quad \mathrm{sgn}(j)(|j|-1)]$. But this is precisely d_i^{i-1} since everything is the same as combing $b_{i+1} \cdots b_{n-1}$ past $[i \quad j]$ but is shifted one place to the left so that the result, instead of being d_i, is d_i^{i-1}. Therefore $a' = c'$. Purification can then be completed to show that the pure form of a is the pure form. The essential point is that it does not matter in what order rotation and combing are carried out; if both are done often enough, the outcome is the same.

The result of Lemma 3 and Theorem 2 is our goal.

THEOREM 3. *A partially closed braid of B_n, given in combed form by $b_1 b_2 \cdots b_{n-1}$ (b_j j-pure) has as its unique pure form the* 1*-pure part of the combed form of the braid*

$$b_{n-1}^1 (b_{n-2}^1 b_{n-1}^2)(b_{n-3}^1 b_{n-2}^2 b_{n-1}^3)(\cdots)(b_3^1 b_4^2 \cdots b_{n-1}^{n-3})(b_2^1 b_3^2 \cdots b_{n-2}^{n-3} b_{n-1}^{n-2}) b_1.$$

6. **Conclusion.** Theorem 3 is algebraically proved, and its content is entirely algebraic despite its topological motivation. It gives rise to a natural topological conjecture, that, provided a sensed polygonal arc is expressed as a partially closed braid on a minimum number of strands, its pure form is a unique representation. Such a conjecture is, however appealing, false, as I shall undertake to show elsewhere.

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