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ON WEIGHTED SOBOLEV SPACES

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ABSTRACT. We study density and extension problems for weighted Sobolev spaces on bounded (ε,δ) domains $\mathcal{D}$ when a doubling weight w satisfies the weighted Poincaré inequality on cubes near the boundary of $\mathcal{D}$ and when it is in the Muckenhoupt A_p class locally in $\mathcal{D}$. Moreover, when the weights $w_i(x)$ are of the form $\operatorname{dist}(x,M_i)^{\alpha_i}$, $\alpha_i \in \mathbb{R}$, $M_i \subset \mathcal{D}$ that are doubling, we are able to obtain some extension theorems on (ε,∞) domains.

1. **Introduction.** Recently there has been quite a number of works related to weighted Sobolev spaces. For example, Kufner [23] studied various properties of weighted Sobolev spaces on certain domains $\mathcal{D}$ for weights arising from $\operatorname{dist}(\cdot,M)$ with $M \subset \partial\mathcal{D}$. Also, Brown and Hinton [2], [3], [4] and Gutierrez and Wheeden [20] obtained weighted Sobolev interpolation inequalities. Meanwhile, the author [9], [11], [13] has studied the extension and restriction problems on weighted Sobolev spaces. In this paper, we would like to improve some results in [9]. Namely, we will study density problems and extension problems on weighted Sobolev spaces. Note that some of our results overlap some of those in [23] and [17].

By a weight w, we mean a non-negative locally integrable function on $\mathbb{R}^n$. By abusing notation, we will also write w for the measure induced by w. Sometimes we write dw to denote $w\,dx$. We always assume w is doubling, by which we mean $w(2Q) \le Cw(Q)$ for every cube Q, where $2Q$ denotes the cube with the same center as Q and twice its edgelength. All cubes in this paper are assumed to be closed and with edges parallel to the axes. By $w \in A_p$, we mean w satisfies the Muckenhoupt A_p condition, *i.e.*,

$$\frac{1}{|Q|}\Big(\int_Q w\,dx\Big)^{1/p}\Big(\int_Q w^{-1/(p-1)}\,dx\Big)^{1/p'} \le C \quad \text{when } 1<p<\infty, \text{ and}$$

$$\frac{1}{|Q|}\int_Q w(x)\,dx \le C \operatorname*{essinf}_{x\in Q} w(x) \quad \text{when } p=1,$$

for all cubes Q in $\mathbb{R}^n$. Note that w is doubling when it is in A_p. Moreover, when $\mathcal{D}$ is an open set, we will write $w \in A_p^{\mathrm{loc}}(\mathcal{D})$ if for any cube $Q_0 \subset \mathcal{D}$, there exists $C_{Q_0} > 0$ such that

$$\frac{1}{|Q|} w(Q\cap Q_0)^{1/p}\Big(\int_{Q\cap Q_0} w^{\frac{-1}{p-1}}(x)\,dx\Big)^{1/p'} \le C_{Q_0} \quad \text{when } 1<p<\infty, \text{ and}$$

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$$\frac{w(Q \cap Q_0)}{|Q|} \le C_{Q_0} \operatorname*{essinf}_{x \in Q \cap Q_0} w(x) \quad \text{when } p = 1,$$

for all cubes Q in $\mathbb{R}^n$.[1]

Let $\mathcal{D}$ be an open set in $\mathbb{R}^n$. If α is a multi-index, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{Z}_+^n$, we will denote $\sum_{j=1}^n \alpha_j$ by $|\alpha|$ and $D^\alpha = (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_n})^{\alpha_n}$. By $\alpha \ge \beta$, we mean $\alpha_j \ge \beta_j$ for all $1 \le j \le n$. Moreover we write $\alpha > \beta$ if $\alpha \ge \beta$ and $\alpha \ne \beta$. We denote by ∇ the vector $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n})$ and by ∇^m the vector of all possible m^{th} order derivatives for $m \in \mathbb{N}$. A locally integrable function f on $\mathcal{D}$ (we will write $f \in L^1_{\mathrm{loc}}(\mathcal{D})$) has a weak derivative of order α if there is a locally integrable function (denoted by $D^\alpha f$) such that

$$\int_{\mathcal{D}} f(D^\alpha \varphi)\, dx = (-1)^{|\alpha|} \int_{\mathcal{D}} (D^\alpha f)\varphi\, dx$$

for all C^∞ functions φ with compact support in $\mathcal{D}$ (we will write $\varphi \in C_0^\infty(\mathcal{D})$).

If $1 < p < \infty$, p' is always equal to $p/(p-1)$ and $p' = \infty$ when $p = 1$. Q will always be a cube and $l(Q)$ will be its edgelength. Following [22], we say that two cubes touch if a face of one cube is contained in a face of the other. For $1 \le p < \infty$, $k \in \mathbb{N}$, and any weight w, $L^p_{w,k}(\mathcal{D})$ and $E^p_{w,k}(\mathcal{D})$ are the spaces of functions having weak derivatives of all orders α, $|\alpha| \le k$, and satisfying

$$\|f\|_{L^p_{w,k}(\mathcal{D})} = \sum_{0 \le |\alpha| \le k} \|D^\alpha f\|_{L^p_w(\mathcal{D})} = \sum_{0 \le |\alpha| \le k} \Big(\int_{\mathcal{D}} |D^\alpha f|^p\, dw \Big)^{1/p} < \infty,$$

and

$$\|f\|_{E^p_{w,k}(\mathcal{D})} = \sum_{|\alpha| = k} \|D^\alpha f\|_{L^p_w(\mathcal{D})} < \infty$$

respectively. Moreover, in the case when $w \equiv 1$, we will denote $L^p_{w,k}(\mathcal{D})$ and $E^p_{w,k}(\mathcal{D})$ by $L^p_k(\mathcal{D})$ and $E^p_k(\mathcal{D})$ respectively. Also, let $\hat{E}^p_{w,k}(\mathcal{D})$ be the factor space $E^p_{w,k}(\mathcal{D})/\mathcal{P}_{k-1}$ where $\mathcal{P}_l$ is the subspace of polynomials of degree not greater than l. By $f \in L^p_{w,1,\mathrm{loc}}(\mathcal{D})$, we mean $f \in L^p_{w,1}(K^o)$ for all compact sets K in $\mathcal{D}$.

Let $\mathcal{D}$ be an open connected set. It is easy to see that $L^p_{w,k}(\mathcal{D})$ is a Banach space when $w^{-1/p} \in L^{p'}_{\mathrm{loc}}(\mathcal{D})$ [17]. Moreover, the author [9] prove that $\hat{E}^p_{w,k}(\mathcal{D})$ is a Banach space when $w \in A_p$. Note that it is just a weighted version of Theorem 1.1.13.1 in [26]. We will show that indeed the following is true.

THEOREM 1.1. *Let $1 \le p < \infty$ and let w be a doubling weight. If $w^{-1/p} \in L^{p'}_{\mathrm{loc}}(\mathcal{D})$ then $\hat{E}^p_{w,k}(\mathcal{D})$ is a Banach space for any connected open set $\mathcal{D}$.*

DEFINITION 1.2. An open set $\mathcal{D}$ is an (ε, δ) domain if for all $x, y \in \mathcal{D}$, $|x - y| < \delta$, there exists a rectifiable curve γ connecting x, y such that γ lies in $\mathcal{D}$ and

$$l(\gamma) < \frac{|x-y|}{\varepsilon} \tag{1.1}$$

[1] Note that $w \in A_p^{\mathrm{loc}}(\mathcal{D}) \Rightarrow w \in A_p^K$ for all compact sets $K \subset \mathcal{D}$ in the notation of Wolff [35].

$$d(z,\partial \mathcal{D}) > \frac{\varepsilon|x-z|\,|y-z|}{|x-y|} \quad \forall z \in \gamma. \tag{1.2}$$

Here $l(\gamma)$ is the length of γ and $d(z,\partial\mathcal{D})$ is the distance between z and the boundary of $\mathcal{D}$. Moreover, we will write $d(Q,S) = \inf_{x\in Q, y\in S}|x-y|$, $d(Q) = d(Q,\partial\mathcal{D})$ and $d(z) = d(\{z\},\partial\mathcal{D})$.

In 1981, P. Jones [22] extended a famous extension theorem on Lipschitz domains to (ε,δ) domains.

THEOREM 1.3. *If $\mathcal{D}$ is a connected (ε,δ) domain and $1 \le p \le \infty$, then $C^\infty(\mathbb{R}^n)\cap L^p_k(\mathcal{D})$ is dense in $L^p_k(\mathcal{D})$ and $L^p_k(\mathcal{D})$ has a bounded extension operator. Moreover the norm of the extension operator depends only on ε, δ, k, p,* rad$(\mathcal{D})$*, and the dimension n.*

Furthermore he proved that

THEOREM 1.4. *If $\mathcal{D}$ is an (ε,∞) domain in $\mathbb{R}^n$, then $E^n_1(\mathcal{D})$ has a bounded extension operator, i.e., there exists $\Lambda\colon E^n_1(\mathcal{D}) \longrightarrow E^n_1(\mathbb{R}^n)$ such that $\Lambda f|_{\mathcal{D}} = f$ a.e. and $\|\Lambda\|$ is bounded.*

Recently, the author extended Theorems 1.3 and 1.4 to weighted Sobolev spaces when the weight is in A_p [9]. In this paper, we will extend these results further by relaxing the A_p assumption on the weight w to the following conditions on a bounded (ε,δ) domain $\mathcal{D}$:

w is doubling on $\mathbb{R}^n$, $w \in A^{\text{loc}}_p(\mathcal{D})$

w satisfies a local Poincaré inequality on $\mathcal{D}$.

Indeed, we prove that

THEOREM 1.5. *Let $\mathcal{D}$ be a bounded (ε,δ) domain. Let $1 \le p < \infty$ and let w be a doubling weight such that $w \in A^{\text{loc}}_p(\mathcal{D})$. Suppose further that*

$$\|f - f_{Q,w}\|_{L^p_w(Q)} \le C(A)l(Q)\|\nabla f\|_{L^p_w(Q)} \quad \forall f \in L^p_{w,1,\text{loc}}(\mathcal{D}) \tag{1.3}$$

for all cubes $Q \subset \mathcal{D}$ near $\partial\mathcal{D}$ such that $Ad(Q) \le l(Q) \le d(Q)/A$, $A > 0$ where $f_{Q,w} = \int_Q f\,dw/w(Q)$. Then given any $f \in L^p_{w,k}(\mathcal{D})$ (resp. $E^p_{w,k}(\mathcal{D})$) and $\eta > 0$, there exists $f_\eta \in C^\infty(\mathbb{R}^n)$ such that

$$\|f - f_\eta\|_{L^p_{w,k}(\mathcal{D})} < \eta \quad (\textit{resp. } \|\nabla^k(f - f_\eta)\|_{L^p_w(\mathcal{D})} < \eta).$$

Moreover, with the help of [11, Theorems 1.1 and 1.2] and the previous theorem, we show that:

THEOREM 1.6. *Let $\mathcal{D}$ be a bounded (ε,δ) domain. Let $1 \le p < \infty$ and w a doubling weight. If $w \in A^{\text{loc}}_p(\mathcal{D})$, $w^{-1/p} \in L^{p'}_{\text{loc}}(\mathbb{R}^n)$ and (3.3) holds, then there exists an extension operator Λ on $L^p_{w,k}(\mathcal{D})$ such that*

$$\|\Lambda f\|_{L^p_{w,k}(\mathbb{R}^n)} \le C\|f\|_{L^p_{w,k}(\mathcal{D})}.$$

Moreover, if in addition that $\mathcal{D}$ is a bounded (ε,∞) domain, then there exists an extension operator Λ' on $E^p_{w,k}(\mathcal{D})$ such that

$$\|\nabla^k\Lambda' f\|_{L^p_w(\mathbb{R}^n)} \leq C\|\nabla^k f\|_{L^p_w(\mathcal{D})}.$$

REMARK 1.7. (a) Let $M \subset \partial\mathcal{D}$ and $1 \leq p < \infty$. It is easy to see that if $w(x) = \operatorname{dist}(x,M)^\alpha$, $\alpha \in \mathbb{R}$, then it follows from the non-weighted Poincaré inequality that

$$\|f - f_Q\|_{L^p_w(Q)} \leq Cl(Q)\|\nabla f\|_{L^p_w(Q)} \quad \forall f \in L^p_{w,1,\mathrm{loc}}(\mathcal{D}) \tag{1.4}$$

for all cubes Q with $l(Q)$ comparable to $d(Q)$. Moreover, it is clear that $w \in A^{\mathrm{loc}}_p(\mathcal{D})$. Hence it follows from Theorem 1.5 that $C^\infty(\mathbb{R}^n) \cap L^p_{w,k}(\mathcal{D})$ is dense in $L^p_{w,k}(\mathcal{D})$ when $w(x) = \operatorname{dist}(x,M)^\alpha$ is doubling (note that (1.4) implies (1.3)). Thus when w is doubling and $\mathcal{D}$ is a bounded (ε,δ) domain, we obtain those density theorems in [23].

(b) Furthermore, if $w(x) = s\big(\operatorname{dist}(x,M)\big)$ where s is a positive and continuous function on positive real numbers that satisfies certain properties described in Kufner [23] or [17], similar conclusion can be obtained by Theorem 1.5 if we know that w is doubling.

(c) We do not know exactly when will the weights w defined as above will be doubling. However, in the case that M is just a finite subset of $\partial\mathcal{D}$, it is easy to see that $\operatorname{dist}(x,M)^\alpha$ is doubling if and only if $\alpha > -n$. For more details, refer to [15].

REMARK 1.8. (a) Let w be as in Remark 1.7. If in addition that $w^{-1/p} \in L^{p'}_{\mathrm{loc}}(\mathbb{R}^n)$, then we can apply Theorem 1.6 to get extension operator for $L^p_{w,k}(\mathcal{D})$ or $E^p_{w,k}(\mathcal{D})$. This overlaps some results in [17].

(b) The assumption that $w^{-1/p} \in L^{p'}_{\mathrm{loc}}(\mathbb{R}^n)$ in Theorem 1.6 is somewhat too strong. Indeed, we need only to assume that $w^{-1/p} \in L^{p'}(\mathcal{D})$. For the details, see [10]. Note that when $\mathcal{D}$ is a bounded (ε,∞) domain, $w \in A^{\mathrm{loc}}_p(\mathcal{D})$ and (3.3) holds, it follows from [14, Corollary 1.5] that $f \in E^p_{w,k}(\mathcal{D})$ if and only if $f \in L^p_{w,k}(\mathcal{D})$.

Finally, when the weights are of the form as in Remark 1.7(a), we are able to obtain extension theorems similar to Theorems 1.4 and 1.5 in [9]; see Remark 4.3.

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2. **Preliminaries.** In what follows, C denotes various positive constants, they may differ even in a same string of estimates. Moreover, sometimes, we will use $C(\alpha,\beta,\ldots)$ instead of C to emphasize that the constant is depending on $\alpha,\beta,\ldots$. Following [22], we say that two cubes touch if a face of one cube is contained in a face of the other. In particular, the union of two touching cubes of equal size is a rectangle.

First, let us state a theorem on polynomials.

THEOREM 2.1 ([9, LEMMA 2.3]). *Let F, Q be cubes such that $F \subset Q$ and $|F| > \gamma|Q|$. If w is a doubling weight, $1 \leq q < \infty$, and p is a polynomial of degree m, then*

$$\|p\|_{L^q_w(E)} \leq C(\gamma,m,n,w)\left(\frac{w(E)}{w(F)}\right)^{1/q}\|p\|_{L^q_w(F)}$$

for all measurable sets $E \subset Q$.

Next, the following lemma is indeed a special case of a result in [12].

LEMMA 2.2 ([12, THEOREM 2.1]). *Let f be a measurable function on $\mathbb{R}^n$ and let w be a doubling weight. Also, let $1 \le p \le \infty$, $k \in \mathbb{N}$ and $L > 0$. For each cube Q in $\mathbb{R}^n$, let $a(f,Q)$ be a polynomial of degree k associated to f on Q for each cube Q. Suppose that $\{Q_i\}_{i=0}^{l}$ is a sequence of cubes such that $Q_i \cap Q_{i+1}$ contains a cube Q^i with $|Q^i| \ge L \max\{|Q_i|, |Q_{i+1}|\}$ for each $i = 0, 1, \ldots, l-1$. Then*

$$\|f - a(f,Q_0)\|_{L^p_w(Q_l)} \le C \sum_i \|f - a(f,Q_i)\|_{L^p_w(Q_i)} \tag{2.1}$$

where C depends only on L, l, w, k, p and the dimension n.

PROOF OF THEOREM 1.1. We will modify the proof of [26, Theorem 1.1.13.1] and [9, Theorem 4.9].

Let Q_0 be a Whitney cube in $\mathcal{D}$ and let $\{\Omega_i\}$ be a sequence of open connected sets which are the interiors of finite unions of touching Whitney cubes of $\mathcal{D}$ (when $\mathcal{D} = \mathbb{R}^n$, just take $\{\Omega_i\}$ be a sequence of nested cubes) such that $Q_0 \subset \Omega_i$, $\bar{\Omega}_i \subset \Omega_{i+1}$, $\bigcup_i \Omega_i = \mathcal{D}$.

Given any Cauchy sequence $\{u_j\} \subset E^p_{w,k}(\mathcal{D})$, and any cube Q in $\mathcal{D}$, let $P(Q,u_j)$ be the unique polynomial of degree $< k$ such that $\int_Q D^\beta\big(u_j - P(Q,u_j)\big)\,dx = 0$ for all $|\beta| < k$. Since

$$\begin{aligned}\left\|D^\beta\big(u_j - u_l - P(Q,u_j - u_l)\big)\right\|_{L^1(Q)} &= \left\|D^\beta\Big(u_j - u_l - \big(P(Q,u_j) - P(Q,u_l)\big)\Big)\right\|_{L^1(Q)} \\ &\le C l(Q)^{k-|\beta|}\|\nabla^k(u_j - u_l)\|_{L^1(Q)}\end{aligned}$$

for all cubes Q in $\mathcal{D}$ by the unweighted Poincaré inequality, we have if $P_j = P(Q_0,u_j)$,

$$\begin{aligned}\left\|D^\beta\big(u_j - u_l - (P_j - P_l)\big)\right\|_{L^1(\Omega_i)} &\le C(\Omega_i)\|\nabla^k(u_j - u_l)\|_{L^1(\Omega_i)} \\ &\le C(\Omega_i)\|\nabla^k(u_j - u_l)\|_{L^p_w(\Omega_i)}\|w^{-1/p}\|_{L^{p'}(\Omega_i)} \\ &\le C(\Omega_i)\|\nabla^k(u_j - u_l)\|_{L^p_w(\Omega_i)},\end{aligned}$$

by the previous lemma, the Hölder inequality and the assumption on w. Hence if $v_j = u_j - P_j$, then $\{D^\beta v_j\}$ is a Cauchy sequence in $L^1(\Omega_i)$ for any i and $|\beta| \le k$. Thus it follows that for each i and β with $|\beta| < k$, there exists $h_{i,\beta} \in L^1(\Omega_i)$ such that $\|D^\beta v_j - h_{i,\beta}\|_{L^1(\Omega_i)} \to 0$ as $j \to \infty$. (When $|\beta| = k$, clearly there exists $h_\beta \in L^p_w(\mathcal{D})$ such that $\|D^\beta v_j - h_\beta\|_{L^p_w(\mathcal{D})} \to 0$ as $L^p_w(\mathcal{D})$ is complete.) Using subsequences, it is clear that $h_{i+1,\beta} = h_{i,\beta}$ a.e. on Ω_i. If we define h_β on $\mathcal{D}$ by setting $h_\beta = h_{i,\beta}$ on Ω_i, it follows that for each compact set $K \subset \mathcal{D}$ we have $h_\beta \in L^1(K)$ and $D^\beta v_j \to h_\beta$ in $L^1(K)$ for all $|\beta| \le k$ (for $|\beta| = k$, just use the Hölder inequality and the fact that $w^{-1/p} \in L^{p'}_{\mathrm{loc}}(\mathcal{D})$). Thus if $\varphi \in C_0^\infty(\mathcal{D})$, then (let us write h_β as h when $\beta = 0$)

$$\int_{\mathcal{D}} hD^\beta\varphi\,dx = \lim_{j\to\infty}\int_{\mathcal{D}} v_jD^\beta\varphi\,dx = \lim_{j\to\infty}(-1)^{|\beta|}\int_{\mathcal{D}}(D^\beta v_j)\varphi\,dx = (-1)^{|\beta|}\int_{\mathcal{D}} h_\beta\varphi\,dx.$$

Hence $D^\beta h = h_\beta$ exists. Moreover $D^\alpha h = \lim D^\alpha u_j$ when $|\alpha| = k$ since $D^\alpha u_j = D^\alpha v_j$. This completes the proof of the theorem.

COROLLARY 2.3. *Let $\mathcal{D}$ be an open connected set, let $\{u_j\}$ be a Cauchy sequence in $E^p_{w,k}(\mathcal{D})$ and let u be a function in $E^p_{w,k}(\mathcal{D})$ such that*

$$\|\nabla^k(u_j - u)\|_{L^p_w(\mathcal{D})} \longrightarrow 0.$$

Then there exists a sequence of polynomials $\{P_j\}$ of degree $< k$ with $u_j - P_j \longrightarrow u$ in $L^1(K)$ for all compact sets K in $\mathcal{D}$.

PROOF. By the previous proof, we know $v_j = u_j - P_j \longrightarrow h$ in $L^1(K)$ for each compact set K in $\mathcal{D}$, and $\nabla^k u_j \longrightarrow \nabla^k h$ in $L^p_w(\mathcal{D})$. Since also $\nabla^k u_j \longrightarrow \nabla^k u$ in $L^p_w(\mathcal{D})$, we see that $\nabla^k(u - h) = 0$, so $u - h = P$ for some polynomial P of degree $< k$. Thus $u_j - P_j + P \longrightarrow h + P = u$ in $L^1(K)$.

Now we will state a well-known lemma; see for example, Theorem III.2 in [31].

LEMMA 2.4. *Let $k(x)$ be nonnegative and integrable on $\mathbb{R}^n$ and suppose $k(x)$ depends only on $|x|$ and decreases as $|x|$ increases. Then for all non-negative measurable functions f,*

$$\sup_{t>0} |f * k_t(x)| \leq C\|k\|_{L^1(\mathbb{R}^n)} Mf(x)$$

with C independent of x, f and k. Here $k_t(y) = t^{-n}k(y/t)$ and Mf is the Hardy-Littlewood maximal function of f.

Similar to A_p weights [27], [18], we have the following results.

LEMMA 2.5. *Let $1 < p < \infty$, and $w \in A_p^{\mathrm{loc}}(\mathcal{D})$. Then*

$$\|M(f\chi_K)\|_{L^p_w(K)} \leq C_K \|f\|_{L^p_w(K)} \tag{2.2}$$

for all compact sets K in $\mathcal{D}$.

PROOF. We will only prove it for the case when w is doubling.[2] It suffices to show that (2.2) holds for $K = Q_0$ for all cubes Q_0 in $\mathcal{D}$ such that $3Q_0 \subset \mathcal{D}$.

Let $\mu = \chi_{3Q_0}$, $\nu = \chi_{3Q_0} w$ and $\tilde{w} = \chi_{Q_0} w$. Note that $(\frac{d\mu}{d\nu})^{p'-1} = \chi_{3Q_0} w^{1-p'}$. Let $M_\mu h(x) = \sup \int_F h(y)\, d\mu / \mu(F)$ where the supremum is taken over all cubes F containing x. Let Q be any cube. We will now show that $\nu, \tilde{w}$ and M_μ satisfies the S_p condition [29]. Let $x \in Q_0 \cap Q$, we now consider two cases:

CASE (i) $Q \subset 3Q_0$. Then there exists a cube $F \subset Q$ and $x \in F$ such that $M_\mu \chi_{Q \cap 3Q_0} w^{1-p'}(x) \leq C \int_F w^{1-p'}\, dy / |F|$. Thus

$$M_\mu(\chi_{Q\cap 3Q_0} w^{1-p'})(x) \leq C\Big(\frac{1}{|F|}\int_F w\, dy\Big)^{1-p'} \quad \text{since } w \in A_p^{\mathrm{loc}}(\mathcal{D})$$

$$= C\Big(\frac{1}{w(F)}\int_F w^{-1} w\, dy\Big)^{p'-1} \leq C\big(M_w(\chi_{Q\cap 3Q_0} w^{-1})(x)\big)^{p'-1}. \tag{2.3}$$

[2] The idea of this proof was provided by the referee.

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Hence

$$\begin{aligned}\int_Q [M_\mu(\chi_{Q\cap 3Q_0} w^{1-p'})(x)]^p \, d\tilde{w}(x) &= \int_{Q\cap Q_0} [M_\mu(\chi_{Q\cap 3Q_0} w^{1-p'})(x)]^p w(x)\,dx \\ &\le C \int_{Q\cap 3Q_0} [M_w(\chi_{Q\cap 3Q_0} w^{-1})(x)]^{p'} w(x)\,dx \\ &\le \int_{Q\cap 3Q_0} (w^{-1})^{p'} w(x)\,dx \\ &= \int \chi_Q \Big(\frac{d\mu}{d\nu}\Big)^{p'-1} \nu(x)\,dx \qquad (2.4)\end{aligned}$$

since w is doubling[3] on $\mathbb{R}^n$; see for example [21].

CASE (ii). Q is not contained in $3Q_0$. Since there is nothing to prove when $Q\cap Q_0 = \emptyset$, we may assume $3^n|Q\cap 3Q_0| \ge |3Q_0|$. Thus

$$\begin{aligned}\int_Q [M_\mu(\chi_{Q\cap 3Q_0} w^{1-p'})(x)]^p \, d\tilde{w}(x) &\le \int_{Q_0} [M_\mu(\chi_{3Q_0} w^{1-p'})(x)]^p w(x)\,dx \\ &\le C\int_{3Q_0} w^{1-p'}(x)\,dx \le \int_{Q\cap 3Q_0} w^{1-p'}(x)\,dx\end{aligned}$$

since $w \in A_p^{\mathrm{loc}}(\mathcal{D})$. Hence by Theorem A of [29], we have

$$\begin{aligned}\|M(\chi_{Q_0} f)\|_{L^p_w(Q_0)} &= \|M_\mu(\chi_{Q_0} f)\|_{L^p_w(Q_0)} = \|M_\mu(\chi_{Q_0} f)\|_{L^p_{\tilde{w}}(\mathbb{R}^n)} \\ &\le \|\chi_{Q_0} f\|_{L^p_\nu(\mathbb{R}^n)} = C\|f\|_{L^p_w(Q_0)}\end{aligned}$$

and hence (2.2) holds for $K = Q_0$.

LEMMA 2.6. *Let $1 \le p < \infty, w \in A_p^{\mathrm{loc}}(\mathcal{D})$ and let $\xi \in C_0^\infty$ be a non-negative decreasing radial function with support in $\{x \in \mathbb{R}^n : |x| \le 1\}$ and $\int \xi(x)\,dx = 1$. Then for $f \in L^p_w(\mathcal{D})$, $f * \xi_t \longrightarrow f$ in $L^p_w(K)$ as $t \longrightarrow 0$ for all compact sets K in $\mathcal{D}$. Moreover, if $f \in L^p_{w,k}(\mathcal{D})$ then $f * \xi_t \longrightarrow f$ in $L^p_{w,k}(K)$ for all compact sets K in $\mathcal{D}$.*

PROOF. When $1 < p < \infty$, it follows from Lemmas 2.4 and 2.5 and the Lebesgue dominated convergence theorem. Now if $p = 1$, given any compact set $K \subset \mathcal{D}$, let us first choose a continuous function g such that

$$\|f - g\|_{L^1_w(K^s)} \le \eta \qquad (2.5)$$

where $K^s = \{x + y : |y| \le s, x \in K\}$, and s is chosen so that $K^s \subset \mathcal{D}$. Next since g is continuous, there exists $L > 0$ such that $|g(x) - g(y)| < \eta$ for $x, y \in K^s$ and $|x - y| \le L$. Next if $sB = \{x \in \mathbb{R}^n : |x| \le s\}$ and $0 < t < s$,

$$\begin{aligned}\|f * \xi_t - f\|_{L^1_w(K)} &\le \int_K \int_{sB} |f(x-y) - f(x)|\xi_t(y)\,dy w(x)\,dx \\ &\le \int_K \int_{sB} |f(x-y) - g(x-y)|\xi_t(y)\,dy w(x)\,dx \\ &\quad + \int_K \int_{sB} |g(x-y) - g(x)|\xi_t(y)\,dy w(x)\,dx \\ &\quad + \int_K \int_{sB} |g(x) - f(x)|\xi_t(y)\,dy w(x)\,dx \\ &= I + II + III.\end{aligned}$$

[3] However, the theorem can be proved without assuming w is doubling *i.e.*, assuming only $w \in A_p^{\mathrm{loc}}(\mathcal{D})$.

However, $II \leq w(K)\eta$ when $0 < t < s \leq L$ and

$$III = \int_K |g(x) - f(x)| w(x)\, dx \leq \eta$$

by (2.5). Finally, note that

$$\begin{aligned} I &\leq \int_K \int_{K^s} |f(y) - g(y)| \xi_t(x-y)\, dy w(x)\, dx \\ &\leq \int_{K^s} \int_K \xi_t(x-y) w(x)\, dx |f(y) - g(y)|\, dy \\ &\leq C \int_{K^s} M(w\chi_K)(y) |f(y) - g(y)|\, dy \\ &\leq C \|f - g\|_{L^1_w(K^s)} \leq C(K)\eta. \end{aligned}$$

Lemma 2.6 now follows from the fact that $D^\alpha(f * \xi_t) = (D^\alpha f) * \xi_t$.[4]

THEOREM 2.7. *Let* $1 \leq p < \infty$ *and* $w \in A_p^{\mathrm{loc}}(\mathcal{D})$. *Then for all compact sets* K *in* $\mathcal{D}$,

$$\|f - a(f, Q)\|_{L^p_w(Q)} \leq C(K) l(Q) \|\nabla f\|_{L^p_w(Q)} \tag{2.6}$$

for all $f \in L^p_{w,1,\mathrm{loc}}(\mathcal{D})$ *and cube* $Q \subset K$ *where* $a(f, Q) = \int_Q f\, dx / |Q|$ *or* $\int_Q f\, dw / w(Q)$.

PROOF. Let K be any compact set in $\mathcal{D}$. First, note that it suffices to show that (2.6) holds with $a(f, Q) = f_Q = \int_Q f\, dx / |Q|$. However,

$$|f(x) - f_Q| \leq \frac{1}{|Q|} \int_Q |f(x) - f(y)|\, dy \leq C \int_Q \frac{|\nabla f(y)|}{|x - y|^{n-1}}\, dy$$

for $x \in Q, f \in C^\infty(\mathbb{R}^n)$ (see [33, Proposition 4.2]). Hence if $f \in C^\infty(\mathbb{R}^n)$ it suffices to show that

$$\left\| \int_Q \frac{g(y)}{|\cdot - y|^{n-1}}\, dy \right\|_{L^p_w(Q)} \leq C(K) l(Q) \|g\|_{L^p_w(Q)} \tag{2.7}$$

for all cubes $Q \subset K$. However, in the case $1 < p < \infty$, (2.7) is just a consequence of Lemma 2.5. Moreover, the case $p = 1$ follows immediately from the fact that $w \in A_1^{\mathrm{loc}}(\mathcal{D})$. Finally, with the help of Lemma 2.6, by similar argument as the proof of Theorem 4.3 in [9], our assertion follows.

Next we will state a theorem which is similar to [26, Theorem 1.1.2.1] and [9, Theorem 4.2]. Since it can be proved by very similar method as the proof of [9, Theorem 4.2] with the help of Lemma 2.6 and Theorem 2.7, we will omit the proof.

THEOREM 2.8. *Let* $\mathcal{D}$ *be any open set in* $\mathbb{R}^n$ *and let* $1 \leq p < \infty, w \in A_p^{\mathrm{loc}}(\mathcal{D})$. *If* $f \in E^p_{w,k}(\mathcal{D})$, *then*

$$\int_K |D^\gamma f|^p\, dw < \infty \quad \textit{for all compact sets } K \subset \mathcal{D},\ \forall 0 \leq |\gamma| \leq k.$$

[4] For the case $p = 1$, indeed we just modify the proof of Lemma 8 in [28].

3. **Density theorems.** Let $\mathcal{D}$ be an (ε,δ) domain, we will decompose $\mathcal{D} = \cup \mathcal{D}_\alpha$ into connected components and define

$$r = \operatorname{rad}(\mathcal{D}) = \inf_\alpha \inf_{x\in\mathcal{D}_\alpha} \sup_{y\in\mathcal{D}_\alpha} |x-y|.$$

We will assume $r > 0$ in most cases. Then for any $x \in \mathcal{D}$, there is a point y in the same component with $|x-y| \geq \frac{3r}{4}$. Note that we always have $r > 0$ when $\mathcal{D}$ is an (ε,∞) domain since $\mathcal{D}$ is then connected.

Let us recall that two cubes touch if a face of one cube is contained in a face of the other. In particular, the union of two touching cubes of equal size is a rectangle. A collection of cubes $\{S_i\}_{i=0}^m$ is called a *chain* if S_i touches S_{i+1} for all i.

Next let us recall some properties of the cubes in the Whitney decomposition of an open set $\mathcal{D}$ [31]. Since these properties are well-known, we will often make use of them without explicitly mentioning them.

$$\begin{aligned} l(Q) &= 2^{-k} \quad \text{for some } k \in \mathbb{Z}, \\ Q_1^o \cap Q_2^o &= \emptyset \quad \text{if } Q_1 \neq Q_2, \\ 1/4 \leq \frac{l(Q_1)}{l(Q_2)} &\leq 4 \quad \text{if } Q_1 \cap Q_2 \neq \emptyset, \\ 1 \leq \frac{d(Q)}{l(Q)} &\leq 4\sqrt{n}. \end{aligned}$$

The purpose of this section is to prove the density theorem.

PROOF OF THEOREM 1.5. Our proof is similar to that of [22] and [9]. Let $\varrho = 2^{-m}, m \in \mathbb{Z}_+$. Let W_1 be the Whitney decomposition of $\mathcal{D}$. Define

$\mathfrak{R}' = \{\text{dyadic cubes } R \text{ with edgelength } \varrho, R \subset \mathcal{D}\}$ and

$\mathfrak{R} = \{R \in \mathfrak{R}' : R \subset S \text{ for some } S \in W_1, l(S) \geq 32n^3\varrho/\varepsilon\}$.

Moreover, for each $R \in \mathfrak{R}$ let $\tilde{R}, \tilde{\tilde{R}}$ be cubes concentric with R with sides parallel to the axes and $l(\tilde{R}) = 1281n^4\varrho/\varepsilon^2$ and $l(\tilde{\tilde{R}}) = 2562n^4\varrho/\varepsilon^2$. For $s > 0$, let $\mathcal{D}_s = \{x \in \mathcal{D} : d(x) \geq s\}$. First, let us make the following two observations.

(I) $\mathcal{D} \subset \bigcup_{R\in\mathfrak{R}} \tilde{R}$ provided $\operatorname{rad}(\mathcal{D}) > 0$ and ϱ is small enough.

(II) Let $\mathcal{D}$ be an (ε,δ) domain with $\operatorname{rad}(\mathcal{D}) > 0$ and let $s = 3203n^5\varrho/\varepsilon^3 < \delta$. Then for all $R_0, R_j \in \mathfrak{R}$ with $\tilde{\tilde{R}}_0 \cap \tilde{\tilde{R}}_j \neq \emptyset$ and $\tilde{R}_0 \cap (\mathcal{D} \setminus \mathcal{D}_{2s}) \neq \emptyset$, there exists a chain $G_{0,j} = \{R_0 = S_1, S_2, \ldots, S_m = R_j\}$ in $\mathfrak{R}'$ connecting R_0, R_j with $m \leq C$ that depends only on ε, δ and n, and $\cup G_{0,j} \subset \mathcal{D} \setminus \mathcal{D}_{3s}$, $d(\cup G_{0,j}) \geq 20n^2\rho$.

(I) is first stated in [22] without proof. Nevertheless, the reader can refer to the proof of Theorem 6.1 in [9]. A similar conclusion as (II) can indeed be found in [22, Lemma 4.1] or [9]. However, since (II) is slightly stronger than the conclusion in [22] or [9], we will prove it.

First note that since $d(R_0, R_j) \leq \sqrt{n}(2561n^4\rho/\varepsilon^2) < \delta$, there exists γ connecting R_0, R_j which satisfies (1.1) and (1.2). Next if $z \in \gamma$, we will show that $d(z, \mathcal{D}_{3s}) > \sqrt{n}\rho$.

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First, we have

$$d(z,R_0)\le l(\gamma)<d(R_0,R_j)/\varepsilon\le 2561n^5\rho/\varepsilon^3,$$
$$d\big(R_0,(\mathcal{D}_{2s})^c\big)\le\sqrt{n}(640n^4\rho/\varepsilon^2)\le 640n^5\rho/\varepsilon^2$$

as $\tilde R_0\cap(\mathcal{D}_{2s})^c\ne\emptyset$. Moreover,

$$\begin{aligned}d(R_0,\mathcal{D}_{3s})&\ge d\big((\mathcal{D}_{2s})^c,\mathcal{D}_{3s}\big)-d\big(R_0,(\mathcal{D}_{2s})^c\big)-\sqrt{n}l(R_0)\\&\ge 3203n^5\rho/\varepsilon^3-640n^5\rho/\varepsilon^2-\sqrt{n}\rho\\&\ge 2562n^5\rho/\varepsilon^3.\end{aligned}$$

Next, without loss of generality, we may assume that $d(z,R_0)\le d(z,R_j)$. We now consider two cases:

CASE (i). $d(z,R_0)\le 42n^2\varrho/\varepsilon$. Then $d(z)\ge 32n^3\varrho/\varepsilon-42n^2\varrho/\varepsilon\ge 22n^2\varrho/\varepsilon$. (Note that we may restrict ourself to the case $n\ge 2$.)

CASE (ii). $d(z,R_0)>42n^2\varrho/\varepsilon$. Then by (1.2),

$$d(z)\ge\frac{\varepsilon d(z,R_0)d(z,R_j)}{d(R_0,R_j)}\ge 21n^2\varrho.$$

Finally let us note that an appropriate subcollection of $\{R\in\Re':R\cap\gamma\ne\emptyset\}$ will provide us the required chain. Moreover, $m\le C$ as $l(\gamma)\le d(R_0,R_j)/\varepsilon$.

Now, given $f\in L^p_{w,k}(\mathcal{D})$, we will let $P_j=P(R_j)$ be the unique polynomial of degree $k-1$ such that

$$\int_{R_j}D^\alpha\big(f-P(R_j)\big)\,dw=0,\quad 0\le|\alpha|\le k-1.$$

Next let $R_0,R_j\in\Re$, R_0,R_j be as in (II). Suppose that $G_{0,j}$ is the chain connecting R_0,R_j guaranteed by (II). If $P_0=P(R_0)$ and $P_j=P(R_j)$, similar to the proof of [9, Lemma 6.3], by the triangle inequality, (1.3), Lemma 2.2 and the fact that $\varepsilon^3d(R)/10000n^5\le l(R)\le 20n^2d(R)$ for all $R\in\cup G_{0,j}$, we can show that

$$\|D^\alpha(P_0-P_j)\|_{L^p_w(R_0)}\le C\varrho^{k-|\alpha|}\|\nabla^kf\|_{L^p_w(\cup G_{0,j})}\quad\forall 0\le|\alpha|\le k\tag{3.1}$$

where C is independent of f, R_0, R_j and ϱ.

Next given $\eta>0$, let us choose $s>0$ such that $\|f\|_{L^p_{w,k}(\mathcal{D}\setminus\mathcal{D}_{3s})}\le\eta$. We then choose $\psi\in C^\infty$ such that $\chi_{\mathcal{D}_{2s}}\le\psi\le\chi_{\mathcal{D}_s}$ and $|D^\alpha\psi|\le C(\alpha)s^{-|\alpha|}$.

Recall that by Lemma 2.6, there exists $\xi\in C_0^\infty$ such that $\int\xi\,dx=1$ and

$$\|f-f*\xi_t\|_{L^p_{w,k}(\mathcal{D}_s)}\longrightarrow 0\quad\text{as } t\longrightarrow 0 \text{ for } f\in L^p_{w,k}(\mathcal{D}),\ \text{where } \xi_t(x)=t^{-n}\xi\Big(\frac{x}{t}\Big).$$

Thus we can choose $0<t<s/2$ such that

$$\|D^\alpha(f-f*\xi_t)\|_{L^p_w(\mathcal{D}_s)}=\|D^\alpha f-(D^\alpha f)*\xi_t\|_{L^p_w(\mathcal{D}_s)}\le\eta s^{k-|\alpha|},\quad 0\le|\alpha|\le k.\tag{3.2}$$

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For each $R_j \in \Re$, let us choose $\varphi_j \in C^\infty$ with $0 \le \varphi_j \le \chi_{\tilde{\tilde{R}}_j}$ such that $\sum_{R_j \in \Re} \varphi_j \equiv 1$ on $\bigcup_{R_j \in \Re} \tilde{R}_j$, $0 \le \sum_{R_j \in \Re} \varphi_j \le 1$ and $|D^\alpha \varphi_j| \le C\varrho^{-|\alpha|}$.

Fixing t and s, let $g_0 = \sum_{R_j \in \Re} P_j \varphi_j$, $g_1 = g_0(1-\psi)$ and $g_2 = (f * \xi_t)\psi$. Then clearly $g_0, g_1, g_2 \in C^\infty(\mathbb{R}^n)$. We now show that $\|f - (g_1 + g_2)\|_{L^p_{w,k}(\mathcal{D})} \le C\eta$. First, we will show that $\|f - (g_1 + g_2)\|_{L^p_{w,k}(\mathcal{D}_{2s})} \le C\eta$. Let us note that since $g_1 \equiv 0$ on $\mathcal{D}_{2s}$ and $g_2 = f * \xi_t$ on $\mathcal{D}_{2s}$, for $|\alpha| \le k$ we have

$$\left\| D^\alpha\big(f - (g_1 + g_2)\big)\right\|_{L^p_w(\mathcal{D}_{2s})} = \|D^\alpha(f - f * \xi_t)\|_{L^p_w(\mathcal{D}_{2s})} \le C\eta \quad \text{by (3.2).}$$

Next write

$$\begin{aligned} D^\alpha\big(f - (g_1 + g_2)\big) &= D^\alpha\big(\psi(f - f * \xi_t)\big) + D^\alpha\big((1-\psi)(f - g_0)\big) \\ &= \sum_{\beta \le \alpha} C_{\alpha,\beta} D^{\alpha-\beta}\psi D^\beta(f - f * \xi_t) + \sum_{\beta \le \alpha} C_{\alpha,\beta} D^{\alpha-\beta}(1-\psi) D^\beta(f - g_0) \\ &= A + B. \end{aligned}$$

Since $|D^{\alpha-\beta}\psi| \le Cs^{-|\alpha-\beta|}$, $0 \le \beta \le \alpha$ and $\psi \equiv 0$ on $(\mathcal{D}_s)^c$, we have $\|A\|_{L^p_w(\mathcal{D} \setminus \mathcal{D}_{2s})} \le C\eta$ by (3.2).

To complete the proof, we need only to prove that $\|B\|_{L^p_w(\mathcal{D}\setminus\mathcal{D}_{2s})} \le C\|\nabla^k f\|_{L^p_w(\mathcal{D}\setminus\mathcal{D}_{3s})}$. To this end, first note that if $\tilde{R}_0 \cap (\mathcal{D} \setminus \mathcal{D}_{2s}) \ne \emptyset$, $\tilde{\tilde{R}}_0 \cap \tilde{\tilde{R}}_j \ne \emptyset$ then by the triangle inequality and (3.1),

$$\begin{aligned} \sum_{R_j \in \Re} \left\| D^\beta\big((P_0 - P_j)\varphi_j\big)\right\|_{L^p_w(R_0)} &\le C \sum_{\tilde{\tilde{R}}_0 \cap \tilde{\tilde{R}}_j \ne \emptyset} \sum_{\gamma \le \beta} l(R_0)^{-|\gamma|} \|D^{\beta-\gamma}(P_0 - P_j)\|_{L^p_w(R_0)} \\ &\le C \sum_{\tilde{\tilde{R}}_0 \cap \tilde{\tilde{R}}_j \ne \emptyset} \varrho^{k-|\beta|} \|\nabla^k f\|_{L^p_w(\cup G_{0,j})}. \end{aligned} \tag{3.3}$$

Also, note that

$$|D^\beta(f - g_0)| = \left| D^\beta\Big(f - \sum P_j \varphi_j\Big)\right| \le |D^\beta(f - P_0)| + \left| D^\beta \sum_{R_j \in \Re} (P_0 - P_j)\varphi_j \right|. \tag{3.4}$$

We now consider two cases:

CASE (i). $\beta < \alpha$. Then $D^{\alpha-\beta}(1-\psi) = 0$ on $\mathcal{D} \setminus \mathcal{D}_s$ and hence

$$\begin{aligned} &\|D^{\alpha-\beta}(1-\psi) D^\beta(f - g_0)\|^p_{L^p_w(\mathcal{D}\setminus\mathcal{D}_{2s})} \\ &\quad \le Cs^{-|\alpha-\beta|p} \sum_{R_0 \in \Re, R_0 \cap (\mathcal{D}_s \setminus \mathcal{D}_{2s}) \ne \emptyset} [\varrho^{k-|\beta|} \|\nabla^k f\|_{L^p_w(R_0)}]^p \\ &\qquad + Cs^{-|\alpha-\beta|p} \sum_{R_0 \in \Re, R_0 \cap (\mathcal{D}_s \setminus \mathcal{D}_{2s}) \ne \emptyset} \sum_{\tilde{\tilde{R}}_0 \cap \tilde{\tilde{R}}_j \ne \emptyset} [\varrho^{k-|\beta|} \|\nabla^k f\|_{L^p_w(\cup G_{0,j})}]^p \end{aligned}$$

by (3.4) and (3.3) since $\mathcal{D}_s \setminus \mathcal{D}_{2s} \subset \bigcup_{R_0 \in \Re} R_0$. Next note that $\|\sum_{R_0 \in \Re} \sum_{\tilde{\tilde{R}}_j \cap \tilde{\tilde{R}}_0 \ne \emptyset} \chi_{\cup G_{0,j}}\|_{L^\infty} \le C$ where C is independent of ϱ. Moreover by (II), if $R_0 \cap (\mathcal{D}_s \setminus \mathcal{D}_{2s}) \ne \emptyset$, $\tilde{\tilde{R}}_j \cap \tilde{\tilde{R}}_0 \ne \emptyset$, then $\cup G_{0,j} \subset \mathcal{D} \setminus \mathcal{D}_{3s}$, and in particular $R_0 \subset \mathcal{D} \setminus \mathcal{D}_{3s}$. Hence if $\alpha > \beta$ (then $|\beta| < k$),

$$\|D^{\alpha-\beta}(1-\psi) D^\beta(f - g_0)\|_{L^p_w(\mathcal{D}\setminus\mathcal{D}_{2s})} \le Cs^{-|\alpha-\beta|} \varrho^{k-|\beta|} \|\nabla^k f\|_{L^p_w(\mathcal{D}\setminus\mathcal{D}_{3s})} \le C\eta.$$

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CASE (ii). $\beta = \alpha$. First observe that for each $R_0 \in \Re$, $\tilde{R}_0 \cap (\mathcal{D} \setminus \mathcal{D}_{2s}) \neq \emptyset$, similar to (3.3) we have

$$\sum_{R_j \in \Re} \left\| D^\alpha\big((P_0 - P_j)\varphi_j\big)\right\|_{L^p_w(\tilde{R}_0)} \leq C \sum_{R_j \in \Re, \tilde{\tilde{R}}_0 \cap \tilde{\tilde{R}}_j \neq \emptyset} \varrho^{k-|\alpha|} \|\nabla^k f\|_{L^p_w(\cup G_{0,j})}$$

by Lemma 2.1. Thus

$$\begin{aligned}\left\|D^\alpha \sum P_j\varphi_j\right\|_{L^p_w(\tilde{R}_0)} &\leq \|D^\alpha P_0\|_{L^p_w(\tilde{R}_0)} + \left\|D^\alpha \sum (P_j - P_0)\varphi_j\right\|_{L^p_w(\tilde{R}_0)}\\ &\leq C\|D^\alpha P_0\|_{L^p_w(R_0)} + C \sum_{R_j \in \Re, \tilde{\tilde{R}}_0 \cap \tilde{\tilde{R}}_j \neq \emptyset} \varrho^{k-|\alpha|}\|\nabla^k f\|_{L^p_w(\cup G_{0,j})}\\ &\leq C\|D^\alpha f\|_{L^p_w(R_0)} + C\varrho^{k-|\alpha|}\|\nabla^k f\|_{L^p_w(R_0)}\\ &\quad + C\varrho^{k-|\alpha|} \sum_{R_j \in \Re, \tilde{\tilde{R}}_0 \cap \tilde{\tilde{R}}_j \neq \emptyset} \|\nabla^k f\|_{L^p_w(\cup G_{0,j})}.\end{aligned}$$

Note that again by (II), if $\tilde{R}_0 \cap (\mathcal{D} \setminus \mathcal{D}_{2s}) \neq \emptyset$ and $\tilde{\tilde{R}}_0 \cap \tilde{\tilde{R}}_j \neq \emptyset$ then $\cup G_{0,j} \subset \mathcal{D} \setminus \mathcal{D}_{3s}$, and in particular $R_0 \subset \mathcal{D} \setminus \mathcal{D}_{3s}$. Hence by the previous estimate,

$$\begin{aligned}\|D^\alpha(f - g_0)\|^p_{L^p_w(\mathcal{D}\setminus\mathcal{D}_{2s})} &\leq C\|D^\alpha f\|^p_{L^p_w(\mathcal{D}\setminus\mathcal{D}_{2s})}\\ &\quad + \sum_{R_0 \in \Re, \tilde{R}_0 \cap (\mathcal{D}\setminus\mathcal{D}_{2s}) \neq \emptyset} C\left\|D^\alpha \sum_{R_j \in \Re} P_j\varphi_j\right\|^p_{L^p_w(\tilde{R}_0)}\\ &\leq C\|D^\alpha f\|^p_{L^p_w(\mathcal{D}\setminus\mathcal{D}_{2s})} + C\|D^\alpha f\|^p_{L^p_w(\mathcal{D}\setminus\mathcal{D}_{3s})}\\ &\quad + C\varrho^{(k-|\alpha|)p}\|\nabla^k f\|^p_{L^p_w(\mathcal{D}\setminus\mathcal{D}_{3s})} \leq C\eta^p\end{aligned}$$

since $\| \sum_{R_0 \in \Re} \sum_{\tilde{\tilde{R}}_j \cap \tilde{\tilde{R}}_0 \neq \emptyset} \chi_{\cup G_{0,j}} \|_{L^\infty} < C$. Thus $\left\|D^\alpha\big(f - (g_1 + g_2)\big)\right\|_{L^p_w(\mathcal{D}\setminus\mathcal{D}_{2s})} \leq C\eta$.

Finally, if $f \in E^p_{w,k}(\mathcal{D})$, let us note that by Theorem 2.8, we have $f \in L^p_{w,k}(\mathcal{D}_s)$. We can then construct $g_1 + g_2$ as before since (3.2) still hold. One can just check through the proof and see that $g_1 + g_2$ satisfies our assertion.

4. Extension theorems.

First, let us state an extension theorem from [11].

THEOREM 4.1 ([11, THEOREMS 1.1 AND 1.2]). *Let $\mathcal{D}$ be an (ε, δ) domain. Let $1 \leq p < \infty$ and let w be a doubling weight such that*

$$\|f - f_{Q,w}\|_{L^p_w(Q)} \leq C_0 l(Q)\|\nabla f\|_{L^p_w(Q)} \quad \forall f \in \mathrm{Lip_{loc}}(\mathbb{R}^n) \tag{4.1}$$

for all cubes Q in $\mathcal{D}$ where $f_{Q,w} = \int_Q f\,dw/w(Q)$. Then there exists an extension operator Λ on $\mathcal{D}$ (i.e., $\Lambda f = f$ on $\mathcal{D}$ a.e.) such that

$$\|\Lambda f\|_{L^p_{w,k}(\mathbb{R}^n)} \leq C\|f\|_{L^p_{w,k}(\mathcal{D})}$$

for all $f \in \mathrm{Lip}^{k-1}_{\mathrm{loc}}(\mathbb{R}^n)$ $(= \{f : D^\alpha f \in \mathrm{Lip_{loc}}(\mathbb{R}^n)$ for all $|\alpha| < k\})$ where C depends only on ε, δ, $\mathrm{rad}(\mathcal{D})$, p, w, k, C_0 and n. Moreover, if $\mathcal{D}$ is an (ε, ∞) domain, then there exists another extension operator Λ' on $\mathcal{D}$ such that

$$\|\nabla^k \Lambda' f\|_{L^p_w(\mathbb{R}^n)} \leq C\|\nabla^k f\|_{L^p_w(\mathcal{D})}$$

for all $f \in \mathrm{Lip}^{k-1}_{\mathrm{loc}}(\mathbb{R}^n)$ where C depends only on ε, p, w, k, C_0 and n.

REMARK 4.2. Checking through the proof of Theorem 1.1 in [11], let us note that indeed we need only to assume (4.1) holds for all cubes Q near $\partial\mathcal{D}$ such that $l(Q)$ is comparable to $d(Q)$ for the first part. However, for the second part, we need to assume in addition that $\mathcal{D}$ is bounded.

With the help of the preceding theorem and the density theorem in the previous section, we can now prove our extension theorem.

PROOF OF THEOREM 1.6. First given $f \in L^p_{w,k}(\mathcal{D})$, by Theorem 1.5, there exists a sequence $\{f_j\} \subset C^\infty(\mathbb{R}^n)$ such that $f_j \longrightarrow f$ in $L^p_{w,k}(\mathcal{D})$. Next since $L^p_{w,k}(\mathbb{R}^n)$ is a Banach space, the first part of the theorem now follows from the preceding theorem (see Remark 4.2). Now let $f \in E^p_{w,k}(\mathcal{D})$. By Theorem 1.5 there exists $\{f_j\} \subset C^\infty(\mathbb{R}^n)$ such that $\|\nabla^k f_j - \nabla^k f\|_{L^p_w(\mathcal{D})} \longrightarrow 0$. Then $\{\Lambda' f_j\}$ is a Cauchy sequence in $E^p_{w,k}(\mathbb{R}^n)$ by the preceding theorem. Since $E^p_{w,k}(\mathbb{R}^n)$ is complete by Theorem 1.1, there exists $g \in E^p_{w,k}(\mathbb{R}^n)$ such that $\nabla^k\Lambda' f_j \longrightarrow \nabla^k g$ in $L^p_w(\mathbb{R}^n)$. Since $\Lambda' f_j = f_j$ on $\mathcal{D}$, we obtain $\|\nabla^k g - \nabla^k f\|_{L^p_w(\mathcal{D})} = 0$. Hence there exists a polynomial P of degree $< k$ such that $g = f + P$ a.e. on $\mathcal{D}$. Define $\Lambda' f = g - P$. Then $\Lambda' f = f$ a.e. on $\mathcal{D}$. Also, $\nabla^k\Lambda' f = \nabla^k g$ and consequently $\nabla^k\Lambda' f_j \longrightarrow \nabla^k\Lambda' f$ in $L^p_w(\mathbb{R}^n)$. The proof of the theorem is now complete by passing to the limit.

REMARK 4.3. (a) Let $\mathcal{D}$ be a bounded (ε,∞) domain with $r = \mathrm{rad}(\mathcal{D})$ and let Ω be a bounded open set containing $\mathcal{D}$. Let W_2 be the collection of cubes in the Whitney decomposition of $(\mathcal{D}^c)^o$ and define

$$W_3 = \Big\{Q \in W_2 : l(Q) \le \frac{\varepsilon r}{16nL}\Big\}, \quad L = 2^{-m},\ m \in \mathbb{Z}_+,$$

where L is chosen so that $\Omega \subset (\bigcup_{Q\in W_3} Q) \cup \mathcal{D}$. Finally, when the weights are of the form as in Remark 1.7(a), we have better extension theorems.

THEOREM 4.4. *Let* $1 \le p_i < \infty$, $w_i = \mathrm{dist}(x, M_i)^{\alpha_i}$, $\alpha_i \in \mathbb{R}$, $M_i \subset \partial\mathcal{D}$ *such that* w_i *is doubling for* $i = 0, 1, \ldots, N$. *Let* Ω *be a bounded open set containing an* (ε,∞) *domain* $\mathcal{D}$ *and let L and r be defined as above. Suppose that* $k_i = 0$ *for* $0 \le i \le N_1$, $k_i = k > 0$ *for* $N_2 < i \le N$ *and* $0 < k_i < k$ *otherwise. Then there exist extension operators* Λ *and* Λ' *on* $\mathcal{D}$ *such that*

$$\|\Lambda f\|_{L^{p_i}_{w_i}(\mathbb{R}^n)} \le C_i\|f\|_{L^{p_i}_{w_i}(\mathcal{D})} \quad \textit{for } 0 \le i \le N_1$$

$$\|\nabla^{k_i}\Lambda f\|_{L^{p_i}_{w_i}(\Omega)} \le C_i\|\nabla^{k_i} f\|_{L^{p_i}_{w_i}(\mathcal{D})} \quad \textit{for } N_1 < i \le N$$

$$\|\nabla^{k_i}\Lambda' f\|_{L^{p_i}_{w_i}(\Omega)} \le C_i\|\nabla^{k_i} f\|_{L^{p_i}_{w_i}(\mathcal{D})} \quad \textit{for } 0 \le i \le N_2$$

$$\|\nabla^{k}\Lambda' f\|_{L^{p_i}_{w_i}(\mathbb{R}^n)} \le C_i\|\nabla^{k} f\|_{L^{p_i}_{w_i}(\mathcal{D})} \quad \textit{for } N_2 < i \le N$$

for all $f \in \mathrm{Lip}^{k-1}_{\mathrm{loc}}(\mathbb{R}^n)$. *Here* C_i *depends only on* ε, p_i, w_i, k_i, n, L *and* $\max_i k_i$. *(Unfortunately L usually depends on r, but there are cases where L is independent of r and consequently* C_i *is independent of r.)*

THEOREM 4.5. *Let* $1 \leq p_i < \infty$, $w_i = \mathrm{dist}(x, M_i)^{\alpha_i}$, $\alpha_i \in \mathbb{R}$, $M_i \subset \partial \mathcal{D}$ *such that* w_i *is doubling for* $i = 0, 1, \ldots, N$. *If* $\mathcal{D}$ *is an unbounded* (ε, ∞) *domain, then there exists an extension operator on* $\mathcal{D}$ *such that*

$$\|\nabla^{k_i} \Lambda f\|_{L^{p_i}_{w_i}(\mathbb{R}^n)} \leq C_i \|\nabla^{k_i} f\|_{L^{p_i}_{w_i}(\mathcal{D})}$$

for all i and $f \in \mathrm{Lip}^{k-1}_{\mathrm{loc}}(\mathbb{R}^n)$. *Here* C_i *depends only on* ε, w_i, p_i, $k_i n$ *and* $\max_i k_i$.

PROOF OF THEOREMS 4.4 AND 4.5. If $w(x) = \mathrm{dist}(x, M)^{\alpha}$ for $M \subset \mathcal{D}$, $\alpha \in \mathbb{R}$, let us make the following two observations:

$$\|f - f_Q\|_{L^p_w(Q)} \leq C(A) l(Q) \|\nabla f\|_{L^p_w(Q)} \tag{4.2}$$

$$\frac{1}{|Q|} \|f\|_{L^1(Q)} \leq C(A) w(Q)^{-1/p} \|f\|_{L^p_w(Q)} \tag{4.3}$$

for all cubes Q in $\mathcal{D}$ such that $Al(Q) \leq d(Q) \leq l(Q)/A$ for $A > 0$. We can now check through the proof of Theorems 1.4 and 1.5 in [9] using (4.2) and (4.3) as the substitute of the condition that $w \in A_p$ to obtain Theorems 4.4 and 4.5.

(b) In Theorem 4.4, if we assume in addition that $w^{-1/p} \in L^{p'}_{\mathrm{loc}}(\mathbb{R}^n)$, we can indeed replace $\mathrm{Lip}^{k-1}_{\mathrm{loc}}(\mathbb{R}^n)$ by $\cap E^{p_i}_{w_i,k_i}(\mathcal{D})$ as $C^\infty(\mathbb{R}^n) \cap \left(\bigcap E^{p_i}_{w_i,k_i}(\mathcal{D})\right)$ is dense in $\cap E^{p_i}_{w_i,k_i}(\mathcal{D})$. For the details, check through the proof of Theorem 6.1 in [9].

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