SEMIGROUPS IN WHICH ALL SUBSEMIGROUPS ARE LEFT IDEALS

NAOKI KIMURA, TAKAYUKI TAMURA, AND RUDOLPH MERKEL

1. Introduction. A semigroup S is a λ- [ρ-, σ-] semigroup if and only if each subsemigroup of S is a left [right, two-sided] ideal of S. Since the concept of ρ-semigroup is the dual of that of λ-semigroup, the results for ρ-semigroups are generally not stated explicitly. σ-semigroups are treated as a special case of λ-semigroups; in fact, a semigroup S is a σ-semigroup if and only if it is a λ-semigroup and a ρ-semigroup. The purpose of this paper is to determine the structure of λ- [ρ-, σ-] semigroups.

In Section 2, the idempotents of a λ-semigroup S are used to obtain a natural decomposition of S as the disjoint union of unipotent λ-semigroups. In Section 3, the structure theorem of unipotent λ-semigroups is proved. The structure of a general λ-semigroup follows in Section 4. The structure theorem of σ-semigroups in Section 5 is an application of the results of Section 3 and the dual theorems. Throughout this paper, $X \subset Y$ stands for $X \subseteq Y$.

The definitions imply

LEMMA 1. *If S is a λ- [ρ-, σ-] semigroup, then any subsemigroup of S as well as any homomorphic image of S is of the same type.*

2. Decomposition of a λ-semigroup into unipotent λ-semigroups. $\langle a\rangle$ is the subsemigroup of semigroup S generated by $a \in S$.

LEMMA 2. *S is a λ-semigroup if and only if $Sa \subset \langle a\rangle$ for all $a \in S$.*

Proof. By the definition of λ-semigroup, $Sa \subset S\langle a\rangle \subset \langle a\rangle$. Conversely, let T be a subsemigroup of S and $a \in T$. $Sa \subset \langle a\rangle \subset T$ so that T is a left ideal.

LEMMA 3. *$|\langle a\rangle| \leqslant 3$ for all $a \in S$; $\langle a\rangle$ contains an idempotent. If E is the set of idempotents of S, then every $e \in E$ is a right zero of S.*

Proof. Suppose $\langle a\rangle$ is not finite. Then

$$\langle a\rangle = \{a^n : n \text{ is a positive integer, } a^{n_1} \neq a^{n_2},\ n_1 \neq n_2\}$$

and $\langle a^2\rangle = \{a^{2k} : k \text{ is a positive integer}\}$ is a subsemigroup of $\langle a\rangle$. By Lemma 1, $a^3 = aa^2 \in \langle a\rangle\langle a^2\rangle \subset \langle a^2\rangle$, which is a contradiction. Thus, $\langle a\rangle$ is finite for $a \in S$. By (**1**, Theorem 1.9), $\langle a\rangle$ contains an idempotent.

Let $e \in E$. By Lemma 2, $Se \subset \langle e\rangle = \{e\} = \{ee\} \subset Se$. Thus, E is the set of right zeros of S and E is a right zero semigroup.

Let $a \in S$. Suppose p is the smallest positive integer such that $a^p = e \in E$

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and $p \geqslant 4$. By **(1**, Theorem 1.9**)**, $\langle a \rangle$ contains a cyclic subgroup K_a in which $a^p = e$ is the identity. Suppose $y \in K_a$. Since e is the identity in K_a and e is a right zero in S, $y = ye = e$. Thus, $K_a = \{e\}$ and $\langle a \rangle$ has period 1 and index p; that is,

$$\langle a \rangle = \{a, a^2, \ldots . a^{p-1}, a^p = e\}.$$

$T = \{a^2, a^4, a^5, \ldots, a^p\}$ is a subsemigroup of $\langle a \rangle$. Therefore

$$a^3 = aa^2 \in \langle a \rangle T \subset T,$$

which is a contradiction. Hence, $p \leqslant 3$.

LEMMA 4. *$xy = y$ if and only if $y \in E$.*

Proof. Suppose there is a $y \in S$ such that $xy = y$ for some $x \in S$. Then $x \in X = \{z \in S{:}zy = y\} \neq \emptyset$. Since X is a subsemigroup of S, it is a left ideal. Since X is a left ideal, $yx \in SX \subset X$. Since X is a subsemigroup, we have $x(yx) \in X$. By the definition of X, $y^2 = (xy)^2 = \{x(yx)\}y = y$. Thus, y lies in E and is a right zero. The converse is obvious.

Let e_x be the idempotent determined by $x \in S$. By Lemma 3, $\phi{:}S \to E$, $x\phi = e_x$, is a homomorphism. Clearly, ϕ is a mapping. First, $x\phi = x^2\phi$. From $x^p = e_x$ it follows that

$$e_x = e_x{}^2 = (x^p)^2 = (x^2)^p.$$

By Lemma 2, $xy \in Sy \subset \langle y \rangle$. By Lemmas 3 and 4, $xy = e_y$ or y^2. Thus,

$$(xy)\phi = \begin{Bmatrix} e_y\, \phi \\ y^2\, \phi \end{Bmatrix} = e_y = e_x\, e_y = (x\phi)(y\phi).$$

For $e \in E$, let $S(e) = \phi^{-1}(e)$. Since $S(e)$ is a subsemigroup of S, it is a left ideal. $S(e)$ is unipotent. Let $x \in S(e)$ so that $x^p = e$ for some positive integer p. Then $ex = x^px = xx^p = xe = e$. Thus, e is the zero in $S(e)$.

$S(e)$ is the maximal unipotent subsemigroup of S with e as its idempotent. Let M be a unipotent subsemigroup of S containing e. Suppose $x \in M$. Then $\langle x \rangle \subset M$. By Lemma 3, $\langle x \rangle$ contains an idempotent d. Since M is unipotent, $d = e$. Thus, $x \in S(e)$ and $M \subset S(e)$.

In summary we obtain

THEOREM 1. *If S is a λ-semigroup, then S is the union of the disjoint left ideals $S(e)$. In terms of the definitions of* **(1**, *p.* 25**)**, *a λ-semigroup S is the union of a band B, B a right zero semigroup, of unipotent λ-semigroups $S(e)$, and this is the greatest decomposition such that the factor semigroup is a band.*

DEFINITION 1. $S_p = \{x \in S{:}|\langle x \rangle| = p\}$; $S_p(e) = S_p \cap S(e)$.

LEMMA 5.

$$S(e) = S_1(e) \cup S_2(e) \cup S_3(e),$$
$$S = S_1 \cup S_2 \cup S_3, \textit{ disjoint union;}$$
$$S_1(e) = \{e\}, \qquad S_1 = E.$$

LEMMA 6. $xyz = e_z \in E$, $x, y, z \in S$.

Proof.

$$xyz = x(yz) = \begin{cases} xe_z = e_z, \\ xz^2 = \begin{cases} e_z z = e_z, \\ z^4 = e_z. \end{cases} \end{cases}$$

3. The structure of unipotent λ-semigroups. Let e be the unique idempotent of a unipotent λ-semigroup S; e is the zero of S. Moreover, by Lemma 5,

$$S = S_1(e) \cup S_2(e) \cup S_3(e),$$
$$S_1(e) = \{e\}, \quad S_2(e) = \{x \in S: x \neq e, x^2 = e\},$$
$$S_3(e) = \{x \in S: x^2 \neq e, x^3 = e\}.$$

We define

$$A = \{x \in S: x^2 \neq e\},$$
$$B = \{x \in S: x \neq e, x^2 = e, x = y^2 \text{ for no } y \in S\},$$
$$C = \{x \in S: x \neq e, x = y^2 \text{ for some } y \in S\}.$$

From these definitions we have

LEMMA 7.

$$S = A \cup B \cup C \cup \{e\}, \textit{ disjoint union},$$
$$A = S_3, \qquad B \cup C = S_2, \qquad C = \{x \in S: x = y^2, y \in A\}.$$

DEFINITION 2. *Let $I = \{0, 1\}$ be a set with two elements. Let $\mu: (A \cup B) \times A \to I$ and $\nu: A \to C$ be functions defined respectively by*

$$(x, y)\mu = \begin{cases} 1 & \textit{if } xy \neq e \\ 0 & \textit{if } xy = e \end{cases} \qquad \textit{and} \quad x\nu = x^2.$$

When $A = \emptyset$, μ and ν are null functions. μ and ν are well-defined functions such that $(a, a)\mu = 1$ and ν is surjective. Moreover, for all $x, y \in S$,

$$xy = \begin{cases} y\nu & \text{if } (x, y)\mu = 1, \\ e & \text{otherwise} \end{cases}$$

defines the product in S.

Define $xy = e$ if $A = \emptyset$. μ is obviously well defined. Since $a^2 \neq e$ for $a \in A$, $(a, a)\mu = 1$. By Lemma 7, ν is a surjection.

Let $x, y \in S$. By Lemma 2, $xy \in Sy \subset \langle y \rangle$. By Lemma 3, $xy = y$ or y^2 or e. By Lemma 4, $xy = y$ if and only if $y = e$.

Now, suppose $y \neq e$. Then $xy = y^2$ or e. By Lemma 7, $y \in B \cup C = S_2$ or A. If $y \in S_2$, then $y^2 = e$. There are three cases for $y \in A$:

(i) $x = e$. Then $xy = ey = e$.

(ii) $x \in C$. Then $x = z^2$, $z \in S$. By Lemma 6, $xy = z^2 y = e$.

(iii) $x \in A \cup B$. There are two subcases:

(α) $xy = e$. By the definition of μ, this is equivalent to $(x, y)\mu = 0$.

(β) $xy \neq e$. Again, by the definition of μ, this is equivalent to $(x, y)\mu = 1$. In this case we have, by the definition of ν, $xy = y^2 = y\nu$. Similarly, for the converse, $xy = y\nu$.

THEOREM 2. *Let S be a unipotent λ-semigroup with zero e. Then there is a family $\{A, B, C\}$ of disjoint subsets of S and, if $A \neq \emptyset$, there are functions $\mu: (A \cup B) \times A \to I$ and $\nu: A \to C$ such that:*

(1) *$S = A \cup B \cup C \cup \{e\}$, disjoint union;*

(2) *$(a, a)\mu = 1, a \in A$;*

(3) *ν is surjective, and so $|A| \geqslant |C|$;*

(4) *$xy = y\nu$ if $(x, y)\mu = 1$;*

(5) *$xy = e$ otherwise.*

Conversely, let A, B, C, $\{e\}$ be pairwise disjoint sets with $|A| \geqslant |C|$. Let $S = A \cup B \cup C \cup \{e\}$; if $A \neq \emptyset$, let $\mu: (A \cup B) \times A \to I$ be any function such that $(a, a)\mu = 1, a \in A$, and let $\nu: A \to C$ be any surjection. Define the binary operation m on S by

$$(x, y)m = \begin{cases} y\nu & \text{if } (x, y)\mu = 1, \\ e & \text{otherwise}. \end{cases}$$

Then the groupoid (S, m) is a unipotent λ-semigroup.

Proof. Only the converse remains to be proved. m is single-valued. Let $x, y, z \in S$. $(y, z)m = e$ or $z\nu$, where $(y, z)m = z\nu$ if $(y, z)\mu = 1$. Thus, $(y, z)m \in C \cup \{e\}$. Therefore $(x, (y, z)m)m = e$. Similarly, $(x, y)m \in C \cup \{e\}$ and $((x, y)m, z)m = e$. Hence, associativity holds.

Since $(e, e)m = e$, e is an idempotent. Suppose $x \in S$ is idempotent; that is, $(x, x)m = x$. Then $x = (x, x)m = (x, (x, x)m)m = e$. Hence, S is unipotent and $e \in \langle x \rangle$ for every $x \in S$.

Finally, let T be any subsemigroup of S and let $x \in T$, $y \in S$. Then

$$(y, x)m = e \in \langle x \rangle \subset T \quad \text{or} \quad (y, x)m = x\nu = (x, x)m \in \langle x \rangle \subset T.$$

Therefore $ST \subset T$ and S is a λ-semigroup.

DEFINITION 3. *The 6-tuple $\mathfrak{S} = (A, B, C, e, \mu, \nu)$ satisfying the conditions of Theorem 2 is called the structure set of λ-semigroup S.*

THEOREM 3. *Let $\mathfrak{S} = (A, B, C, e, \mu, \nu)$ and $\mathfrak{S}' = (A', B', C', e', \mu', \nu')$ be the structure sets of two unipotent λ-semigroups S and S' and let $\Sigma: S \to S'$ be a mapping. Then Σ is a homomorphism if and only if*

(1) *$B\Sigma \subset B' \cup C' \cup \{e'\}$,*

(2) *$C\Sigma \subset C' \cup \{e'\}$,*

(3) *$e\Sigma = e'$,*

(4) *$(x\nu)\Sigma = (x\Sigma)\nu'$ for every $x \in \Sigma^{-1}(A')$,*

(5) $(y, x)\mu = (y\Sigma, x\Sigma)\mu'$ *for* $x \in \Sigma^{-1}(A')$, $y \in \Sigma^{-1}(A' \cup B')$,

(6) $(y, x)\mu = 0$ *for* $x \in \Sigma^{-1}(A')$, $y \in A \cup B$, $y \notin \Sigma^{-1}(A' \cup B')$,

(7) $x\Sigma \notin A'$ *implies* $x^2\Sigma = e'$.

Proof. Note that (1), (2), and (3) are equivalent to

(1′) $\Sigma^{-1}(A') \subset A$,

(2′) $\Sigma^{-1}(B') \subset A \cup B$,

(3′) $\Sigma^{-1}(C') \subset A \cup B \cup C$.

Let Σ be a mapping of S into S' which satisfies the seven conditions of Theorem 3.

(i) By Theorem 2 and (1′), $x \notin A$ implies $yx = e$, $y \in S$, and $x\Sigma \notin A'$. By (3), $(yx)\Sigma = e\Sigma = e'$. By Theorem 2, $x\Sigma \notin A'$ implies $(y\Sigma)(x\Sigma) = e'$, $y\Sigma \in S'$.

(ii) By Theorem 2 and (1′), (2′), $x \in A$ and $y \notin A \cup B$ imply $yx = e$ and $y\Sigma \notin A' \cup B'$. By (3), $(yx)\Sigma = e\Sigma = e'$. By Theorem 2, $y\Sigma \notin A' \cup B'$ implies $(y\Sigma)(x\Sigma) = e'$.

(iii) Suppose $x \in A$ and $y \in A \cup B$. By Theorem 2, $yx = x\nu$ if $(y, x)\mu = 1$; $yx = e$ if $(y, x)\mu = 0$.

(α) Let $(y, x)\mu = 0$. By (3), $(yx)\Sigma = e\Sigma = e'$. If $x\Sigma \in A'$ and $y\Sigma \in A' \cup B'$, then, by (5), $0 = (y, x)\mu = (y\Sigma, x\Sigma)\mu'$. By Theorem 2, $(y\Sigma, x\Sigma)\mu' = 0$ implies $(y\Sigma)(x\Sigma) = e'$.

If $x\Sigma \notin A'$ or $y\Sigma \notin A' \cup B'$, then $(y\Sigma, x\Sigma)\mu'$ is not defined. But, by the definition of products in S', $(y\Sigma)(x\Sigma) = e'$.

(β) Let $(y, x)\mu = 1$. We consider three cases:

(β_1) $y\Sigma \in A' \cup B'$ and $x\Sigma \in A'$. By (5), $1 = (y, x)\mu = (y\Sigma, x\Sigma)\mu'$. By Theorem 2 and (4), $(y\Sigma)(x\Sigma) = (x\Sigma)\nu' = (x\nu)\Sigma = (yx)\Sigma$.

(β_2) $x\Sigma \notin A'$. By Theorem 2, $(y\Sigma)(x\Sigma) = e'$. $(yx)\Sigma = (x\nu)\Sigma = x^2\Sigma$. By (7), $x^2\Sigma = e'$.

(β_3) $y\Sigma \notin A' \cup B'$ and $x\Sigma \in A'$. By (6), $(y, x)\mu = 0$. This contradicts the assumption that $(y, x)\mu = 1$. Hence, this case does not occur.

Therefore Σ is a homomorphism.

Conversely, assume that Σ is a homomorphism. Since S' is unipotent and $e\Sigma = (ee)\Sigma = (e\Sigma)(e\Sigma)$, $e\Sigma = e'$. This proves (3).

Suppose $x \in B$. Then $x^2 = e$ and $e' = e\Sigma = x^2\Sigma = (x\Sigma)^2$. Thus, $x\Sigma \notin A'$ so that $x\Sigma \in B' \cup C' \cup \{e'\}$ and (1) holds.

Suppose $x \in C$. Then there is a $y \in A$ such that $x = y^2$. Thus, $x\Sigma = y^2\Sigma = (y\Sigma)^2$. Hence, $x\Sigma \in C'$ or $x\Sigma = e'$. This proves (2).

Since (1), (2), (3) hold now, we may use (1′), (2′), (3′) if this is helpful.

Suppose $x \in \Sigma^{-1}(A')$; that is, $x\Sigma \in A'$. By (1′), $x \in A$. Thus, both $x\nu$ and $(x\Sigma)\nu'$ are defined.

Hence, $(x\nu)\Sigma = x^2\Sigma = (x\Sigma)^2 = (x\Sigma)\nu'$. Thus, (4) holds.

Suppose $y \in \Sigma^{-1}(A' \cup B')$; that is, $y\Sigma \in A' \cup B'$. Since

$$\Sigma^{-1}(A' \cup B') = \Sigma^{-1}(A') \cup \Sigma^{-1}(B'),$$

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$y \in A \cup B$ by (1′) and (2′). Thus, both $(y, x)\mu$ and $(y\Sigma, x\Sigma)\mu'$ are defined. $(y, x)\mu = 1$ implies $yx = x\nu$. By (4),

$$(y\Sigma)(x\Sigma) = (yx)\Sigma = (x\nu)\Sigma = (x\Sigma)\nu' \neq e'.$$

$(x\Sigma)\nu' \neq e'$ since $\nu': A' \to C'$ is a surjection and $e' \notin C'$. Thus, $(y\Sigma, x\Sigma)\mu' = 1$ so that $(y, x)\mu = (y\Sigma, x\Sigma)\mu'$. $(y, x)\mu = 0$ implies $yx = e$. Thus, $e' = e\Sigma = (yx)\Sigma = (y\Sigma)(x\Sigma)$. Hence $(y\Sigma, x\Sigma)\mu' = 0$ and $(y, x)\mu = (y\Sigma, x\Sigma)\mu'$. Now (5) holds.

Suppose $y \in A \cup B$ but $y \notin \Sigma^{-1}(A' \cup B')$. Then $y\Sigma \notin A' \cup B'$. By Theorem 2, $(yx)\Sigma = (y\Sigma)(x\Sigma) = e'$. If $(y, x)\mu = 1$, then $yx = x\nu$. By (4), $(yx)\Sigma = (x\nu)\Sigma = (x\Sigma)\nu' \neq e'$, which is a contradiction. Hence, $(y, x)\mu = 0$ and (6) follows.

By Theorem 2, $x\Sigma \notin A'$ implies $(x\Sigma)^2 = e'$. Since Σ is a homomorphism, $e' = (x\Sigma)^2 = x^2\Sigma$. Thus, (7) holds.

THEOREM 4. *Let (A, B, C, e, μ, ν) and $(A', B', C', e', \mu', \nu')$ be the structure sets of two unipotent λ-semigroups S and S'. Then S and S' are isomorphic if and only if there is a bijection $\Sigma: S \to S'$ such that:*

(1) $A\Sigma = A'$,

(2) $B\Sigma = B'$,

(3) $C\Sigma = C'$,

(4) $e\Sigma = e'$,

(5) $(x\nu)\Sigma = (x\Sigma)\nu', x \in A$,

(6) $(y, x)\mu = (y\Sigma, x\Sigma)\mu', x \in A, y \in A \cup B$,

(7) $x\Sigma \notin A'$ *implies* $x^2\Sigma = e'$.

Proof. Suppose $S \cong S'$ under Σ. By Theorem 3,

$$B\Sigma \subset B' \cup C' \cup \{e'\}, \qquad C\Sigma \subset C' \cup \{e'\}, \qquad e\Sigma = e'.$$

Since Σ is a bijection, Σ^{-1} is a mapping of S' onto S such that (1′), (2′), (3′) become respectively

$$A'\Sigma^{-1} \subset A, \qquad B'\Sigma^{-1} \subset A \cup B, \qquad C'\Sigma^{-1} \subset A \cup B \cup C.$$

Furthermore, applying Theorem 3 to Σ^{-1}, we obtain

$$B'\Sigma^{-1} \subset B \cup C \cup \{e\}, \qquad C'\Sigma^{-1} \subset C \cup \{e\}, \qquad e'\Sigma^{-1} = e,$$

$$A\Sigma \subset A', \qquad B\Sigma \subset A' \cup B' \qquad C\Sigma \subset A' \cup B' \cup C'.$$

$A'\Sigma^{-1} \subset A$ implies $A' \subset A\Sigma$, which together with $A\Sigma \subset A'$ gives $A\Sigma = A'$. This proves (1). $A\Sigma = A'$, $B\Sigma \subset A' \cup B'$, Σ is an injection imply $B\Sigma \subset B'$. Similarly, $B'\Sigma^{-1} \subset B$, which then gives $B' \subset B\Sigma$. Thus, $B\Sigma = B'$ and (2) holds. The proof that (3) holds is similar. (4) is obvious. Since $A'\Sigma^{-1} = A$, (5) holds. Again, $A'\Sigma^{-1} = A$ and $(A' \cup B')\Sigma^{-1} = A \cup B$ imply (6). Finally, we note that Condition (6) of Theorem 3 cannot occur since $y \in A \cup B$ and

$y \notin (A' \cup B')\Sigma^{-1}$ are contradictory. (7) is verified as in Theorem 3. Thus, Σ satisfies all the conditions of Theorem 3, which reduce to those of this corollary.

Conversely, suppose the seven conditions of the corollary hold and Σ is a bijection. Then the first five conditions and Condition (7) of Theorem 3 clearly hold. Condition (6) of Theorem 3 is vacuously true. Theorem 3 now implies that Σ is a homomorphism. Since Σ is a bijection, Σ is an isomorphism.

4. The structure of general λ-semigroups. Let S be a λ-semigroup. By Theorem 1, S is the disjoint union of the $S(e)$, $e \in E$. If we index E with the set J (E itself is not used in order to avoid confusion), then

$$S = \bigcup_{j \in J} S(e_j), \qquad E = \{e_j \in S: e_j^2 = e_j, j \in J\}.$$

Since $S(e_j)$ is a unipotent λ-semigroup, Theorem 2 applies. Let $(A_j, B_j, C_j, e_j, \mu_j, \nu_j)$ be the structure set of $S(e_j)$. We investigate the behaviour of the product $xy \in S$, where $x \in S(e_i)$, $y \in S(e_j)$, $i \neq j$.

LEMMA 8. *If $x \in S(e_i)$, $y \in S(e_j)$, $i \neq j$, then:*

(i) *$xy = y$ if and only if $y = e_j$,*

(ii) *$xy = e_j$ if $y \notin A_j$,*

(iii) *$xy = e_j$ if $y \in A_j$ and $x \notin A_i \cup B_i$,*

(iv) *$xy = e_j$ or y^2 if $x \in A_i \cup B_i$, $y \in A_j$.*

Proof. (i) By Lemma 4, $xy = y$ if and only if $y = e_j$.

(ii) By (i), $y \in B_j \cup C_j$. By Lemma 7, $y^2 = e_j$. By Lemma 2, $xy \in \langle y \rangle = \{y, e_j\}$. By (i), $xy = e_j$.

(iii) $x \notin A_i \cup B_i$ implies $x \in C_i \cup \{e_i\}$. Thus, $x = e_i$ or $x \in C_i$. By Lemma 7, $x \in C_i$ implies $x = z^2$, $z \in A_i$. By Lemma 6,

$$xy = \begin{Bmatrix} e_i y = e_i^2 y \\ z^2 y \end{Bmatrix} = e_j.$$

(iv) Since $xy \in \langle y \rangle = \{y, y^2, y^3 = e_j\}$ and, by (i), $xy \neq y$, $xy = y^2$ or e_j.

DEFINITION 4. *Let $S = \bigcup S(e_j)$ be a λ-semigroup. Then there is a family of functions $\mathfrak{F}^* = \{\mu_j: \mu_j: (A_j \cup B_j) \times A_j \to I, j \in J\}$. For each $j \in J$ define additional functions $\mu_{ij}: (A_i \cup B_i) \times A_j \to I$, $i \in J$, by*

$$(x, y)\mu_{ij} = \begin{cases} 1 & \textit{if } xy = y^2, \\ 0 & \textit{otherwise.} \end{cases}$$

By Lemma 8, μ_{ij}, $i, j \in J$, is well defined. Also $\mu_{jj} = \mu_j$, $j \in J$. Moreover, if $x \in S(e_i)$, $y \in S(e_j)$, then

$$xy = \begin{cases} y\nu_j & \text{if } (x, y)\mu_{ij} = 1, \\ e_j & \text{otherwise.} \end{cases}$$

An immediate consequence of Definition 4 is

THEOREM 5. *Let S be a λ-semigroup with the set of idempotents $E = \{e_j : j \in J\}$. Then there exist families of subsets of S and families of functions as follows:*

$$\mathfrak{F}_a = \{A_j : j \in J\}, \qquad \mathfrak{F}_b = \{B_j : j \in J\}, \qquad \mathfrak{F}_c = \{C_j : j \in J\},$$
$$\mathfrak{F}_\mu = \{\mu_{ij} : i, j \in J\}, \qquad \mathfrak{F}_\nu = \{\nu_j : j \in J\},$$

where the A_j's, B_j's, C_j's, $\{e_j\}$'s are pairwise disjoint,

$$\mu_{ij} : (A_i \cup B_i) \times A_j \to I, \qquad (x, x)\mu_{jj} = 1 \text{ for } x \in A_j,$$

$\nu_j : A_j \to C_j$ is surjective, such that, for

$$x \in S(e_i) = A_i \cup B_i \cup C_i \cup \{e_i\}, \; y \in S(e_j) = A_j \cup B_j \cup C_j \cup \{e_j\},$$

$$xy = \begin{cases} y\nu_j & \text{if } (x, y)\mu_{ij} = 1, \\ e_j & \text{otherwise.} \end{cases}$$

THEOREM 6. *Let J be any non-empty set. Let S be a set which is the disjoint union of four sets A^*, B^*, C^*, E, where A^*, B^*, C^* are disjoint unions of*

$$\mathfrak{F}_a = \{A_j : j \in J\}, \qquad \mathfrak{F}_b = \{B_j : j \in J\}, \qquad \mathfrak{F}_c = \{C_j : j \in J\},$$

respectively, and $E = \{e_j : j \in J\}$. Assume $|A_j| \geqslant |C_j|$ for each $j \in J$. For every pair $(i, j) \in J \times J$, let

$$\mu_{ij} : (A_i \cup B_i) \times A_j \to I$$

be any function satisfying $(a, a)\mu_{jj} = 1$ for $a \in A_j$. Also let $\nu_j : A_j \to C_j$ be any surjection. Put

$$S(e_j) = A_j \cup B_j \cup C_j \cup \{e_j\}.$$

Then $S = \bigcup_{j \in J} S(e_j)$ is a λ-semigroup with multiplication m defined by

$$(x, y)m = \begin{cases} y\nu_j & \text{if } (x, y)\mu_{ij} = 1, \\ e_j & \text{otherwise} \end{cases}$$

for $x \in S(e_i)$, $y \in S(e_j)$; and $S(e_j)$ is the maximal unipotent subsemigroup with idempotent e_j and structure set $(A_j, B_j, C_j, e_j, \mu_{jj}, \nu_j)$.

Proof. The proof is similar to that of Theorem 2 and we note that if $z \in S(e_k)$, then, by the definition of m,

$$((x, y)m, z)m = e_k = (x, (y, z)m)m.$$

Another view of λ-semigroups uses the concept of elementary semigroup (Definition 6).

THEOREM 7. *Let $S = \bigcup_{\alpha \in \Gamma} S_\alpha$, $S_{\alpha_1} \cap S_{\alpha_2} = \emptyset$, $\alpha_1 \neq \alpha_2$, such that*

$$S_\alpha = \bigcup_{\beta \in \Delta_\alpha} A_{\alpha\beta}, \qquad A_{\alpha\beta_1} \cap A_{\alpha\beta_2} = \{0_\alpha\}, \qquad \beta_1 \neq \beta_2, \beta_1, \beta_2 \in \Delta_\alpha, \alpha \in \Gamma.$$

Let β_0 be a fixed index element such that $\beta_0 \in \bigcap_{\alpha \in \Gamma} \Delta_\alpha$. Let $F = \{f_\alpha : \alpha \in \Gamma\}$ be a family of functions

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$$f_\alpha:\Delta_\alpha\backslash\{\beta_0\} \to \bigcup \{A_{\alpha\beta}\backslash\{0_\alpha\}:\beta \in \Delta_\alpha\backslash\{\beta_0\}\}$$

such that $\beta f_\alpha \in A_{\alpha\beta}$ *for* $\beta \in \Delta_\alpha\backslash\{\beta_0\}$. *Further, let*

$$B_{\alpha\beta_0} = A_{\alpha\beta_0}\backslash\{0_\alpha\}, \quad \alpha \in \Gamma; \qquad B_{\alpha\beta} = A_{\alpha\beta}\backslash\{\beta f_\alpha, 0_\alpha\}, \quad \beta \neq \beta_0, \alpha \in \Gamma.$$

For $x \in A_{\gamma\delta}$, $y \in A_{\alpha\beta}$, $\gamma, \alpha \in \Gamma$, $\delta \in \Delta_\gamma$, $\beta \in \Delta_\alpha$, *define a binary operation in S by*

$$xy = \begin{cases} \beta f_\alpha & \text{if } x = y \in B_{\alpha\beta},\ \gamma = \alpha,\ \delta = \beta,\ \beta \neq \beta_0, \\ \beta f_\alpha \text{ or } 0_\alpha & \text{if } x \in \bigcup_{\delta\in\Delta_\gamma} B_{\gamma\delta},\ y \in B_{\alpha\beta},\ \beta \neq \beta_0, \\ 0_\alpha & \text{otherwise.} \end{cases}$$

Then S is a λ*-semigroup and conversely any* λ*-semigroup has such a structure.*

Proof. Suppose $x, y, z \in S$, $x \in A_{\alpha\beta}$, $y \in A_{\gamma\delta}$, $z \in A_{\kappa\tau}$. Then, by the definition of multiplication in S, $xy = \delta f_\gamma$ or 0_γ. Since 0_γ and $\delta f_\gamma \notin \bigcup_{\delta\in\Delta_\gamma} B_{\gamma\delta}$, another application of the definition of multiplication yields $(xy)z = 0_\kappa$. Similarly,

$$x(yz) = \begin{cases} x(\tau f_\kappa) = 0_\kappa, \\ x0_\kappa = 0_\kappa. \end{cases}$$

Thus, S is a semigroup.

Let T be a subsemigroup of semigroup S. Suppose $z \in T$, $y \in S$. By hypothesis, $x \in A_{\alpha\beta} \subset S_\alpha$, $y \in A_{\gamma\delta} \subset S_\gamma$, $\alpha, \gamma \in \Gamma$, $\beta \in \Delta_\alpha$, $\delta \in \Delta_\gamma$. By the definition of multiplication in S, $yx = \beta f_\alpha$ or 0_α. Since T is a subsemigroup, $0_\alpha = x^3 \in T$. Also, if $x \in B_{\alpha\beta}$, $\beta \neq \beta_0$, then $\beta f_\alpha = x^2 \in T$. Thus, $yx \in T$, and T is a λ-semigroup.

For the converse we note that S_α, $\alpha \in \Gamma$, is a unipotent λ-semigroup $S(e_i)$, $i \in J$; $A_{\alpha\beta}$, $\alpha \in \Gamma$, $\beta \in \Delta_\alpha$, $\beta \neq \beta_0$, is an elementary semigroup of the type T_c (Definitions 5 and 6); 0_α, $\alpha \in \Gamma$, is a right zero, say e_i, $i \in J$; $A_{\alpha\beta_0}$, $\alpha \in \Gamma$, is a null semigroup of the type $\bigcup_{b\in B} T_b$, $B \subset S(e_i)$ say (Definition 5). Also $\beta f_\alpha = c \in T_c$. Thus, S is the union of $\mathfrak{S}$ and $\mathfrak{F}$, where $\mathfrak{S} = \bigcup_{\alpha\in\Gamma,\beta_0\neq\beta\in\Delta_\alpha} A_{\alpha\beta}$ is a union of elementary semigroups and $\mathfrak{F} = \bigcup_{\alpha\in\Gamma} A_{\alpha\beta_0}$ is a union of null semigroups.

5. The structure of σ-semigroups. Let S be a σ-semigroup. Since S is both a λ-semigroup and a ρ-semigroup, S is unipotent and the unique idempotent is zero.

An application of Theorem 2 for unipotent λ-semigroups to σ-semigroup S gives a family $\{A, B, C\}$ of disjoint subsets of S and, for $A \neq \emptyset$, functions $\mu:(A \cup B) \times A \to I$, $\nu:A \to C$ such that:

(1) $S = A \cup B \cup C \cup \{e\}$,

(2) $(a, a)\mu = 1$,

(3) ν is surjective,

(4) $xy = y\nu$ if $(x, y)\mu = 1$,

(5) $xy = e$ otherwise.

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Suppose $(x, y)\mu = 1$. Then, by (4), $xy = y\nu = y^2 \neq e$. Since $\langle x\rangle$ is an ideal in the σ-semigroup S, $xy \in \langle x\rangle$. By Lemma 3, $\langle x\rangle \subset \{x, x^2, x^3 = e\}$. $xy = x^3 = e$ contradicts $xy \neq e$. By Lemma 6, $xy = x$ implies $x = xy = (xy)y = e$. This contradicts $xy \neq e$ too. Thus, $xy = x^2$. By Lemma 7, $x^2 = xy = y^2 \neq e$ implies $x^2 \in C$ and $x \in A$. Thus, $x\nu = x^2$ is defined and $x\nu = y\nu$.

By the dual of Theorem 2, let $S = A' \cup B' \cup C' \cup \{e\}$ be the ρ-semigroup decomposition of S. By Lemma 7 and its dual, both A and A' are characterized as the set $\{a \in S{:}a^2 \neq e\}$. Thus, $A' = A$. Again, by Lemma 7 and its dual, $C = \{x^2{:}x \in A\}$, $C' = \{y^2{:}y \in A'\}$. Thus, $C' = C$. Hence, $B' = B$ also.

By (4) and (5), if $x \in A$, $y \in B$, then $xy = e$. From the duals of (4) and (5), $xy = e$ if $x \in B$, $y \in A$. This result implies that $\mu{:}(A \cup B) \times A \to I$ may be replaced by $\mu{:}A \times A \to I$, where the same symbol is used for a function and one of its restrictions.

We summarize in

THEOREM 8. *Let S be a σ-semigroup. Then S is unipotent and there is a family $\{A, B, C\}$ of disjoint subsets of S, $|A| \geqslant |C|$, and functions $\mu{:}A \times A \to I$, $(x, x)\mu = 1$, $xy = e$ if $x\nu \neq y\nu$, and ν is a surjection. The operation in S is defined by*

$$xy = \begin{cases} y\nu & \textit{if } (x, y)\mu = 1, \\ e & \textit{otherwise.} \end{cases}$$

Conversely, if S satisfies $S = A \cup B \cup C \cup \{e\}$, disjoint union, and there are functions $\mu{:}A \times A \to I$, $(x, x)\mu = 1$, $(x, y)m = e$ for $x\nu \neq y\nu$, and $\nu{:}A \to C$ is a surjection, then (S, m) is a σ-semigroup for m defined by

$$(x, y)m = \begin{cases} y\nu & \textit{if } (x, y)\mu = 1, \\ e & \textit{otherwise.} \end{cases}$$

Proof. The proof of the converse is similar to the proofs of Theorems 2 and 6.

DEFINITION 5. *For each $a \in A$, $T_a = \{a, c, e{:}a^2 = c \in C\} = \langle a\rangle$. For each $b \in B$, $T_b = \{b, e\} = \langle b\rangle$. For each $c \in C$, $T_c' = \{c, e\} = \langle c\rangle$ and $T_c = A_c \cup \{c, e\} = A_c \cup T_c'$, where $A_c = \{a \in S{:}a^2 = a\nu = c\} = \nu^{-1}(c) \subset A$.*

Clearly, $A = \bigcup_{c \in C} A_c$; T_a, T_b, T_c', T_c are σ-(sub)semigroups of S; $S = \bigcup_{d \in B \cup C} T_d$.

DEFINITION 6. *An elementary semigroup S is a σ-semigroup such that $B = \emptyset$, $|C| = 1$. An elemental semigroup S is an elementary semigroup such that $|A| = 1$. A nil-semigroup S is a σ-semigroup such that $|B| = 1$, $A = C = \emptyset$.*

COROLLARY 1. *All elemental semigroups are isomorphic. If S is an elemental semigroup, then $|S| = 3$. For any $a \in A$, T_a is an elemental (sub)semigroup.*

All nil-semigroups are isomorphic. A nil-semigroup is a null semigroup of order 2. For any $b \in B$, T_b is a nil-semigroup, for any $c \in C$, T_c' is a nil-semigroup.

LEMMA 9. *If S is an elementary semigroup, then S is the union of elemental semigroups; that is, $S = \bigcup_{a \in A} T_a$. Moreover, $S = A_c \cup \{c, e\}, x^2 = c$ for all $x \in A$. S need not be finite.*

If S is a null semigroup, $|S| \geqslant 2$, then S is the 0-disjoint union of nil-semigroups; that is, $S = \bigcup_{b \in B} T_b$.

THEOREM 9. *A semigroup S is a σ-semigroup if and only if S is the 0-disjoint union of a collection $\mathfrak{C}$ of elementary semigroups and a collection $\mathfrak{B}$ of nil-semigroups,*

$$S = \bigcup_{d \in D} T_d, \qquad T_d \in \mathfrak{C} \cup \mathfrak{B},$$

such that $T_i \cap T_j = \{e\}, xy = e, x \in T_i, y \in T_j, i \neq j, i, j \in D$. Either $\mathfrak{C}$ or $\mathfrak{B}$ may be empty but not both.

Proof. Let $S = A \cup B \cup C \cup \{e\}$ be a σ-semigroup. Let $D = B \cup C$. By Definitions 5 and 6, $\{T_d : d \in D\}$ is a set of elementary semigroups and nil-semigroups. By definition 5, $T_i \cap T_j = \{e\}$ if $i \neq j, i, j \in D$. By the statement following Definition 5, $S = \bigcup_{d \in D} T_d$. Moreover, since $xy = e$ for $x\nu \neq y\nu$, $x \in T_i$, $y \in T_j$, $i \neq j$, $i, j \in D$, we have $xy = e$.

Conversely, if $S = \bigcup_{d \in D} T_d$ is a groupoid satisfying the given properties, then S is a semigroup because $(xy)z = e = x(yz)$ if x, y, z are not all in the same elementary semigroup or nil-semigroup, and $(xy)z = x(yz)$ if x, y, z are all in the same elementary semigroup or nil-semigroup since these are already associative.

Let S' be a subsemigroup of S. Certainly, $e \in S'$. Suppose $x \in S'$, $x \neq e$; then $x \in T_d$, $d \in D$, T_d is an elementary semigroup or T_d is a nil-semigroup. If T_d is nil, then $T_d = \langle x \rangle \subset S'$. If T_d is elementary, then $\langle x \rangle$ is a subsemigroup of T_d. Thus, S' is a 0-disjoint union of subsemigroups of the T_d or $S' = \{e\}$. Conversely, any such 0-disjoint union is a subsemigroup of S. Thus, let $S' = \bigcup_{d' \in D'} T_{d'}$, where $T_{d'}$ is nil or elemental. Hence, $T_i T_{j'} = T_{j'} T_i = \{e\}$ if $i, j \in D, i \neq j$, and $T_{j'}$ is an elemental subsemigroup of elementary semigroup T_j or is some nil-semigroup; $T_j T_{j'} \subset T_{j'}$, $T_{j'} T_j \subset T_{j'}$ because $T_{j'}$ is a subsemigroup of a σ-semigroup T_j. Therefore S' is an ideal and S is a σ-semigroup.

Theorem 9 gives a practical way of constructing all non-isomorphic σ-semigroups of order n if n is a small positive integer.

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University of Saskatchewan, University of Oklahoma,
and University of California, Davis