J. Aust. Math. Soc. 79 (2005), 349–360

MODULES WHICH ARE INVARIANT UNDER MONOMORPHISMS OF THEIR INJECTIVE HULLS

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(Received 5 December 2003; revised 30 November 2004)

Communicated by J. Du

Abstract

In this paper certain injectivity conditions in terms of extensions of monomorphisms are considered. In particular, it is proved that a ring R is a quasi-Frobenius ring if and only if every monomorphism from any essential right ideal of R into $R_R^{(\mathbb{N})}$ can be extended to R_R. Also, known results on pseudo-injective modules are extended. Dinh raised the question if a pseudo-injective CS module is quasi-injective. The following results are obtained: M is quasi-injective if and only if M is pseudo-injective and M^2 is CS. Furthermore, if M is a direct sum of uniform modules, then M is quasi-injective if and only if M is pseudo-injective. As a consequence of this it is shown that over a right Noetherian ring R, quasi-injective modules are precisely pseudo-injective CS modules.

2000 *Mathematics subject classification*: primary 16D50, 16D70.

1. Introduction

Throughout the paper rings are associative with identity and modules are unitary (right) modules. Let M and N be two right R-modules over a ring R. M is called *(pseudo-)N-injective* if, for any submodule A of N, every homomorphism (monomorphism) in $\mathrm{Hom}_R(A, M)$ can be extended to an element of $\mathrm{Hom}_R(N, M)$. M is called *quasi-injective (pseudo-injective)* if it is (pseudo-)M-injective. M and N are called relatively injective if M is N-injective and N is M-injective. A submodule K of M is said to be a complement in M of a submodule B if K is a maximal submodule among those that have zero intersection with B. Complement submodules of M coincide with the submodules of M which do not have any proper essential extension in M. Also, if A is a complement in M and B is a complement in A, then B is a complement in M.

https://doi.org/10.1017/S1446788700010946 Published online by Cambridge University Press

A CS module is one in which complement submodules are direct summands. M is called a continuous module if it is a CS module and submodules of M isomorphic to direct summands of M are again direct summands. If M is continuous and A and B are two direct summands of M with $A \cap B = 0$, then $A \oplus B$ is also a direct summand of M. The hierarchy is as follows:

$$\text{Injective} \Longrightarrow \text{quasi-injective} \Longrightarrow \text{continuous} \Longrightarrow \text{CS}.$$

For other properties of complements and CS/continuous modules and the proofs of the above mentioned properties, the reader is referred to [3] and [10].

In this paper, a weaker form of pseudo-N-injectivity is considered, and it is proved, in particular, that a ring R is quasi-Frobenius if and only if monomorphisms from essential right ideals of R into $R^{(\mathbb{N})}$ can be extended to R_R. Also it is shown that a module M is invariant under monomorphisms of its injective hull if and only if every monomorphism from any essential submodule of M can be extended to M. This extension property is used to characterize when semi-prime/right nonsingular rings are SI (see [6]).

Pseudo-injectivity has been studied by several authors such as Dinh, Jain, Singh, Teply, Tuganbaev and others (see [2, 8, 9, 13–15]). It was first introduced by Jain and Singh [8]. Teply [14] constructed examples of pseudo-injective modules which are not quasi-injective. In [2] Dinh raised the question if a pseudo-injective CS module is quasi-injective. He stated in [2] that the answer is affirmative if we assume further that M is nonsingular. In this paper we prove the following: M is quasi-injective if and only if M is pseudo-injective and M^2 is CS. Every uniform pseudo-injective module is quasi-injective. Consequently, over a right Noetherian ring R, quasi-injective modules are precisely pseudo-injective CS modules.

2. Essentially pseudo-N-injectivity

In this section we consider a weaker form of pseudo-N-injectivity.

DEFINITION 2.1. Let M and N be two modules. M is said to be *essentially pseudo-N-injective* if for any essential submodule A of N, any monomorphism $f : A \to M$ can be extended to some $g \in \operatorname{Hom}(N, M)$. M is called *essentially pseudo-injective* if M is essentially pseudo-M-injective.

Obviously any pseudo-N-injective module is essentially pseudo-N-injective, but the converse is not true in general.

EXAMPLE 1. Let p be a prime. The $\mathbb{Z}$-module $\mathbb{Z}/p^2\mathbb{Z}$ is not pseudo-($\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z}$)-injective since the obvious isomorphism $\iota : p\mathbb{Z}/p^3\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z}$ can not be extended

https://doi.org/10.1017/S1446788700010946 Published online by Cambridge University Press

to any element of Hom($\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z}$, $\mathbb{Z}/p^2\mathbb{Z}$), but it is essentially pseudo-($\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z}$)-injective.

The following proposition provides a characterization of essentially pseudo-N-injectivity.

PROPOSITION 2.2. *Let M and N be two modules and $X = M \oplus N$. The following conditions are equivalent*:

(i) *M is essentially pseudo-N-injective.*
(ii) *For any complement K in X of M with $K \cap N = 0$, $M \oplus K = X$.*

PROOF. (i) $\Rightarrow$ (ii) Let K be a complement in X of M with $K \cap N = 0$, and $\pi_M : M \oplus N \to M$ and $\pi_N : M \oplus N \to N$ be the obvious projections. Note that $M \oplus K = M \oplus \pi_N(K)$ so that $\pi_N(K)$ is essential in N.

Now define $\theta : \pi_N(K) \to \pi_M(K)$ as follows: For $k \in K$ with $k = m + n$ $(m \in M, n \in N)$, $\theta(n) = m$. Then θ is a monomorphism by the $K \cap N = 0$ assumption. Hence θ can be extended to some $g : N \to M$, since M is essentially pseudo-N-injective. Now let $T = \{n+g(n) : n \in N\}$. It is easy to see that $M \oplus T = X$. Also, T contains K essentially by modularity. Since K is a complement, this implies $T = K$. Now the conclusion follows.

(ii) $\Rightarrow$ (i) Assume (ii). Let A be an essential submodule of N and $f : A \to M$ be a monomorphism. Let $H = \{a - f(a) : a \in A\}$. Obviously, $H \cap N = 0$. Also note that $M \oplus H = M \oplus \pi_N(H) = M \oplus A$, which is essential in X. Let K be a complement in X of M containing H. By the previous argument and modularity H is essential in K, so that $K \cap N = 0$. By assumption we have $M \oplus K = X$. Now let $\phi : M \oplus K \to M$ be the obvious projection. Then the restriction $\phi_{|N}$ is the desired extension of f. The proof is now complete. □

PROPOSITION 2.3. *If M is essentially pseudo-N-injective, every direct summand of M is essentially pseudo-N-injective.*

PROOF. Let $X = M \oplus N$ and assume $M = M_0 \oplus A$. Let K be a complement in $M_0 \oplus N$ of M_0 with $K \cap N = 0$. Then $M \oplus K$ is essential in X. Since K is a complement submodule, the preceding argument implies that K is also a complement in X of M. Now by Proposition 2.2 $M \oplus K = X$. Then $M_0 \oplus K = M_0 \oplus N$, which yields the conclusion again by Proposition 2.2. □

The next example shows that essentially pseudo-N-injectivity is not inherited by direct sums.

EXAMPLE 2. Let F be a field and

$$R = \begin{pmatrix} F & F \oplus F \\ 0 & F \end{pmatrix}.$$

Consider the R-modules

$$N = \begin{pmatrix} F & F \oplus F \\ 0 & 0 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 0 & 0 \oplus F \\ 0 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & F \oplus 0 \\ 0 & 0 \end{pmatrix}.$$

Then S_1 and S_2 are both essentially pseudo-N-injective. But since the identity map of $S_1 \oplus S_2$ obviously can not be extended to an element of $\mathrm{Hom}(N, S_1 \oplus S_2)$, $S_1 \oplus S_2$ is not essentially pseudo-N-injective.

PROPOSITION 2.4. *Let M and N be two modules. Then the following conditions are equivalent*:

(i) *M is N-injective.*

(ii) *M is essentially pseudo-N/L-injective for every submodule L of N.*

PROOF. (i) $\Rightarrow$ (ii) follows from [10, Proposition 1.3].

(ii) $\Rightarrow$ (i) Assume M is essentially pseudo-N/L-injective for every submodule L of N. Let $X = M \oplus N$, $A \subseteq X$ with $A \cap M = 0$ and K be a complement in X of M containing A. Also let $T = K \cap N$. Since $(M \oplus K)/K$ is essential in X/K, then $(M \oplus K)/T$ is essential in X/T, and $K/T \cap N/T = 0$. Thus it is easy to see that K/T is a complement in X/T of $(M \oplus T)/T$. Now by assumption and Proposition 2.2 we have $(M \oplus T)/T \oplus K/T = X/T$. Hence $M \oplus K = X$. Then by [3, Lemma 7.5] M is N-injective. □

COROLLARY 2.5. *M is injective if and only if M is essentially pseudo-N-injective for any cyclic module N.*

COROLLARY 2.6. *A nonsingular module M is injective if and only if it is essentially pseudo-N-injective for any nonsingular cyclic module N.*

PROOF. Let A be any cyclic module and B be an essential submodule of A. Let $f : B \to M$ be a monomorphism. Then A is obviously nonsingular, so that f can be extended to some $g : A \to M$ by assumption. Now the result follows by Corollary 2.5. □

The following result generalizes [2, Theorem 2.2] and [9, Theorem 1].

THEOREM 2.7. *If $M \oplus N$ is essentially pseudo-N-injective, then M is N-injective.*

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PROOF. Call $X = M \oplus N$. Let A and K be as in the proof of Proposition 2.4. Let $\pi : M \oplus N \to N$ be the obvious projection. Then $M \oplus K = M \oplus \pi(K)$ and thus $\pi(K)$ essential in N. Note that $K \cong \pi(K)$. Pick any isomorphism $f : \pi(K) \to K$. By assumption f can be extended to some monomorphism $g : N \to X$. Then $g(\pi(K)) = K$ is essential in $g(N)$. But since K is a complement in X, we must have $K = g(N)$, whence $\pi(K) = N$. Thus $M \oplus K = X$. Now the result follows by [3, Lemma 7.5]. □

COROLLARY 2.8. *M is quasi-injective if and only if M^2 is essentially pseudo-M-injective.*

Osofsky proved in [12] that a ring R is semisimple Artinian if and only if every cyclic right (left) R-module is injective.

COROLLARY 2.9. *A ring R is semisimple Artinian if and only if every countably generated right R-module is essentially pseudo-injective.*

PROOF. Let M be a cyclic right R-module. Then $(M \oplus R)^{(\mathbb{N})} \cong (M \oplus R)^{(\mathbb{N})} \oplus (M \oplus R)^{(\mathbb{N})}$, which is countably generated, whence essentially pseudo-injective. Thus $(M \oplus R^{(\mathbb{N})})^2$ is essentially pseudo-$(M \oplus R^{(\mathbb{N})})$-injective. Then by Theorem 2.7, $(M \oplus R^{(\mathbb{N})})$ is quasi-injective, whence R_R-injective. Therefore M is injective. Now the conclusion follows by Osofsky's theorem. □

COROLLARY 2.10 ([2, Theorem 2.2]). *If $M \oplus N$ is pseudo-injective, then M and N are relatively injective.*

In what follows $E(M)$ stands for the injective hull of M and we will consider M as a submodule of $E(M)$. We will also use the notation $E_N(M)$ for the submodule of $E(M)$ generated by all the isomorphic copies of N. Note that $E_N(M)$ is invariant under monomorphisms of $\mathrm{End}(E(M))$ and that $E_{R_R}(M)$ contains all elements of M with zero right annihilator in R.

PROPOSITION 2.11. *M is essentially pseudo-N-injective if and only if $E_N(M) \subseteq M$.*

PROOF. Assume $E_N(M) \subseteq M$ and let B be an essential submodule of N, and $f : B \to M$ be a monomorphism. There exists some monomorphism $g : N \to E(M)$ such that $g_{|B} = f$. By assumption $g(N) \subseteq M$. Thus g is the desired extension of f, whence M is essentially pseudo-N-injective.

Conversely assume that M is essentially pseudo-N-injective. We will use the same argument as in [10, Lemma 1.13]: Let $h : N \to E(M)$ be a monomorphism. Let $A = h^{-1}(M)$. Then A is essential in N. Thus, by assumption, the restriction $h_{|A}$ extends to some $\theta : N \to M$. Now assume $h(n) \neq \theta(n)$ for some $n \in N$. Then

$x = h(n) - \theta(n) \neq 0$. Since M is essential in $E(M)$, there exists some $r \in R$ such that $0 \neq xr = h(nr) - \theta(nr) \in M$. But then $h(nr) \in M$ so that $nr \in A$. This is a contradiction since $\theta_{|A} = h_{|A}$. Now the conclusion follows. □

COROLLARY 2.12. *M is essentially pseudo-injective if and only if it is invariant under monomorphisms in* End$(E(M))$.

COROLLARY 2.13. *Let $\{A_i\}$ be a family of submodules of a module N, $B = \Sigma A_i$ and assume M is essentially pseudo-A_i-injective for each i. Then M is essentially pseudo-B-injective.*

PROOF. Let $f : B \to E(M)$ be a monomorphism. Then $f(B) = \Sigma f(A_i)$. By assumption and Proposition 2.11, $f(B)$ is contained in M. Now the conclusion follows again by Proposition 2.11. □

The converse of the Corollary 2.13 does not hold in general.

EXAMPLE 3. Let p be a prime. It is easy to see that the $\mathbb{Z}$-module $\mathbb{Z}/p^2\mathbb{Z}$ is not essentially pseudo-$\mathbb{Z}/p^3\mathbb{Z}$-injective, but it is trivially essentially pseudo-$(\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z})$-injective.

COROLLARY 2.14. *Let E be an injective module and A be any submodule of E. Then $X = \Sigma\{C \mid C \leq E, C \cong A\}$ is essentially pseudo-injective.*

PROOF. First note that $E(X)$ is a summand of E. As in the proof of Corollary 2.13, for any monomorphism $f : X \to E(X)$, $f(X)$ is contained in X. The conclusion follows by Proposition 2.11. □

Goodearl defined a right SI-ring to be one over which every singular right module is injective ([6]). Such rings are precisely right nonsingular rings over which singular right modules are semi-simple (see [3]).

THEOREM 2.15. *Let R be a ring which is either right nonsingular or semi-prime. The following conditions are equivalent*:

(i) *R is a right SI-ring.*

(ii) *Any two cyclic singular right R-modules are relatively essentially pseudo-injective.*

(iii) *For any two cyclic singular right R-modules B and C, $E_B(C) \subseteq C$.*

PROOF. (i) $\Rightarrow$ (ii) Trivial.

(ii) $\Leftrightarrow$ (iii) The statement follows from Proposition 2.11.

(ii) $\Rightarrow$ (i) Assume (ii). Then cyclic singular right R-modules are relatively injective by Proposition 2.4. So if C and M are singular right R-modules and C is cyclic, then C is M-injective by the above argument and [10, Proposition 1.4]. This implies, by [3, Corollary 7.14], that all singular right R-modules are semi-simple.

Now, if R is right nonsingular, the conclusion immediately follows by the preceding remark and the above argument. Else, assume that R is semi-prime. Since singular modules are semi-simple, $Z(R_R)^2 = 0$, whence $Z(R_R) = 0$. Now the conclusion follows by the above argument. □

3. Pseudo-injectivity

PROPOSITION 3.1 ([16, Corollary 2.9]). *Let M and N be two modules and $X = M \oplus N$. The following conditions are equivalent:*

(i) *M is pseudo-N-injective.*

(ii) *For any submodule A of X with $A \cap M = A \cap N = 0$, there exists a submodule T of X containing A with $M \oplus T = X$.*

PROOF. (i) $\Rightarrow$ (ii) Assume (i) and let A satisfy the assumptions of (ii). Also let π_M and π_N be as in the Proposition 2.2, and define $\theta : \pi_N(A) \to \pi_M(A)$ as follows: $\theta(\pi_N(a)) = \pi_M(a)$, for $a \in A$. Then, by assumption, θ extends to some $g \in \operatorname{Hom}(N, M)$. Let $T = \{n + \theta(n) \mid n \in N\}$. Then we have $M \oplus T = X$ and $A \subseteq T$, as required.

(ii) $\Rightarrow$ (i) Assume (ii). Let B be a submodule of N and $f : B \to M$ be a monomorphism. Call $A = \{b - f(b) \mid b \in B\}$. Then $A \cap M = A \cap N = 0$. Now, by assumption, there exists a submodule T of X containing A with $M \oplus T = X$. Let $\pi : M \oplus T \to M$ be the obvious projection. Then the restriction $\pi_{|N}$ is the desired extension of f. □

Jain and Singh proved in [8, Theorem 3.7] that for a nonsingular module M with finite uniform dimension, the following conditions are equivalent: (i) M is pseudo-injective; (ii) M is invariant under any monomorphism (isomorphism in the terminology of [8]) of $\operatorname{End}(E(M))$ (that is, M is essentially pseudo-injective by Corollary 2.12). The following result extends it to any module with finite uniform dimension.

THEOREM 3.2. *Let M be a module with finite uniform dimension. Assume that for any two essential submodules D and E of M, every isomorphism $h : D \to E$ can be extended to some $g \in$* End(M). *Then every monomorphism from any submodule of M into M can be extended to a monomorphism of M.*

https://doi.org/10.1017/S1446788700010946 Published online by Cambridge University Press

In particular, a module with finite uniform dimension is pseudo-injective if and only if it is essentially pseudo-injective.

PROOF. Let M be as in the former assumption, A be a submodule of M, and $f : A \to M$ be a monomorphism. Call $B = f(A)$. Pick, by Zorn's Lemma, two submodules A' and B' of M such that $A \oplus A'$ and $B \oplus B'$ are essential in M. Now, $E(M) = E(A) \oplus E(A') = E(B) \oplus E(B')$ and $E(A) \cong E(B)$. Then by [10, Theorem 1.29] and since M has finite uniform dimension, we have $E(A') \cong E(B')$. Thus A' and B' have isomorphic essential submodules $U \subseteq A'$ and $V \subseteq B'$. Then $A \oplus U$ and $B \oplus V$ are essential submodules of M. And since U and V are isomorphic to each other, there exists an isomorphism $\theta : A \oplus U \to B \oplus V$ such that $\theta_{|A} = f$. By assumption θ extends to some monomorphism $g \in \mathrm{End}(M)$. Obviously, $g_{|A} = f$. Therefore, the conclusion follows. □

Note that, in [1, Theorem 2.1], Alamelu gives a proof that M is pseudo-injective if and only if M is invariant under monomorphisms of $\mathrm{End}(E(M))$, where M is an arbitrary module over a commutative ring (here the commutativity assumption is irrelevant to the proof). However, the proof is incorrect. In summary, the proof states that for a module M which is invariant under monomorphisms of its injective hull, and for any monomorphism $f : N \to M$ where N is a submodule of M, f can be extended to a monomorphism $f'' : E(M) \to E(M)$. This is not correct as the following example shows: Let M be any directly infinite injective module with $M = N \oplus B$, where $M \cong N$ and B is nonzero. Also let $f : N \to M$ be any isomorphism. Obviously, f cannot be extended to a monomorphism in $\mathrm{End}(E(M))$.

In [4] and [5] Er studied the modules in which isomorphic copies of complements are again complements. These are called SICC-modules in [5]. The following result was proved in [8] for nonsingular modules, but the proof works for an arbitrary pseudo-injective module as well.

LEMMA 3.3 ([8, Lemma 3.1]). *If M is pseudo-injective, then submodules of M isomorphic to complements in M are again complements.*

PROOF. Let K be a complement in M and A be a submodule of M with an isomorphism $f : A \to K$. Then f extends to some $g \in \mathrm{End}(M)$ by assumption. Pick, by Zorn's Lemma, a complement A' in M essentially containing A. Then the restriction $g_{|A'}$ is obviously a monomorphism. Hence $K = g(A)$ is essential in $g(A')$. Since K is a complement this implies $K = g(A')$, whence $A = A'$. The conclusion follows. □

REMARK. Modules in which submodules isomorphic to complements are complements always decompose into relatively injective summands by [5, Lemma 4]. So

Corollary 2.10 also follows from that result and Lemma 3.3. It is proved in [2, Corollary 2.8] that a pseudo-injective CS module is continuous. This result also follows from Lemma 3.3 and the definition of CS.

Dinh [2] raised the question whether a CS module M which is pseudo-injective is quasi-injective, and stated in [2] that the answer is affirmative when M is furthermore nonsingular. Now we present some partial answers to Dinh's question.

THEOREM 3.4. *M is quasi-injective if and only if M is pseudo-injective and M^2 is CS.*

PROOF. Assume M is pseudo-injective and M^2 is CS. Let M_1 and M_2 be two isomorphic copies of M and $X = M_1 \oplus M_2$. Note that M is continuous by the preceding remark.

First let A be any complement in X with $A \cap M_1 = 0$ and $A \cap M_2$ essential in A. There exist submodules V and V' of M_2 such that $V \oplus V' = M_2$ and V contains $A \cap M_2$ essentially. Also since M^2 is CS by assumption, we have $A \oplus A' = X$ for some submodule A' of X. Since V is a direct summand of a continuous module, V is continuous (see [10]), whence it has exchange property by [10, Theorem 3.4]. Since $V \cap A$ is essential in A, we have $V \cap A' = 0$. Thus we must have $V \oplus A' = X$. Hence A is isomorphic to a summand, namely V of M_2.

Now let C be a submodule of X such that $C \cap M_1 = 0$ and pick, by Zorn's Lemma, a complement K in X of M_1 containing C. Again by Zorn's Lemma, choose a complement K_1 in K of $K \cap M_2$ and a complement K_2 in K of K_1 containing $K \cap M_2$. Note that $K \cap M_2$ is essential in K_2 and that K_1 and K_2 are complements in X by [3, 1.10]. By Proposition 3.1 there exists some submodule T of X containing K_1 with $M_1 \oplus T = X$. Then $T \cong M$ and K_1 is a complement in T, whence K_1 is isomorphic to a complement in M_2. Also by the preceding paragraph K_2 is isomorphic to a complement of M_2 too. Now consider the usual projection $\pi : M_1 \oplus M_2 \to M_2$. We have $M_1 \oplus (K_1 \oplus K_2) = M_1 \oplus (\pi(K_1) \oplus \pi(K_2))$, where $\pi(K_i) \cong K_i$. Hence by continuity of M_2 and the above argument, $\pi(K_1) \oplus \pi(K_2)$ is a summand of M_2. Now, since K is a complement of M_1, $M_1 \oplus K = M_1 \oplus \pi(K)$ is essential in X. Then $\pi(K)$ is essential in M_2. Also, by choice of K_i, $K_1 \oplus K_2$ is essential in K. Then $\pi(K_1) \oplus \pi(K_2)$ is essential in $\pi(K)$, hence in M_2. This implies that $M_2 = \pi(K_1) \oplus \pi(K_2) = \pi(K)$. Thus $M_1 \oplus K = X$. Now it follows by [3, Lemma 7.5] that M_1 is M_2-injective. The proof is now complete. □

The following is a key result.

LEMMA 3.5. *Let $M = \bigoplus_{i \in I} M_i$ be a direct sum of uniform modules M_i. M is quasi-injective if and only if it is pseudo-injective. In particular, any uniform pseudo-injective module is quasi-injective.*

PROOF. First let M be a uniform pseudo-injective module. Let A be a submodule of M and $f : A \to M$ be a nonzero homomorphism. If $\mathrm{Ker}(f) = 0$, then f can be extended to an element of $\mathrm{End}(M)$ by assumption. So assume $\mathrm{Ker}(f) \neq 0$. Let $\delta = i_A - f$, where $i_A : A \to M$ is the inclusion map. Since $\mathrm{Ker}(f) \neq 0$ and M is uniform, $\mathrm{Ker}(\delta) = 0$. Then by pseudo-injectivity assumption δ can be extended to some $g \in \mathrm{End}(M)$. Now $1 - g$ is obviously an extension of f. Thus M is quasi-injective.

Now let $M = \bigoplus_{i \in I} M_i$ be a direct sum of uniform modules M_i and assume that M is pseudo-injective. Then, by Corollary 2.10, $M(I - i)$ is M_i-injective for all $i \in I$. Now by the preceding paragraph and since direct summands of pseudo-injectives are obviously pseudo-injective, each M_i is quasi-injective. Therefore M is quasi-injective. □

THEOREM 3.6. *Over a right Noetherian ring R, a right R-module M is quasi-injective if and only if M is a pseudo-injective CS-module.*

PROOF. Let M be a pseudo-injective CS module. Then M is a direct sum of uniform submodules by [11]. Now the result follows by Lemma 3.5. □

Before proving the next result, note that R is called a right countably Σ-CS ring if $R_R^{(\mathbb{N})}$ is a CS module.

THEOREM 3.7. *The following conditions are equivalent for a ring R:*

(i) *R is a quasi-Frobenius ring.*

(ii) *Every projective right R-module is essentially pseudo-R_R-injective.*

(iii) *$R_R^{(\mathbb{N})}$ is essentially pseudo-R_R-injective.*

(iv) *R is a right countably Σ-CS ring with finite uniform dimension and R_R is essentially pseudo-injective.*

PROOF. The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are obvious, and (i) $\Rightarrow$ (iv) follows from the fact that every injective module is CS, and (iii) $\Rightarrow$ (i) follows by Theorem 2.7.

(iv) $\Rightarrow$ (i) Since R_R has finite uniform dimension, then R_R is pseudo-injective by Theorem 3.2. Then by Theorem 3.4 R is a right self-injective ring with finite uniform dimension. Hence R is a semiperfect right countably Σ-CS ring. This implies by [7] that R is Artinian. Now the conclusion follows. □

The following results were proved in [5, Theorem 2, Corollary 4, Theorem 3, Theorem 4] for modules in which submodules isomorphic to complements are complements. Each pseudo-injective module satisfies this property by Lemma 3.3, whence we have the following corollaries.

COROLLARY 3.8. *Any decomposition of a pseudo-injective module into indecomposable submodules complements summands.*

COROLLARY 3.9. *An essentially pseudo-injective module with finite uniform dimension has the internal cancellation property.*

Recall that every right R-module over a right Noetherian ring R is locally Noetherian.

COROLLARY 3.10. *If M is a locally Noetherian pseudo-injective module, then $M = A \oplus B$, where A is a maximal quasi-injective summand, B has no quasi-injective summands, and A and B have no nonzero isomorphic submodules.*

COROLLARY 3.11. *A locally Noetherian Dedekind-finite pseudo-injective module has internal cancellation property.*

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