A provocative tale of unwinding

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1. *Introduction*

One fine spring morning, at some point, I was simply tearing a few sheets of paper towel off a fresh roll. As the sheets came off and the roll rotated about the vertical axle of the holder (Figure 1), I lazily pondered the geometry lurking in the situation. I realised that sometimes I would let the roll rotate as the unwound part moved straight until tearing, while at other times, I might hold the roll in place and unwind the sheets around the roll. More often, it would be some spontaneous combination of the two modes of unwinding.

FIGURE 1: The paper towel roll rotates and unwinds

Further rumination about this commonplace act happened to rekindle a happy memory from my college days. The memory of figuring out a simple-

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looking yet provocative physics problem in the classic book of problems by I. E. Irodov [1].

In its slightly paraphrased version, the Irodov problem involves a vertical cylinder fixed on a frictionless horizontal surface and a string that is tightly wound around the cylinder. The free end of the string is attached to a particle (which we call the bob in the rest of the paper) in motion, causing the string to keep unwinding while its unwound part remains straight and taut at all times. While I considered the various thought-provoking aspects of this problem, I wondered, in accordance with the paper towel roll scenario, what would happen if the cylinder in the Irodov problem is hinged smoothly at its axis instead of being fixed in place. That is when intriguing details started to unfold which also involves a due amount of calculus.

The next section is a review of a version of the original Irodov problem, and then we delve into its aforementioned modification in the section that follows thereafter.

2. *A review of the Irodov problem*

The top view of the system at the initial instant $t = 0$ is shown in Figure 2a. The cylinder of radius R is fixed in place with its centre at point C. The massless, inextensible, ideal string of indefinitely large available length is tightly wound around the cylinder counterclockwise. The position of the bob of mass m initially coincides with the point P on the circumference of the cylinder, where both points P and C lie on the x-axis. The initial velocity u_0 of the bob is in the positive x-direction, as shown.

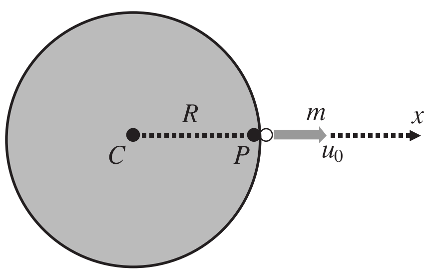

FIGURE 2a: The top view of the system at $t = 0$ with the cylinder of radius R fixed on a frictionless horizontal plane, the initial velocity u_0 of the bob of mass m being in the positive x-direction.

The string unwinds as time progresses while the bob moves accordingly over the frictionless plane. Figure 2b shows the snapshot of the configuration at some arbitrary later time t. The unwound part of the string is the straight line segment tangent to the cylinder at the point Q and connecting to the bob. Since the wound part of the string cannot slide along the circumference of the cylinder, the length s of that unwound part of the string must be equal to the circular arc length PQ. Hence, if the angle

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subtended by PQ at the centre C is ϕ, then

$$s = R\phi. \tag{1}$$

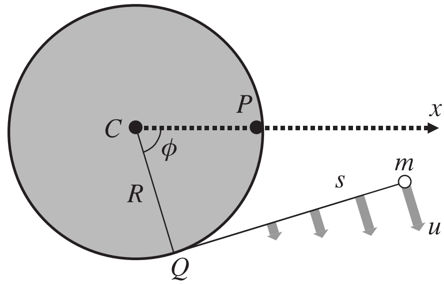

FIGURE 2b: The configuration of the system at a later time: The unwound part of the string is the straight line segment tangent to the point Q and connecting to the bob. The length s of the unwound part is equal to the circular arc length PQ. In addition, for the unwound part, the point Q serves as the instantaneous centre of rotation, with the length of the bold arrows showing the velocities should be proportional to the distance from Q.

Regarding the motion of the unwound part of the string, the point Q where the string detaches from the cylinder, acts as the instantaneous centre of rotation. Hence the velocity of any point on the unwound part is perpendicular to the string as shown in Figure 2b with the bold arrows, its magnitude being directly proportional to the distance from Q. Since the said direction of the velocities is normal to the unwound part of the string, we refer to this mode of unwinding of the string as the 'normal mode' of unwinding. The resulting locus of the bob during the continued unwinding process is an infinite spiral which is known as the 'involute of the circle' [2], plotted in a zoomed-out view in Figure 3, which also shows the circumference of the cylinder at the centre.

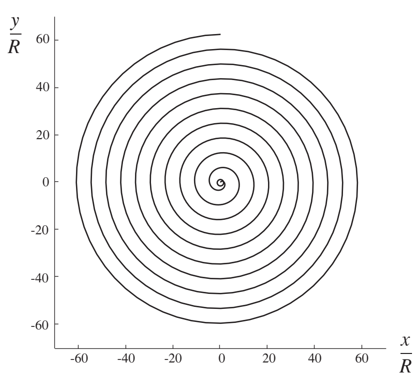

FIGURE 3: The locus of the bob, which is an involute spiral of a circle. The centre of the circular circumference is at the origin, with the circle appearing very small due to the zoomed-out view.

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Regarding the dynamics of the system, the only external force on the bob is the tension exerted by the string. This tension will be directed along the string at any point, and therefore perpendicular to the velocity of the bob. Consequently, the work done on the bob is zero, and thus, according to the work-KE theorem in mechanics, the kinetic energy of the bob remains unchanged. Hence

$$\tfrac{1}{2}mu^2 = \tfrac{1}{2}mu_0^2,$$

From which it follows that

$$u = u_0, \qquad (2)$$

i.e. the speed of the bob also remains unchanged.

Now, let us refer back to Figure 2b. Since the unwound part of the string, being in the direction tangent to the cylinder at point Q, is perpendicular to the radial direction CQ, the angle through which the alignment of the unwound part of the string changes during any given time interval is the same as the increment of ϕ. However, at any given instant, since the point of detachment Q serves as the instantaneous centre of rotation for the unwound part of the string including the bob, the speed u of the bob can be written as

$$u = s\frac{d\phi}{dt}, \qquad (3)$$

since $\frac{d\phi}{dt}$ is the instantaneous angular speed of the unwound part of the string.

Using (1), (2) and (3), we arrive at

$$u_0 = \frac{1}{R}s\frac{ds}{dt}, \qquad \text{or} \qquad u_0\,dt = \frac{1}{R}s\,ds.$$

Integrating, and using the initial condition that $s = 0$ at $t = 0$, we obtain the variation of the unwound length s with time:

$$s = \sqrt{2u_0Rt}. \qquad (4)$$

3. *The modified model with the smoothly hinged cylinder*

Finally, we consider the modified system where the cylinder is now smoothly hinged at its axis rather than being fixed in place. The configuration at the start of the unwinding process still corresponds to Figure 2a, with the cylinder initially stationary and the bob having an initial velocity u_0 in the positive x-direction. However, as the string unwinds, the tension in its unwound part exerts an external torque on the cylinder in the counterclockwise direction, thereby causing an angular acceleration in the same direction. A snapshot at some arbitrary later instant is depicted in Figure 4.

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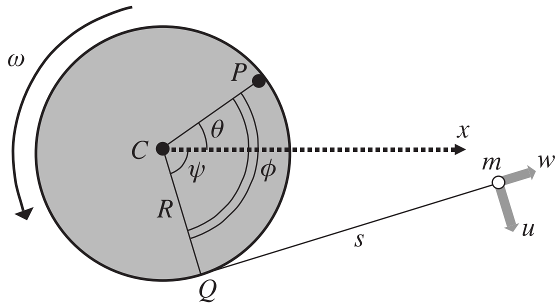

FIGURE 4: The configuration of the system at a later time: The unwound part of the string is the straight line segment tangent to the point Q and connecting to the bob. The length s of the said unwound part is equal to the circular arc length PQ. However, the point P on the circumference of the cylinder has moved to a new position relative to its initial position on the x-axis due to the counterclockwise rotation of the cylinder about its axis.

As shown in Figure 4, due to the counterclockwise motion acquired by the cylinder, the point P, fixed on the circumference of the cylinder, has moved from its initial position on the x-axis. However, the unwound part of the string, the straight line segment tangent to the instantaneous point of detachment Q and connecting to the bob, still has length s that equals the arc length PQ at the moment. Hence if the angle subtended by PQ at the centre C is still denoted by ϕ, as shown, relation (1) still holds.

Now, in terms of the angle θ, the counterclockwise angle turned through by the cylinder, the angular velocity of the cylinder, ω, is given by

$$\omega = \frac{d\theta}{dt}. \tag{5}$$

For this modified system, the previously defined *normal mode* of unwinding will now combine with a new *parallel mode* of unwinding, which is introduced entirely due to the angular motion of the cylinder itself. Hence in addition to the normal components of velocity on the unwound part of the string, there will be a component of velocity on every point on the string that is directed along the string. This 'parallel' component has a magnitude $R\omega$, the same as the instantaneous speed of any point on the circumference of the rotating cylinder, precisely because the wound part of the string cannot slide relative to the cylinder. For the bob in particular, the normal component u and the parallel component w are as shown in Figure 4, where, once again

$$w = R\omega, \tag{6}$$

while the instantaneous speed of bob is given by:

$$v = \sqrt{u^2 + w^2}. \tag{7}$$

One consequence is that the angle through which the alignment of the unwound part of the string changes during any given time interval is the same as the increment of ψ (instead of ϕ), where ψ is the angle made by the

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radial line CQ with the positive x-direction, as shown in Figure 4. This means the angular speed of the unwound part of the string is $\frac{d\psi}{dt}$, which lets us write the normal component of the velocity of the bob as

$$u = s\frac{d\psi}{dt}. \tag{8}$$

Now, as is evident from the Figure

$$\psi = \phi - \theta. \tag{9}$$

Using (5) and (9) in (8), we obtain

$$u = s\left(\frac{d\phi}{dt} - \omega\right). \tag{10}$$

At this point, we ask: What is the way to infer the detailed dynamics of the system as the string keeps unwinding? The obvious quantities of interest are s, ω and u. For all these quantities, we are interested learning how they evolve with time.

First of all, let us consider the net mechanical energy content of the cylinder-string-bob system. There is an external force exerted by the hinge on the axis of the cylinder. However, since the axis does not ever undergo any displacement, the work done by that hinge force is zero. In addition, there are no dissipative internal forces within the system either. Therefore the net mechanical energy of the system, which consists solely of the kinetic energy of the cylinder and the bob, will remain conserved.

Considering the kinetic energy $\frac{1}{2}I\omega^2$ of the cylinder and the kinetic energy $\frac{1}{2}mv^2$ of the bob, we write:

$$\frac{1}{2}I\omega^2 + \frac{1}{2}mv^2 = \frac{1}{2}mu_0^2, \tag{11}$$

where the right-hand side above is the initial kinetic energy, and I is the moment of inertia of the cylinder about its axis. Making use of (6) and (7), we obtain from (11) after slight simplification:

$$\left(I + mR^2\right)\omega^2 + mu^2 = mu_0^2. \tag{12}$$

Next, we consider the angular momentum of the system (about the centre C) of the cylinder-string-bob system and the net external torque on it. Again, the only external force on the system is the force exerted by the hinge, and since it acts directly on the axis located at the centre C itself, its torque about the same point in zero. Consequently the net angular momentum of the system about the centre C will remain conserved.

Let us refer back to Figure 2a. Since at $t = 0$, the velocity of the bob as well as its radial position CP are both in the same direction (which happens to be the positive x-direction), the angular momentum of the bob about the centre C is zero. As for the cylinder, it is at rest at this initial instant, so its angular momentum will be trivially zero. Hence the net angular momentum of the system that we start with is zero, and since it does not change with time, this net angular momentum about the centre C will be zero at all times.

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Now let us refer to Figure 4 again. At this point, the angular momentum of the cylinder is simply $I\omega$ counterclockwise. As for the bob, its parallel component of velocity $w = R\omega$ has a moment arm equal to R, and therefore has a contribution to the angular momentum of $mwR = mR^2\omega$, in the counterclockwise direction. The normal component of velocity u has a moment arm equal to s, and therefore has a contribution to the angular momentum in the amount of mus, in the clockwise direction. Adding all the contributions to the angular momentum of the cylinder-string-bob system, we write

$$\left(I + mR^2\right)\omega - mus = 0. \tag{13}$$

Now, for the sake of compactness, we express I as

$$I = \mu mR^2, \tag{14}$$

where μ is a positive number. Next, without losing generality, we define the following dimensionless quantities:

$$s^* = \frac{s}{R}; \tag{15}$$

$$t^* = \frac{tu_0}{R}; \tag{16}$$

$$u^* = \frac{u}{u_0}; \tag{17}$$

$$w^* = \frac{w}{u_0}; \tag{18}$$

and $$\omega^* = \frac{\omega R}{u_0}. \tag{19}$$

It is worth mentioning that the angular variables ϕ, θ, ψ are dimensionless already. Moreover, the way s^* is defined in (15), a comparison with (1) shows it to be one and the same with ϕ, i.e.

$$s^* \equiv \phi. \tag{20}$$

Making use of (14), (17) and (19) in (12), after simplifying we write:

$$\eta\omega^{*2} + u^{*2} = 1, \tag{21}$$

where $$\eta = \mu + 1. \tag{22}$$

Similarly, making use of (14), (15), (17), (19) and (22) in (13), we write:

$$\eta\omega^* - u^*s^* = 0. \tag{23}$$

At the same time, making use of (15), (17), (19) and (20) in (10), we obtain the expression

$$u^* = s^*\left(\frac{ds^*}{dt^*} - \omega^*\right). \tag{24}$$

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Using (24) in (23), we obtain after rearranging

$$\omega^* = \frac{s^{*2}}{s^{*2} + \eta}\left(\frac{ds^*}{dt^*}\right). \tag{25}$$

Substituting (25) back into (24) gives us upon rearranging

$$u^* = s^*\left(\frac{\eta}{s^{*2} + \eta}\right)\frac{ds^*}{dt^*}. \tag{26}$$

Now, using (26) in (21), after substantial rearranging we obtain

$$\sqrt{\eta}\frac{s^*}{\sqrt{s^{*2} + \eta}}\frac{ds^*}{dt^*} = 1 \tag{27}$$

or
$$\sqrt{\eta}\frac{s^*}{\sqrt{s^{*2} + \eta}}\,ds^* = dt^*.$$

Integrating the above with the initial condition that $s^* = 0$ at $t^* = 0$, after rearranging we obtain

$$s^* = \frac{1}{\sqrt{\eta}}\sqrt{(t^* + \eta)^2 - \eta^2}. \tag{28}$$

Next, using (28) in (25), we obtain

$$\omega^* = \frac{1}{\sqrt{\eta}}\frac{\sqrt{(t^* + \eta)^2 - \eta^2}}{t^* + \eta}. \tag{29}$$

Finally, using (28) in (26), we arrive at

$$u^* = \frac{\eta}{t^* + \eta}. \tag{30}$$

Regarding the parallel component of the velocity of the bob, from (6), (18) and (19), we can easily conclude that $w^* \equiv \omega^*$, thus letting us write from (29)

$$w^* = \frac{1}{\sqrt{\eta}}\frac{\sqrt{(t^* + \eta)^2 - \eta^2}}{t^* + \eta}. \tag{31}$$

It is worth noting that the behaviour of the system pertaining to the original Irodov problem with the cylinder fixed in place, i.e. the one detailed in Section 2, is recovered as $\eta \to \infty$, as expected.

Before we move on to the next section, we look at one more quantity that is worth our attention, namely the orientation of the straight, unwound part of the string. Since $\omega^* = \frac{d\theta}{dt^*}$, it follows from (25) that

$$d\theta = \frac{s^{*2}}{s^{*2} + \eta}\,ds^*.$$

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Integrating the above with the initial condition $\theta^* = 0$ when $s^* = 0$, we obtain

$$\theta = s^* - \sqrt{\eta}\tan^{-1}\left(\frac{s^*}{\sqrt{\eta}}\right). \tag{32}$$

Then, with the use of (9) and (20) in (32), we arrive at

$$\psi = \sqrt{\eta}\tan^{-1}\left(\frac{s^*}{\sqrt{\eta}}\right). \tag{33}$$

4. *The salient features of the findings*

It is quite straightforward to verify using (28), (29), (31) and (30), respectively, that $s^*_\infty = \infty$, $\omega^*_\infty \equiv w^*_\infty = \frac{1}{\sqrt{\eta}}$ and $u^*_\infty = 0$, where the subscript '∞' refers to the limiting value of a quantity as $t^* \to \infty$.

One striking feature of our results is a transition in the mode of unwinding. We recall that at $t^* = 0$, the (dimensionless) normal and parallel components of the velocity of the bob are $u^* = 1$ and $w^* = 0$, respectively, commensurate with a purely normal mode. However, we note that as $t^* \to \infty$, the values of the two quantities approach $u^* = 0$ and $w^* = \frac{1}{\sqrt{\eta}}$, respectively. This implies the conclusion:

Although the unwinding process starts in a purely normal mode, it switches asymptotically to a purely parallel mode in the limit $t^* \to \infty$!

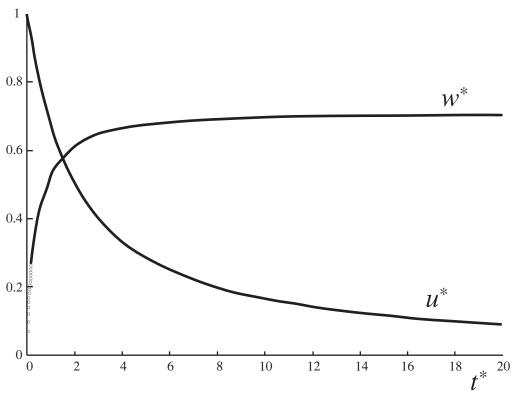

FIGURE 5: The plot of u^* and w^*, according to (30) and (31), respectively, for $\eta = 2$, indicating an asymptotic transition from a purely normal to a purely parallel mode of unwinding.

In Figure 5, we plot u^* and w^*, as obtained from (30) and (31) for $\eta = 2$, Incidentally, this example corresponds to a situation where the cylinder in the system is represented by a uniform cylindrical shell that has the same mass as that of the bob, so that $\mu = 1$ as defined in (14), thus making

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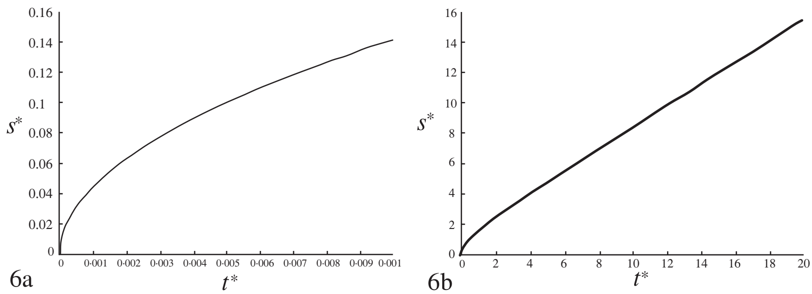
$\eta = 2$ according to (22).

Next we consider the long-time dependence of the rate of unwinding. As deduced from (4), in the case of the cylinder fixed in place, $s^* \propto \sqrt{t^*}$ at all times. However, for our modified system with the hinged cylinder, one can show from (28) without difficulty that, although for small t^* the relation $s^* \propto \sqrt{t^*}$ still holds, for large t^* the nature of the variation becomes $s^* \propto t^*$, i.e. at large enough times the unwinding happens at an effectively constant rate, irrespective of the value of η. This transition is manifest in the plots in Figures 6a and 6b, for the same example value of $\eta = 2$.



FIGURES 6a and 6b: The (dimensionless) length s^* plotted against t^* for (a) small values of t^* and (b) large values of t^*, for $\eta = 2$. In (a), the plot resembles one-half of a horizontally situated parabola around its vertex, corresponding to an effective variation of $t^* \propto s^{*2}$, as expected for small values of t^*. In (b), the linear behaviour is manifest for large values of t^*.

Finally, we focus our attention on the rotation of the unwound part of the string, which brings us to what is perhaps the most fascinating and non-trivial aspect of our findings. Putting $s^* \to \infty$ in (33), we readily deduce that

$$\psi_{\max} = \lim_{s^* \to \infty} \psi = \sqrt{\eta}\,\frac{\pi}{2}. \tag{34}$$

The above relation implies that for any finite value of η, *however large, the maximum angle through which the unwound part of the string rotates is finite, approaching that value asymptotically as time goes on*. This is in stark contrast with the case of the cylinder fixed in place, for which the unwound part of the string rotates through indefinitely large angles while the bob traverses a spiral that is the circle involute.

As an example, for $\eta = 2$, the value of $\psi_{\max}$ as obtained from (34) is $\sqrt{2}\left(\frac{1}{2}\pi\right)$, which is about 127.3°.

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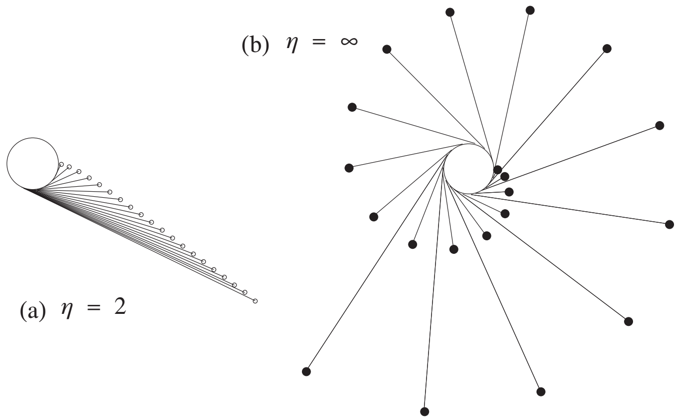

FIGURE 7: The positions of the unwound part of the string, plotted for twenty equally spaced values of s^*, namely $s^* \in \{0.5, 1.0, \ldots, 9.5, 10.0\}$, along with the position of the bob at the end of the string, for (a) $\eta = 2$, and (b) $\eta = \infty$, i.e. with the cylinder effectively fixed in place.

In Figure 7 the positions of the unwound part of the string, along with the positions of the bob at the end of the string, are plotted for twenty equally spaced values of s^* in the course of the unwinding process. The starting configuration is still the one shown in Figure 2a. Figure 7a, which uses $\eta = 2$, it shows how the orientation of the unwound part of the string tends to saturate as the unwinding continues, corresponding to a $\psi_{\max}$ value of 127·3° as obtained from (34). As a contrast, the plots in Figure 7b, which pertains to a cylinder effectively fixed in place, shows an ongoing rotation of the unwound part of the string that is destined to continue rotating through indefinitely large angles, while the bob traverses an infinite spiral locus as mentioned before.

Figure 8 shows the plots of the loci of the bob for (a) $\eta = 2$, (b) $\eta = 4$, (c) $\eta = 16$, (d) $\eta = 36$, (e) $\eta = 100$ and (f) $\eta = \infty$. The case of $\eta = \infty$, which is effectively that of the cylinder fixed in place, reproduces the infinite spiral which is the involute of the circumference of the cylinder. For every other case, with a finite value of η, the locus tends to straighten up at some point, commensurate with the final alignment of the string. The respective $\psi_{\max}$ values in degrees, as obtained from (34) are (a) $\psi_{\max} = 127.3°$, (b) $\psi_{\max} = 180°$, (c) $\psi_{\max} = 360°$, (d) $\psi_{\max} = 540°$, (e) $\psi_{\max} = 900°$ and (f) $\psi_{\max} = \infty$, respectively.

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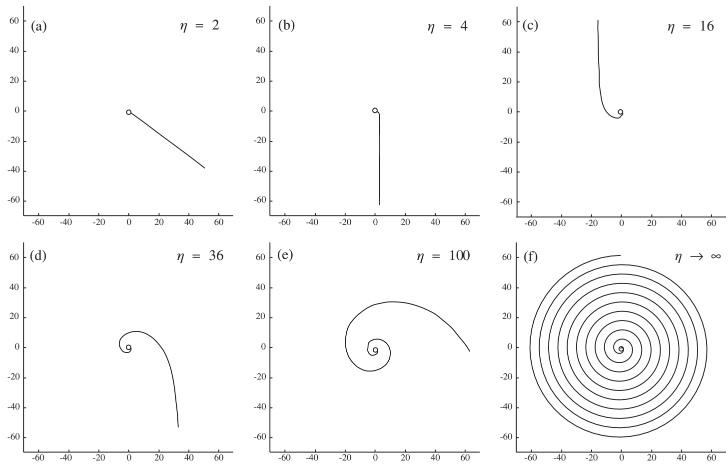

FIGURE 8: The loci of the bob plotted for (a) $\eta = 2$, (b) $\eta = 4$, (c) $\eta = 16$, (d) $\eta = 36$, (e) $\eta = 100$ and (f) $\eta \to \infty$ (cylinder effectively fixed in place).

We leave the reader with a further thought-provoking extension of the problem: If the system is still situated on a frictionless horizontal plane, but this time with the cylinder and the bob both free to move in the course of the unwinding, then how will the details of the unwinding process play out? Assume that just like in the case detailed in this paper, the cylinder is initially at rest and the bob, which is initially touching the cylinder, is imparted a normal velocity to initiate the unwinding process. Here is a hint: the conserved quantities in this case will be the net linear momentum of the cylinder-string-bob system, its net kinetic energy, and the net angular momentum about the centre of mass of the system.

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