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POSITIVE PERTURBATIONS AND UNITARY EQUIVALENCE

C. R. PUTNAM

1. Preliminaries. Let T be a (not necessarily bounded) self-adjoint operator on a Hilbert space $\mathbf{H}$ with the spectral resolution $T = \int_{-\infty}^{\infty} t dE_t$. The set of elements x in $\mathbf{H}$ for which $||E_t x||^2$ is absolutely continuous is a subspace, $\mathbf{H}_a$, of $\mathbf{H}$ which reduces T. (See Halmos [**1**, p. 104]; Kato [**2**, p. 516].) If $\mathbf{H}_a \neq 0$, the restriction of T to $\mathbf{D}_T \cap \mathbf{H}_a$ is called the *absolutely continuous part of* T; in case $\mathbf{H} = \mathbf{H}_a$, T is said to be *absolutely continuous*. Recall also that T is said to be *half-bounded* if for some real number c, either $T \geqq cI$ (that is, $(Tx, x) \geqq c(x, x)$ for all x in $\mathbf{D}_T$) or $T \leqq cI$.

2. Main Results. First we prove the following

THEOREM 1. *A half-bounded self-adjoint operator T has an absolutely continuous part if and only if there exists a bounded operator $D \geqq 0$, $\neq 0$, and a unitary operator U such that*

(2.1) $\quad T + D = UTU^* \quad$ *and* $\quad \sigma(U) \neq \{z : |z| = 1\}$.

Proof. In order to prove the "only if" part of the theorem, suppose first that T is absolutely continuous. The assertion is then an immediate consequence of Theorem 5.15 of Kato [**2**, p. 561]. In fact, for any absolutely continuous self-adjoint operator $T = T_1$ (not necessarily half-bounded) and for any $\alpha > 0$, this result implies the existence of an operator $D_1 \geqq 0$, even of rank 1, and a unitary operator U_1 such that $T_1 + D_1 = U_1 T_1 U_1^*$ and $||U_1 - I|| < \alpha$. In case $\mathbf{H}_a \neq \mathbf{H}$, the "only if" assertion follows by considering the direct sum representation $T = T_1 \oplus T_2$ on $H = \mathbf{H}_a \oplus \mathbf{H}_a^{\perp}$ and putting $D = D_1 \oplus 0$ and $U = U_1 \oplus I$. The "if" part follows immediately from Theorem 2.12.2 of Putnam [**4**, p. 38].

As noted above, the "only if" portion of Theorem 1 is valid for any self-adjoint operator, half-bounded or not. We do not know whether the "if" part holds in general. However, if there exists a $D \geqq 0$, $\neq 0$, for which (2.1) holds and for which $(-\infty, \infty) - \sigma(T)$ contains an open interval of length greater than $||D||$, it follows from [**4**], *loc. cit.*, that T must have an absolutely continuous part.

We mention the following open question:

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(*) Suppose that T is an arbitrary self-adjoint operator and that there exists a bounded $D \geqq 0$, $\neq 0$, and a unitary U satisfying (2.1). Does it follow that such a T always has an absolutely continuous part, even if the complement of $\sigma(T)$ does not contain an open interval of length exceeding $||D||$, as is the case for instance if $\sigma(T) = (-\infty, \infty)$?

THEOREM 2. *There exists a bounded absolutely continuous self-adjoint T and a compact $D \geqq 0$ for which $T + D$ is unitarily equivalent to T and such that $\sigma(U) = \{z : |z| = 1\}$ for every unitary operator U satisfying $T + D = UTU^*$.*

Proof. Let $P = -id/dx$ denote the self-adjoint differential operator on $L^2(-\infty, \infty)$ with domain

$$\mathbf{D}_P = \{f : f \text{ absolutely continuous}, f \text{ and } f' \text{ in } L^2(-\infty, \infty)\};$$

cf. Stone [**5**, p. 441]. Then $P^2 + I = -d^2/dx^2 + 1$ on $L^2(-\infty, \infty)$ has spectrum $[1, \infty)$. If $V(x) = (1 + |x|^c)^{-1}$, where $1 < c = \text{constant} < 2$, the operator of multiplication by $V(x)$ is bounded and non-negative, and, in addition,

$$V \in L(-\infty, \infty) \quad \text{and} \quad \liminf_{b-a\to\infty} (b - a)^{-3} \int_a^b V^{-1}(x)dx = 0.$$

If follows from Theorems 5.16.1 and 5.16.2 of [**4**, pp. 122–123], that $P^2 + I$ and $P^2 + I + V$ are absolutely continuous and unitarily equivalent, and that $\sigma(U) = \{z : |z| = 1\}$ if U is any unitary operator for which

(2.2) $\quad P^2 + I = U(P^2 + I + V)U^*.$

Since $(P^2 + I + V)f - (P^2 + I)f = Vf$ for all f in the domain of P^2, then $(P^2 + I)^{-1} - (P^2 + I + V)^{-1} = (P^2 + I)^{-1}V(P^2 + I + V)^{-1}$, as an equation for bounded operators. (Cf. [**3**, p. 149] for a similar argument.) Also, $P^2 + I + V \geqq P^2 + I$ (as an operator inequality) and hence $(P^2 + I)^{-1} \geqq (P^2 + I + V)^{-1}$; cf. [**4**, pp. 36–37]. Thus, if $T = (P^2 + I + V)^{-1}$ and $D = (P^2 + I)^{-1}V(P^2 + I + V)^{-1}$, we see that $D \geqq 0$, $\neq 0$, and that, by (2.2), $T + D$ and T are unitarily equivalent. In addition, if U is any unitary operator for which $T + D = UTU^*$ then $\sigma(U) = \{z : |z| = 1\}$. Finally, since $V(x) \to 0$ as $|x| \to \infty$, an argument similar to that in [**3**, p. 150], shows that D is compact. This completes the proof of Theorem 2.

The absolutely continuous T of Theorem 2 is of course of a special type. We note the following open question:

(**) If T is any bounded absolutely continuous self-adjoint operator, does there always exist some compact non-negative perturbation D for which $T + D$ is unitarily equivalent to T and such that $\sigma(U) = \{z : |z| = 1\}$ whenever U is a unitary operator satisfying $T + D = UTU^*$?

Clearly, any D in (**) must satisfy $D \neq 0$ since, otherwise, one could satisfy $T + D = UTU^*$ by choosing $U = I$.

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3. An example. Consider the differential operator $P = -id/dx$ on $L^2(-\infty, \infty)$ discussed above. Let $m(x)$ be any real-valued, measurable function on $(-\infty, \infty)$ with the property that $T = P + m$ is self-adjoint on $L^2(-\infty, \infty)$. (A sufficient condition, for instance, is that $m(x)$ be bounded.) Next, let $q(x)$ be a real-valued, measurable function on $(-\infty, \infty)$ satisfying

(3.1) $0 < q(x) <$ constant

If D denotes the self-adjoint operator corresponding to multiplication by $q(x)$, then D is bounded and $D \geqq 0, \neq 0$. Further, $T + D$ is unitarily equivalent to T. In fact, it is easily verified that

$$(3.2) \quad T + D = UTU^* \quad \text{where} \quad U = \exp\left(-i \int_0^x q(t)dt\right) \quad (= \text{unitary}).$$

(See [**2**, pp. 528–529].) Also, a similar argument shows that for any constant c, $P + m + c$ is unitarily equivalent to $P + m$, so that the spectrum of $T = P + m$ is invariant under translations and hence $\sigma(T) = (-\infty, \infty)$. If now $q(x)$ is chosen so as to satisfy $\int_{-\infty}^{\infty} q(t)dt < \pi$, in addition to (3.1), it is clear that $\sigma(U) \neq \{z : |z| = 1\}$ for the unitary U of (3.2).

Consequently, a negative answer to the question (*) would follow if there exists a function $m(x)$ with the property that $P + m$ is self-adjoint on $L^2(-\infty, \infty)$ but has no absolutely continuous part. Of course, it is necessary that such a function m not be summable on some finite interval. Otherwise, $\exp(-i\int_0^x m(t)dt)$ would effect a unitary equivalence of $P + m$ and P (cf. (3.2)), and the latter operator is well-known to be absolutely continuous. We do not know whether the mere self-adjointness of $P + m$ implies that $P + m$ is absolutely continuous or even that it has an absolutely continuous part. Obviously, however, an affirmative answer to (*) would imply the latter assertion.

4. Unitary absolute continuity. The concept of the absolutely continuous part of a unitary operator with the spectral resolution $U = \int_0^{2\pi} e^{it} dE_t$ can be defined in a manner analogous to that for a self-adjoint operator. As a consequence of Theorem 2.12.2 of [**4**, p. 38], we note that if T is any self-adjoint operator (whether or not it has an absolutely continuous part), if D is bounded, $\geqq 0$ and $\neq 0$, and if the complement of $\sigma(T)$ contains an open interval of length exceeding $\|D\|$ (in particular, if T is half-bounded, as in Theorem 1), then any unitary operator U satisfying $T + D = UTU^*$ must have an absolutely continuous part. Further, if

(4.1) 0 is not in the point spectrum of $D (D \geqq 0)$,

then U is absolutely continuous. Consequently, since the operator D constructed in the proof of Theorem 2 satisfies (4.1), it follows that one may require that any U in the statement of that theorem also be absolutely continuous.

Added in Proof. We are indebted to Professor T. Kato for communicating to us a proof that the local integrability of the function m occurring in Section 3 above is also a necessary (as well as a sufficient) condition for the self-adjointness of the operator $P + m$ on $L^2(-\infty, \infty)$.

References

1. P. R. Halmos, *Introduction to Hilbert space and the theory of spectral multiplicity* (Chelsea Pub. Co., New York, 1951.

2. T. Kato, *Perturbation theory for linear operators,* Die Grundlehre der mathematischen Wissenschaften, *132* (Springer, 1966).

3. C. R. Putnam, *A note on inverses of differential operators,* Math. Zeit. *64* (1956), 149–150.

4. ——— *Commutation properties of Hilbert space operators and related topics,* Ergebnisse der Math., *36* (Springer, 1967).

5. M. H. Stone, *Linear transformations in Hilbert space and their applications to analysis,* Amer. Math. Soc. Colloq. Publications, vol. 15, 1932.

Purdue University,
West Lafayette, Indiana