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Synopsis

The underlying template running through all the chapters of this book is the application of the concepts of quantum field theory to the description of indeterminate and random phenomena, be they classical or quantum in origin.

Quantum field theory was initially developed to explain the phenomena of high energy physics and soon spread to condensed matter and solid state physics. The common thread of these applications was that of a quantum system with a large number of degrees of freedom (independent variables). The pioneering work of Wilson (1983) and Witten (1989) showed that quantum field theory is not tied to quantum physics but, instead, has a wide range of applications in many other fields. These ground-breaking developments brought to the forefront what can be called *quantum mathematics*, mentioned earlier in the Preface.

Quantum mathematics refers to the system of mathematical concepts that arise in quantum systems – with some of the leading concepts being that of quantum fields, vacuum expectation values, Hamiltonians, state spaces, operators, correlation functions, Feynman path integrals and Lagrangians.[1] The interpretation of quantum mathematics, in general, does not have any relation to physics and, instead, needs to reflect the specificity of the domain of inquiry to which quantum mathematics is being applied.

Many of the standard books on quantum field theory are written primarily for a readership that is drawn from theoretical physics. There are voluminous and encyclopedic books on quantum field theory – such as the three-volume opus by Weinberg (2010) which runs for more than 1,500 pages – that are meant for professional theorists and researchers, being inaccessible to nonspecialists and beginners.

[1] There is a clash of terminology regarding the term “Lagrangian.” In economics, the term is used for the auxiliary function – for which there is no special term in physics – required when using a Lagrange multiplier for constrained optimization. In physics, the term “Lagrangian” encodes the fundamental model describing a quantum phenomenon, and has an ontological status equal to that of the Hamiltonian. In this book, physics terminology is used.

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Some books on quantum fields are geared toward specific applications, such as the book by Peskin and Schroeder (1995), which is written for applications in high energy theory and phenomenology, or the book by Zinn-Justin (1993) that seeks to explain critical phenomena.

This book eschews the standard approach and provides a quick and concise introduction to quantum field theory, meant for an audience from finance and economics who has neither the patience nor the motivation for reading any of the specialized books. This book is focused on providing a direct route, with a minimum use of formalism, to the leading ideas of quantum fields – from free fields to the concept of renormalization and the renormalization group.

The emphasis of this book is on the underlying mathematics of quantum field theory – which could form the basis of applications of quantum mathematics to disciplines that go beyond theoretical physics. In particular, this book is an introduction to the mathematical formalism of *quantum fields* that one would require for undertaking modeling in finance or in the study of economics.

In the Table of Contents, Chapters on economics and finance are marked by asterisks for ease of reference. Topics from economics and finance have been interwoven with topics of quantum field theory. This interleaving of chapters has the purpose of clearly demonstrating and illustrating how ideas from quantum field theory carry over to economics and finance. The unifying theme of all chapters on economics and finance is that (1) topics are discussed that can be modeled using action functionals, Hamiltonians and path integrals and (2) as discussed in the Preface, topics that can be empirically tested have been included.

The chapters on quantum field theory were taught for many years as an introductory graduate textbook for quantum field theory. Problems have not been included for chapters on quantum field theory since there are many books, such as Radovanovic (2005) and Cheng and Li (2000), that have problems and solutions for different topics of quantum field theory.

1.1 Organization of the book

Figure 1.1 shows the connection of the various chapters of the book, which are grouped into four parts. A reader can navigate the chapters by concentrating on only the chapters of interest.

Part I is the introduction to the two underlying themes that underpin quantum fields, which are the *quantum principle* and *classical field theory*. The mathematical formalism of these two subjects contains the seeds for the mathematics and applications of quantum fields. All the chapters in Part I are based on the

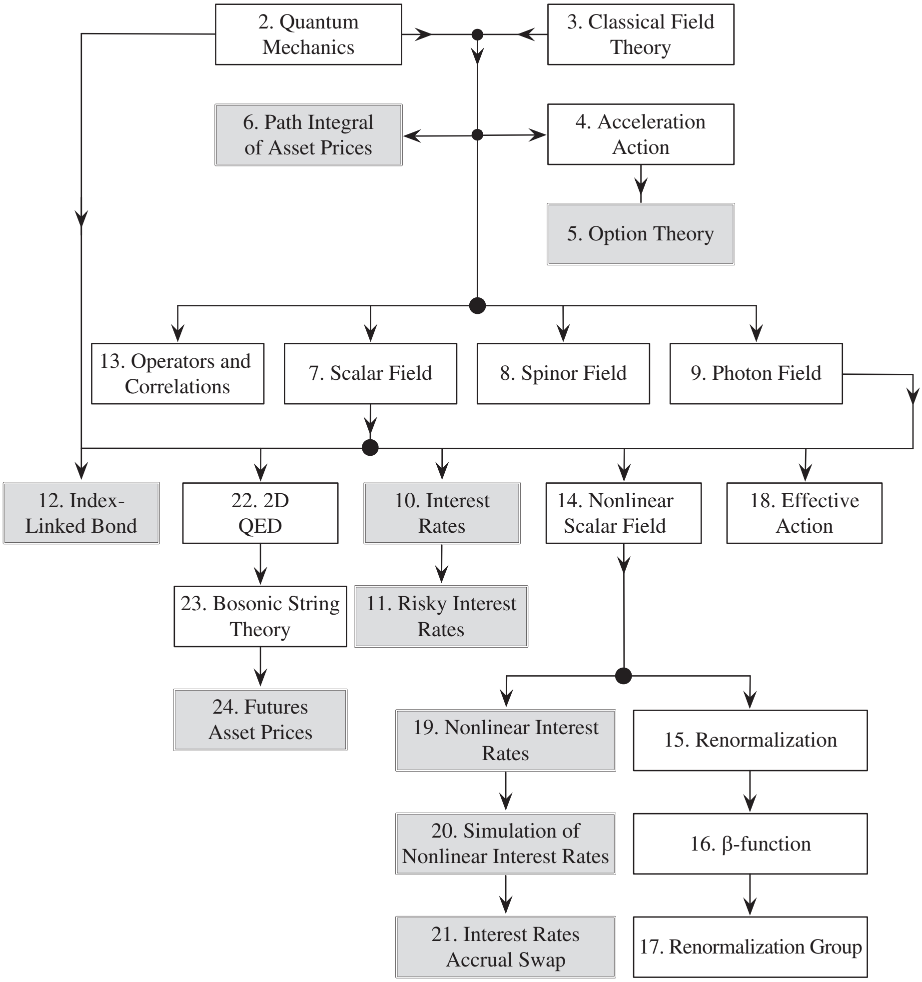

Figure 1.1 The organization of the chapters, with their interconnections. The shaded chapters are on economics and finance.

mathematics of quantum mechanics and foreground the more complex derivations in the next three parts.

Chapter 2 introduces the foundations of quantum mechanics; the quantum principle leads to quantum indeterminacy and to the quantum theory of probability. Quantum probability has emerged in recent years as a major new subfield of decision science and behavioral finance, and the discussion in this chapter is partly to introduce the ideas of quantum probability. A brief discussion of path integrals and Hamiltonians for quantum mechanics prepares the ground for the subsequent analysis.

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In Chapter 3 classical field theory is studied to understand the Lagrangian formulation of classical fields. The Lagrangian and action functional are one of the pillars of quantum mathematics, and classical fields are a precursor to quantum fields. Global and local symmetries of the theory are encoded in the symmetries of the Lagrangian. To illustrate the interplay of symmetry breaking and gauge invariance, symmetry breaking for a nonlinear complex scalar field coupled to the Maxwell field is analyzed; it is shown to lead to both the Landau–Ginzburg formulation of superconductivity and the Higgs mechanism of particle physics. The Lorentz group is analyzed to understand the structure of empty spacetime, and it is shown how the Lorentz group classifies the various types of classical relativistic fields.

Chapter 4 studies the evolution kernel for the quantum mechanical acceleration action, which is a higher derivative action. This chapter gives a derivation of the evolution kernel using the state space and Hamiltonian and not a path integral derivation as given by Baaquie (2014). The acceleration action is a key to the studies of asset prices as well as of forward interest rates. The reason is that asset prices are described by the complex branch of the acceleration action, whereas forward interest rates are described by the real branch. The acceleration action yields a pseudo-Hermitian Hamiltonian and, due its higher derivative kinetic term, yields results quite different from quantum mechanics.

Chapter 5 is on option theory. Central ideas such as the martingale condition and option prices free from arbitrage opportunities are discussed in the quantum mechanical framework. The Black–Scholes equation is given a quantum mechanical derivation with no reference to stochastic calculus. The Black–Scholes equation is generalized to the Baaquie–Yang equation using results from the acceleration action. Options for equities and foreign exchange are derived and empirically tested using market data. It is shown that options provide a more accurate gauge of market instabilities than the volatility of the underlying asset.

In Chapter 6 the formulation of statistical microeconomics is reviewed and a Lagrangian is postulated for modeling asset prices. It is shown how the application of Feynman path integrals arises in the study of asset prices. Empirical evidence is discussed to support the applications of quantum mathematics to the study of asset prices. A Monte Carlo simulation is done to study the nonlinear aspect of the Lagrangian, and confirms the validity of perturbatively studying the nonlinear regime using Feynman diagrams. Multiple commodity prices are analyzed and it is shown that a Lagrangian for multiple commodities provides an accurate description of the empirical correlation function of commodity prices.

Part II focuses on *linear quantum fields* and is a necessary preparation for the study of nonlinear quantum fields. Quantum fields come in many varieties and with a great range of underlying degrees of freedom. The simplest, but not unimportant, case of a quantum field is a scalar field. The free quantum field is studied as a precursor to nonlinear quantum fields. The main difference between a linear and nonlinear quantum field is that the free field does not have any interactions.

Linear fields are important in their own right. The free field comes in many varieties, depending on the nature of the underlying degrees of freedom, with the most important examples being scalar, photon and Dirac fields. One way of decoupling the free field's degrees of freedom is the method of Fourier transform, which resolves the free field into decoupled momentum degrees of freedom. The three most widely used and most useful free quantum fields are the *scalar, spinor* and *vector* quantum fields. These fields have many specific features of great generality and hence need to studied one by one.

Chapter 7 studies the *free scalar* quantum field, which has no self-interaction but nevertheless has many features of a quantum field and is an ideal theoretical laboratory for starting one's study of a system with infinitely many degrees of freedom. Figure 1.1 shows the central position of the free scalar field in developing the more complex and deeper structures of quantum field theory as well as the application of quantum field theory to economics and finance. A scalar quantum field has one degree of freedom for each spacetime point. The scalar field has all the general features of quantum fields and its Lagrangian and Hamiltonian are studied in detail. In particular, the formalism of creation and annihilation operators is carefully analyzed as these are among the most useful mathematical tools for the study of quantum fields. The quantum field in two dimensions is the starting point of this chapter as it is the simplest system quantum field for which the Fock space of states of a quantum field is defined.

Chapter 8 studies the *free spinor* quantum field, of which the Dirac field is a leading example. The Dirac field is based on fermionic degrees of freedom obeying anticommuting fermion statistics. Spinor fields provide a representation of the Lorentz group and are the result of the structure of spacetime. Due to its spinor nature, the quantization of the free Dirac field requires a multicomponent spinor field, having four degrees of freedom at each spacetime point. It is shown how, on the quantization of the Dirac field, two species of particles emerge in its spectrum of states, which are the particle and its *antiparticles*. In fact, the primary motive for studying the Dirac field is to understand the emergence, and the properties, of antiparticles. The relation between the particle and antiparticle states is analyzed and it shown that the Dirac field is invariant under the exchange of particle–antiparticle. Since the Dirac field is a fermionic field, the properties of fermionic variables and the path integral for fermions are briefly reviewed. The Casimir force is evaluated for the Dirac field and leads one to study the boundary conditions for the Dirac field and the associated state space.

Chapter 9 studies the *free photon* field, which is a vector field with the local symmetry of *gauge invariance*. The symmetry of gauge invariance is so important

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that the photon field is also referred to as an Abelian gauge field. To quantize the photon field, one has to choose a gauge. Choosing a gauge is necessary for quantizing both Abelian and non-Abelian Yan–Mills gauge fields. The mathematics required for choosing a gauge is studied in great detail, using both the path integral formalism, which leads to Faddeev–Popov quantization, and the Hamiltonian formalism, which leads to the Coulomb gauge. The state space that results for both the path integral and Hamiltonian quantization are discussed. The Becchi–Rouet–Stora–Tyutin (BRST) symmetry exhibited by the gauge-fixed action in the Faddeev–Popov scheme is utilized to define the state space and is shown to be equivalent to the Gupta–Bleuler quantization for a covariant gauge.

Chapter 10 analyzes interest rates in *finance*. Interest rates are modeled using a two-dimensional stochastic field that is mathematically identical to a two-dimensional Euclidean quantum field. The action, Lagrangian and Hamiltonian for the forward interest rates are modeled using a linear (free) two-dimensional Euclidean quantum field. The Lagrangian is a higher order derivative system, and empirical evidence is briefly reviewed to support the modeling of interest rates by a quantum field. The state space and field Hamiltonian operator are both shown to be time dependent. The martingale condition is derived for the forward interest rates using both the path integral and Hamiltonian formulation.

Chapter 11 continues the study of forward interest rates, with the additional coupling of the risk-free to the risky forward interest rates. It is shown how a spread over the risky rates – the spread being a quantum field in its own right – allows one to extend the formalism. The risky forward interest rates are empirically studied, with reasonable support for the model from market data.

Chapter 12 studies a coupon bond with index-linked stochastic coupons. An asset price, represented by a quantum mechanical degree of freedom, determines the amount of payment of the stochastic coupons. The discounting of future cash flows is determined by the zero coupon bonds modeled by the risk-free forward interest rates, which in turn is modeled by a two-dimensional quantum field. The financial instrument is a synthesis of a quantum mechanical degree with a two-dimensional quantum field, and defines a distinct class of financial instruments.

Part III discusses *nonlinear quantum fields*. The nonlinear properties of quantum fields are, in general, mathematically formidable as well as being fairly intractable – and for the same reason also yield novel and unexpected results.

In Chapter 13, a general derivation is given of the connection of operators and state space with the Feynman path integral; in particular, it is shown that all the time-ordered vacuum expectation values of Heisenberg quantum field operators are given by the correlation functions of the quantum field using the Feynman path integral. The Lehmann–Symanzik–Zimmermann (LSZ) formalism is reviewed to show how the scattering of quantum field states can be reduced to the time-ordered vacuum expectation values of the quantum fields, which in turn can be evaluated using the path integral. These derivations show the centrality of the path integral in the study of quantum fields.

In Chapter 14, the nonlinear scalar quantum field is studied using perturbation theory to understand the divergences of a quantum field. Dimensional regularization is used as an effective cutoff for the quantum field; it is shown that the mass and coupling constant of the quantum field apparently seem to diverge. Feynman diagrams are introduced as a useful bookkeeping device for the terms that appear as one goes to higher and higher order perturbation theory.

Chapter 15 is a key chapter that introduces the idea, as well as a prescription, of renormalization. Four different methods are employed to renormalize the nonlinear scalar field, which are given by bare and renormalized perturbation theory, the background field method and Wilson's thinning of the degrees of freedom. All four methods are shown to yield the same result, but from vastly different perspectives.

The deep and global structures of quantum fields are discussed in Chapters 16 and 17, which address the issues of renormalization and of the renormalization group. The divergences that appear in perturbation theory and the procedure of renormalization are seen to be the natural consequence of the fact that the quantum field describes a system with infinitely many length scales. In Chapter 17 one discovers the rather unexpected connection of quantum field theory to the theory of phase transitions. Recall that quantum field theory was specifically developed to address high energy phenomenon at short distances, whereas phase transitions are determined by the behavior of the system for infinitely separated degrees of freedom.

Another branch of the book leads to the study, in Chapter 18, of effective actions that describe symmetry breaking for nonlinear scalar fields and for scalar quantum electrodynamics. The effective action is evaluated for both cases and it is shown that scalar electrodynamics has spontaneous symmetry breaking that is renormalization group invariant.

Nonlinear scalar fields lead to nonlinear models of interest rates, which is studied in Chapter 19. This chapter concentrates on certain key aspects of the mathematical formalism of nonlinear interest rates. The debt market is driven by Libor simple interest rates. It is shown, due to a nonlinear drift required for fulfilling the martingale condition, that Libor is described by a two- dimensional nonlinear Euclidean quantum field. Due to the higher derivative terms in the Lagrangian, there is no need for renormalizing this nonlinear field. Nonlinear drift is exactly obtained using both the Wilson expansion and the Hamiltonian formulation of the martingale condition. The empirical aspect of nonlinear interest rates has been studied by Baaquie and Yang (2009), Yang (2012) and Baaquie et al. (2014b), and hence is not discussed in this book.

Since perturbation theory is often not effective in studying nonlinear systems, nonlinear interest rates are studied numerically in Chapters 20 and 21– with the intention of exploring the nonlinear structure of these theories. The technique of Monte Carlo simulation is used to evaluate the caplet and swaption price. The interest range accrual swap is studied numerically, and it is shown how to extend the Libor market model to a domain beyond the Libor lattice, to price this instrument.

Part IV analyzes *two-dimensional quantum fields* that are defined on two-dimensional manifolds. The reason for including these topics is because, so far, it is two-dimensional quantum fields that appear in the modeling of finance and economics. The examples chosen are important in their own right, which are quantum electrodynamics in two spacetime dimensions and boson string theory in an ambient 26-dimensional spacetime. These two-dimensional quantum fields provide a glimpse of the rich and diverse variety of quantum fields that exist in two dimensions and could be used for modeling problems in finance and economics. Chapter 24 applies a nonlinear two-dimensional quantum field to the study of futures asset prices. The field theory is free from divergences due to the higher order derivative in the Lagrangia; this feature makes the theory accessible to straightforward calculations since otherwise, the entire machinery of renormalization would have been essential for obtaining finite results.

Chapter 22 discusses an exact solution of quantum electrodynamics in two spacetime dimensions. The exact solution of the Dirac field coupled to the gauge field is reduced to a free field. The coupling of a two-dimensional gauge field to Dirac fermions is studied for various nonperturbative phenomena, such as chiral symmetry breaking and confinement of fermions, both of which are displayed by the interacting system.

Chapter 23 is an elementary introduction to bosonic string theory. Bosonic strings are defined on a two-dimensional manifold and have many features, such as breaking of conformal invariance, that lead to new insights into the behavior of two-dimensional quantum fields. The gauge fixing of the bosonic string leads to the result that the bosonic string is consistent only in 26 spacetime dimensions. Furthermore, the technology of Faddeev–Popov is shown to play a fundamental role in the choosing a gauge for a quantum field having conformal symmetry.

Chapter 24 analyzes futures commodity prices by extending the theory of spot asset prices. Futures commodities prices are modeled by a higher derivative two-dimensional Euclidean quantum field, and empirical evidence in support of this model is discussed. The two-dimensional nonlinear quantum field employed in studying futures asset prices is a higher derivative Lagrangian.

1.2 What is a quantum field?

The principal notion of the entire book is the concept of a quantum field. A preliminary understanding of a quantum field is briefly discussed.

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Relativistic quantum fields arose from the synthesis of quantum mechanics and special relativity. To maintain causality, special relativity – when combined with quantum indeterminacy – requires the existence of antiparticles. The existence of antiparticles makes nonrelativistic quantum mechanics, which has a fixed number of particles (degrees of freedom), inconsistent; particle–antiparticle annihilation and creation from the vacuum needs a state space with an indefinite number of particles.

State space is a linear vector space, and the number of variables required to describe the state space is based on the system's infinitely many degrees of freedom. Recall a degree of freedom, for each instant of time, is an independent variable. For example, a nonrelativistic quantum mechanical particle in three space dimensions has three degrees of freedom. The requirement for an indefinite number of particles for relativistic fields finds its realization in a state space based on infinitely many degrees of freedom. The fact that Feynman path integral is built out of a state space with infinitely many degrees of freedom implies that the path integral is defined on an underlying manifold that is of two or higher dimensions. In contrast, a path integral defined on a one-dimensional manifold describes a quantum mechanical system.

Quantum field theory is appropriate for describing a relativistic quantum system as well as classical random systems with infinitely many coupled degrees of freedom – such as systems undergoing a classical phase transition.

In summary, a quantum field describes a system, either quantum or classical, with infinitely many coupled degrees of freedom.

The question "what is a quantum field?" is revisited in Section 15.14.1. Armed with Wilson's formulation of renormalization, a quantum field will be seen to be a system consisting of infinitely many distinct, but coupled, length scales. Each length scale is described by one of the quantum field's degrees of freedom. The coupling of different length scales is due to nonlinearities of the quantum field.

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