ON THE CONVERGENCE OF SUCCESSIVE APPROXIMATIONS IN THE THEORY OF ORDINARY DIFFERENTIAL EQUATIONS

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1. Let R denote the rectangle: $|t-t_0| \leq a$, $|x-x_0| \leq b$ $(a,b > 0)$ in the (t,x) plane and let $f(t,x)$ be a function of two real variables t and x, defined and continuous on R. If I is the interval $|t-t_0| \leq d$ with $d = \min(a,b/M)$, where $M = \max(|f(t,x)|, (t,x) \in R)$, then every solution $x = x(t)$ of the differential equation $x' = f(t,x)$ defined on I and which satisfies the initial condition $x(t_0) = x_0$, satisfies the integral equation

$$(1.1) \qquad x(t) = x_0 + \int_{t_0}^{t} f(s,x(s))ds,$$

and conversely. In some cases, in order to prove the existence and uniqueness of the solutions of (1.1) on I, one forms the successive approximations

$$(1.2) \qquad x_n(t) = x_0 + \int_{t_0}^{t} f(s,x_{n-1}(s))ds, \; n \geq 2$$

and $x_1(t)$ is a continuous function on I such that $x_1(t_0) = x_0$ and $|x_1(t) - x_0| < b$ for all $t \in I$, then by the choice of I the functions $x_n(t)$ can be defined recursively by (1.2). If the sequence $x_n(t)$, $n = 1,2,..$ converges uniformly on I then its limit is a solution of (1.1) on I. One knows that the condition that f is continuous and the equation (1.1) is uniquely solvable, is not sufficient to guarantee the convergence of the successive approximations [see 1, II, 3].

However, it was shown by E. R. van Kampen [4]

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that the continuity of f together with the Nagumo-Perron uniqueness condition,

$$\text{(N-P)} \quad |f(t,x_1) - f(t,x_2)| \leq k|t-t_0|^{-1}|x_1-x_2|,$$
$$(t,x_1),(t,x_2) \in R \quad \text{and } k \leq 1,$$

are sufficient conditions for the uniform convergence of the successive approximations, but this is no longer true if $k > 1$ ([3]). In this case M.A. Krasnoselskiĭ and S.G. Krein [5] recently showed that if the function f moreover satisfies a Lipschitz condition of order α $(0 < \alpha < 1)$, i.e.,

$$|f(t,x_1) - f(t,x_2)| \leq A|x_1 - x_2|^{\alpha}, \ (t,x_1),\ (t,x_2) \in R$$

and A a constant (independent of t), then the equation (1.1) is uniquely solvable if $k(1-\alpha) < 1$. With an example they also showed that this does not hold if $k(1-\alpha) \geq 1$. The purpose of this note is to show that if f is continuous and satisfies the Krasnoselskiĭ-Krein uniqueness condition

$$\text{(K-K)} \quad \begin{cases} |f(t,x_1) - f(t,x_2)| \leq k|t-t_0|^{-1}|x_1 - x_2| \\ |f(t,x_1) - f(t,x_2)| \leq A|x_1 - x_2|^{\alpha}, \end{cases}$$
$$(t,x_1),(t,x_2) \in R, \quad 0 < \alpha < 1 \text{ and } k(1-\alpha) < 1,$$

the successive approximations are uniformly convergent. We will present two proofs of this fact, which we think are both of interest. The method of the proof we give in section 2 is related to one which was used for similar purposes in [2]. The other proof, which is given in section 3, is quite different, and proves the uniqueness at the same time. At the end of the paper we give an example to show that our theorem is no longer true if $k(1-\alpha) \geq 1$.

2. Let R, f, a, b, M and d be as in 1. As we have to use the fact that the Krasnoselskiǐ-Krein condition implies uniqueness we shall, for the sake of completeness, first present a proof of this fact.

THEOREM (M.A. Krasnoselskiǐ-S.G. Krein[5]):

If $f(t,x)$ satisfies the condition (K-K) (see sec.1), then there exists at most one solution $x = x(t)$ of (1.1) on I for which $x(t_0) = x_0$.

PROOF: First we remark that, if $x_1(t)$ and $x_2(t)$ are two solutions of (1.1) on I, then

$$\lim_{t \to t_0} |t-t_0|^{-k} |x_1(t) - x_2(t)| = 0.$$

Indeed, as

$$|x_1(t) - x_2(t)| \leq \left| \int_{t_0}^{t} |f(s,x_1(s)) - f(s,x_2(s))| ds \right|$$

we obtain

$$|x_1(t) - x_2(t)| \leq 2M|t - t_0|,$$

and hence, by the fact that f satisfies a Lipschitz condition of order α, we have

$$|x_1(t) - x_2(t)| \leq A \left| \int_{t_0}^{t} |x_1(s) - x_2(s)|^{\alpha} ds \right| \leq$$

$$\leq A(2M)^{\alpha} (1+\alpha)^{-1} |t-t_0|^{1+\alpha} < A(2M)^{\alpha} |t-t_0|^{1+\alpha}$$

and by repeating this, we obtain

$$|x_1(t) - x_2(t)| \leq A^{1+\alpha+\dots+\alpha^m} (2M)^{\alpha^{m+1}} |t-t_0|^{1+\alpha+\dots+\alpha^{m+1}}$$

for all m. Hence we have

$$(2.1) \qquad |x_1(t) - x_2(t)| < A^{1/1-\alpha} |t-t_0|^{1/1-\alpha}.$$

And $k(1-\alpha) < 1$ implies

$$\lim_{t \to t_0} |t-t_0|^{-k} |x_1(t)-x_2(t)| \leq \lim_{t \to t_0} A^{1/1-\alpha} |t-t_0|^{(1/1-\alpha)-k} = 0$$

If we assume now that $x_1(t) \neq x_2(t)$ on I, then

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there exists a $t_1 \in I$ such that

$$|t_1-t_0|^{-k}|x_1(t_1) - x_2(t_1)| = \max(|t-t_0|^{-k}|x_1(t)-x_2(t)|, \quad |t-t_0| \le d) = \rho.$$

But then by the first part of the condition (K-K) we obtain the following contradiction

$$\rho \le |t_1-t_0|^{-k}\left|\int_{t_0}^{t_1} |f(s,x_1(s))-f(s,x_2(s))|ds\right| \le |t_1-t_0|^{-k}$$

$$\left|\int_{t_0}^{t_1} k|s-t_0|^{k-1}|s-t_0|^{-k}|x_1(s)-x_2(s)|ds\right|$$

$$< \rho|t_1-t_0|^{-k}\left|\int_{t_0}^{t_1} k|s-t_0|^{k-1}ds\right| = \rho,$$

and the proof is finished.

THEOREM. Let the function $f(t,x)$ be defined on R and continuous there. If f satisfies the condition (K-K), then the successive approximations $(x_n(t))$ $(n = 1,2,\ldots)$ defined by (1.2) converge uniformly on I to the solution $x(t)$ of (1.1) on I.

PROOF: It follows from the definition (1.2) of the successive approximations that they satisfy the inequality

$$(2.2) \qquad |x_n(t_1) - x_n(t_2)| \le M|t_1-t_2|$$

for any t_1, t_2 in the interval I. This implies that the set $(x_n(t))$ $(n = 1,2,\ldots)$ is a set of equicontinuous functions on I. Moreover, letting $t_1 = t_0$ and $t_2 = t$ in (2.2) we obtain

$$|x_n(t)| \le M|t-t_0| + |x_n(t_0)| \le b + x_0$$

and hence the set $(x_n(t))$ is uniformly bounded on I. From Ascoli's theorem (see[1, Ch.1, sec.1]) it follows that there exists a subsequence (x_{n_k}) $(k = 1,2,\ldots)$

which is uniformly convergent to a continuous function $x(t)$ on I as $k \to \infty$. The subsequence $(x_{n_k+1}(t))$ which satisfies

$$x_{n_k+1}(t) = x_0 + \int_{t_0}^{t} f(s, x_{n_k}(s))ds$$

is uniformly convergent on I to a function $\bar{x}(t)$ defined

by $$\bar{x}(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds$$

for f is uniformly continuous on R. We shall show below however, that under the given conditions

$$\lim_{n\to\infty}(x_{n+1}(t) - x_n(t)) = 0 \text{ on I.}$$

If we assume this for a moment then we have also

$$\lim_{k\to\infty}(x_{n_k+1}(t) - x_{n_k}(t)) = 0$$

and this implies $\bar{x}(t) = x(t)$ on I, that is, $x(t)$ is a solution of (1.1). Because of the uniqueness, every subsequence of (x_n) which is convergent will tend to the same solution $x(t)$. This together with Ascoli's theorem implies that every subsequence of $(x_n(t))$ contains a subsequence which converges uniformly to $x(t)$ and this in its turn implies that the sequence $(x_n(t))$ itself is uniformly convergent to $x(t)$ on I. To complete our proof we have to show that

$$\text{(2.3)} \qquad \lim_{n\to\infty}(x_{n+1}(t) - x_n(t)) = 0 \text{ on I.}$$

Because of the facts that $|x_2(t) - x_1(t)| \leq 2M|t-t_0|$ on I and f satisfies a Lipschitz condition of order we have

$$|x_3(t) - x_2(t)| \leq \left| \int_{t_0}^{t} |f(s, x_2(s)) - f(s, x_1(s))| ds \right|$$

$$\leq A(2M)^{\alpha}(1+\alpha)^{-1}|t-t_0|^{1+\alpha} < A(2M)^{\alpha}|t-t_0|^{1+\alpha}$$

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and by repeating

$$(2.4) \quad |x_{m+1}(t)-x_m(t)| \leq A^{1/1-\alpha}(2M)^{\alpha^{m-1}} |t-t_0|^{1+\alpha+..+\alpha^{m-1}}$$
$$< C|t-t_0|^{1+\alpha+.....+\alpha^{m-1}}$$

where $C = A^{1/1-\alpha}\max(2M,1)$. As $k < (1-\alpha)^{-1}$ we can find a positive integer $N(k)$ such that $1+\alpha+.....+\alpha^{n-1} > k$ for all $n \geq N(k)$. Then (2.3) shows that for all $n \geq N(k)$ we have

$$(2.5) \quad |t-t_0|^{-k}|x_{n+1}(t) - x_n(t)| < C|t-t_0|^{\beta}$$

where C has the same meaning as in (2.4) and

$\beta = 1+.....+\alpha^{n-1}-k > 0$. Hence if $n \geq N$ we have

$$(2.6) \quad \lim_{t \to t_0} |t-t_0|^{-k}|x_{n+1}(t) - x_n(t)| = 0.$$

In the remaining part of the proof we shall restrict ourselves to the case that $t_0 \leq t \leq t_0 + d$, as the reasoning is similar for the case $t_0 - d \leq t \leq t_0$.

Put $y_1(t) = (t-t_0)^k \max((s-t_0)^{-k}|x_{N+1}(s) - x_N(s)|, t_0 \leq s \leq t)$

and

$$(2.7) \quad y_{j+1}(t) = \int_{t_0}^{t} k(s-t_0)^{-1}y_j(s)ds \qquad (j = 1,2,...).$$

Then we have

$$(2.8) \qquad 0 \leq y_{j+1}(t) \leq y_j(t) \qquad (j = 1,2,....),$$

$$(2.9) \qquad |x_{N+j}(t) - x_{N+j-1}(t)| \leq y_j(t) \qquad (j = 1,2...),$$

$$(2.10) \qquad \lim(t-t_0)^{-k}y_1(t) = 0.$$

To prove (2.8) we remark that by definition of $y_1(t)$ we have $y_1'(t) \geq k(t-t_0)^{-1}y_1(t)$ and hence $y_1'(t) \geq y_2'(t)$ or $y_1(t) \geq y_2(t)$ and this implies

$$y_3(t) = \int_{t_0}^{t} k(s-t_0)^{-1}y_2(s)ds \leq \int_{t_0}^{t} k(s-t_0)^{-1}y_1(s)ds = y_2(t)$$

and induction will prove (2.8).

For (2.9) we observe that $y_1(t) \geq |x_{N+1}(t)-x_N(t)|$, and this implies that

$$|x_{N+2}(t)-x_{N+1}(t)| \leq \int_{t_0}^{t} |f(s,x_{N+1}(s))-f(s,x_N(s))|ds \leq$$

$$\int_{t_0}^{t} k(s-t_0)^{-1}|x_{N+1}(s)-x_N(s)|ds \leq \int_{t_0}^{t} k(s-t_0)^{-1}y_1(s)ds = y_2(t)$$

and again induction will prove (2.9).

(2.10) follows from (2.5), because (2.5) implies

$$(t-t_0)^{-k}y_1(t) = \max((s-t_0)^{-k}|x_{N+1}(s)-x_N(s)|,\ t_0 \leq s \leq t)$$
$$\leq C(t-t_0)^{\beta},\ \beta > 0.$$

From (2.8) it follows that the sequence $y_j(t)$ is decreasing, and hence it has a limit $y(t) \geq 0$, and by Lebesgue's theorem on dominated convergence we have

$$(2.11) \qquad y(t) = \int_{t_0}^{t} k(s-t_0)^{-1}y(s)ds.$$

Hence $y(t)$ is an integral of the equation

$$(2.12) \qquad y'(t) = k(t-t_0)^{-1}y(t),\ t_0 < t \leq t_0 + d.$$

As $y(t) \leq y_1(t)$ we see by (2.10) that

$$(2.13) \qquad \lim_{t \to t_0}(t-t_0)^{-k}y(t) = 0.$$

Now the only solution of (2.12) which satisfies (2.13) is the zero solution; hence $y(t) = 0$. Then (2.9) implies (2.3) and the proof is completed.

3. Before giving another proof of our theorem we make the following preliminary remarks:

Let $C(I)$ be the set of all continuous functions on the interval I ($|t-t_0| \leq d$). We define

$$(3.1)\ \rho_\lambda(x_1,x_2) = \max(|t-t_0|^{-\lambda}|x_1(t)-x_2(t)|,\ |t-t_0| \leq d,$$
$$\lambda \geq 0,\ x_1,x_2 \in C(I)),$$

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and we put $\rho_0 = \rho$ which is the metric of uniform convergence in C(I). If $\lambda > 0$ then ρ_λ is not necessarily finite for every pair of functions $x_1, x_2 \in C(I)$; however, we shall prove that ρ_λ has the following properties:

(3.2) $\rho_\lambda(x_1,x_2) = \rho_\lambda(x_2,x_1)$ and $\rho_\lambda(x_1,x_3) \leq \rho_\lambda(x_1,x_2) + \rho_\lambda(x_2,x_3)$;

that is, ρ_λ is a metric.

(3.3) $d^{-\lambda}\rho(x_1,x_2) \leq \rho_\lambda(x_1,x_2)$ for all $x_1,x_2 \in C(I)$

and hence $\rho_\lambda(x_1,x_2) = 0$ if $x_1(t) = x_2(t)$ for all $t \in I$,

(3.4) $\quad \lim_{n,m\to\infty} \rho_\lambda(x_n,x_m) = 0$

implies there exists an $x \in C(I)$ such that

$$\lim_{n\to\infty} \rho_\lambda(x_n,x) = 0.$$

We need only to prove (3.4) as (3.3) and (3.2) are evident. From

$$\lim_{n,m\to\infty} \rho_\lambda(x_n,x_m) = 0$$

it follows that there exists a subsequence, which we denote by y_n, such that $\rho_\lambda(y_{n+1},y_n) < 2^{-n}$. It is easy to see that the sequence y_n is uniformly convergent on I, and let its limit be x(t). Then as

$$x(t) - y_n(t) = \sum_{k=n}^{\infty} (y_{k+1} - y_k)$$

we obtain that

$$\rho_\lambda(x,y_n) \leq \sum_{k=n}^{\infty} \rho_\lambda(y_{k+1},y_k) \leq 2^{1-n}$$

and hence $\lim_{n\to\infty} \rho(x,y_n) = 0$.

By (3.2) we have

$$\lim_{n\to\infty} \rho_\lambda(x,x_n) \leq \lim_{n\to\infty}(\rho_\lambda(x,y_n) + \rho_\lambda(y_n,x_n)) = 0,$$

which proves (3.4).

Let now f, R, M and d again be as in 1. $p > 1$ is a number such that $pk(1-\alpha) < 1$ (such a p exists

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as $k(1-\alpha) < 1$). By (2.5) we see that there exists an index $N(pk)$ such that for all $n \geq N(pk)$,

$$(3.5) \quad |t-t_0|^{-pk}|x_{n+1}(t) - x_n(t)| \leq C|t-t_0|^{\gamma},$$

where $\qquad \gamma = 1+\ldots\ldots+\alpha^{n-1} - pk > 0.$

This implies that for all $n \geq N(pk)$, $\rho_\lambda(x_{n+1}, x_n)$ is finite with $\lambda = pk$.

Let $A(x) = x_0 + \int_{t_0}^{t} f(s,x(s))ds$, and assume that $x_1, x_2 \in C(I)$ such that $\rho_\lambda(x_1,x_2) < \infty$ with $\lambda = pk$.

Then, by the first part of the condition (K-K) we have

$$|A(x_1) - A(x_2)| \leq \left| \int_{t_0}^{t} |f(s,x_1(s)) - f(s,x_2(s))|ds \right|$$

$$\leq \left| \int_{t_0}^{t} k|s-t_0|^{-1}|x_1(s) - x_2(s)|ds \right|$$

$$= \left| \int_{t_0}^{t} k|s-t_0|^{pk-1}|s-t_0|^{-pk}|x_1(s) - x_2(s)|ds \right|$$

$$\leq \rho_\lambda(x_1,x_2) \left| \int_{t_0}^{t} k|s-t_0|^{pk-1}ds \right| = k(pk)^{-1}|t-t_0|^{pk}\rho_\lambda(x_1,x_2).$$

Hence $|t-t_0|^{-pk}|A(x_1) - A(x_2)| \leq p^{-1}\rho_\lambda(x_1,x_2)$ with $\lambda = pk$ or

$$(3.6) \quad \rho_\lambda(A(x_1),A(x_2)) \leq q\,\rho_\lambda(x_1,x_2), \ \lambda = pk, \ q = p^{-1} < 1,$$

which shows that the operator A with respect to the metric ρ_λ, $\lambda = pk$, behaves like a contraction. If now l is a positive integer and $n \geq N(pk)$ we have

$$\rho_\lambda(x_{n+\ell},x_n) \leq \sum_{i=1}^{\ell+1} \rho_\lambda(x_{n+i},x_{n+i-\ell})$$
$$\leq q^{n-N}\left(\sum_{i=1}^{\ell-1} q^i\right) \rho_\lambda(A(x_N),x_N),$$

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$\lambda = pk$ or $0 \leq \lim_{n\to\infty}\rho_\lambda(x_{n+\ell}, x_n)$

$\leq \lim_{n\to\infty} q^{n+1-N}(1-q)^{-1}\rho(A(x_N), x_N) = 0.$

Hence by (3.4) there exists a $x \in C(I)$ such that $\lim_{n\to\infty}\rho_\lambda(x_n, x) = 0$. Then

$\rho_\lambda(A(x),x) \leq \rho_\lambda(A(x),x_n) + \rho_\lambda(x,x_n) < q\rho_\lambda(x,x_{n-1}) + \rho(x,x_n)$

for all n, implies $\rho(A(x),x) = 0$, and $A(x) = x$ by (3.3), which means that x is a solution of (1.1). Here we have proved now that the successive approximations (1.2) converge not only uniformly to the solution of (1.1), but even in the stronger sense of the metric ρ_λ. That the solution is unique follows from (3.6). Indeed if x_1, x_2 are two different solutions of (1.1) then we have by (2.1) that $\rho_\lambda(x,x_2) < \infty$ for $\lambda = pk$, and then (3.6) gives rise to the following contradiction:

$$0 < \rho_\lambda(x_1,x_2) = \rho_\lambda(A(x_1),A(x_2)) < q\,\rho_\lambda(x_1,x_2) \quad (q < 1),$$

which proves $x_1 = x_2$.

4. <u>Example</u>: Let $f(t,x)$ be defined by

$$f(t,x) = \begin{cases} 0 & (0 \leq t \leq 1,\ t^{1/1-\alpha} < x < +\infty \\ kt^{\alpha/1-\alpha} - kx/t & (0 \leq t \leq 1,\ 0 \leq x \leq t^{1/1-\alpha}) \\ kt^{\alpha/1-\alpha} & (0 \leq t \leq 1,\ -\infty < x < 0 \end{cases}$$

on the domain $0 \leq t \leq 1$, $-\infty < x < +\infty$, where $k > 0$, $0 < \alpha < 1$. This function f is continuous and bounded by the constant k, and it is not very hard to see that f satisfies the following inequalities:

(i) $|f(t,x_1) - f(t,x_2)| \leq (k/t)|x_1-x_2| \quad (0 < t \leq 1),$

(ii) $|f(t,x_1) - f(t,x_2)| \leq k|x_1-x_2|^\alpha \quad (0 \leq t \leq 1,$
$-\infty < x_1, x_2 < +\infty).$

Let $k(1-\alpha) = \beta$ and consider the differential equation $x' = f(t,x) \quad (0 \leq t \leq 1,\ -\infty < x < +\infty)$ with the initial point (0,0). The successive approximations (1.2) become, for $0 < t < 1$ if $x_1(t) = 0$ $(0 \leq t \leq 1)$, if $\beta < 1$ and $\beta > 1$,

$$x_n(t) = (\beta - \beta^2 + \ldots + (-1)^{n+1}\beta^n)t^{1/1-\alpha}$$

and if $\beta = 1$

$$x_{2n-1}(t) = 0 \quad (n = 1,2,\ldots) \text{ and } x_{2n}(t) = t^{1/1-\alpha}$$
$$(n = 1,2,\ldots).$$

This shows that if $\beta = k(1-\alpha) < 1$ the successive approximations converge uniformly to the unique solution

$$\beta(1+\beta)^{-1}t^{1/1-\alpha}.$$

If $\beta = k(1-\alpha) = 1$, there is no convergence at all; moreover, the functions $x(t) = 0$ and $x(t) = t^{1/1-\alpha}$ are not solutions of the equation, since

$0 \neq f(t,0) = kt^{1/1-\alpha}$ and $(1-\alpha)^{-1}t^{1/1-\alpha} \neq f(t,t^{1/1-\alpha}) = 0$;
if $\beta = k(1-\alpha) > 1$, then the sequence $x_n(t)$ is obviously divergent (the equation however has also in this case the (unique) solution $\beta(1+\beta)^{-1}t^{1/1-\alpha}$). Summing up, this example shows that if $k(1-\alpha) \geq 1$ our theorem is no longer true even if the equation is uniquely solvable.

REFERENCES

1. E.A. Coddington and Norman Levinson, Theory of ordinary differential equations, (New York, 1955).
2. J. Dieudonne, Sur la convergence des approximations successives, Bull. Sci. Math., (2) 69 (1945), 62-72.
3. E.K. Haviland, A note on the convergence of the successive approximations to the solution of an ordinary differential equation, Amer. J. Math., 54 (1932), 632-634.
4. E.R. van Kampen, Notes on systems of ordinary differential equations, Amer. J. Math., 63 (1941), 371-376.

https://doi.org/10.4153/CMB-1958-003-5 Published online by Cambridge University Press

5. M. A. Krasnoselskiǐ and S. G. Krein, _On a class of uniqueness theorems for the equation $y' = f(x,y)$._ Uspehi Mat. Nauk (N.S.) _11_ (1956), no 1 (67), 209–213 (Russian).

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