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DERIVATIONS IN PRIME RINGS

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ABSTRACT. Let R be a prime ring and $d \neq 0$ a derivation of R. We examine the relationship between the structure of R and that of $d(R)$. We prove that if R is an algebra over a commutative ring A such that $d(R)$ is a finitely generated submodule then R is an order in a simple algebra finite dimensional over its center.

In [1] and [2] the relationship between a prime ring R and its subset $d(R) = \{d(x) \mid x \in R\}$, d a derivation of R, is studied. Herstein shows in [1] that if $d(x)\, d(y) = d(y)\, d(x)$ for all $x, y \in R$, R prime and $d \neq 0$, then either R is commutative or R is an order in a simple algebra of characteristic 2 which is 4 dimensional over its center. In [1] Herstein also asks the more general question: If $s_k[x_1, \ldots, x_k] = 0$ is the standard identity of degree k and if $d \neq 0$ is a derivation of a prime ring R such that $s_k[d(x_1), \ldots, d(x_k)] = 0$ for all $x_1, \ldots, x_k \in R$ can we conclude that R must be rather special or must satisfy s_k?

In response to this question Kovacs gives examples in [2] which show:

(A) For any prime number p, there is a prime ring R of characteristic p with derivation $d \neq 0$ such that $s_{4p+1}[d(x_1), \ldots, d(x_{4p+1})] = 0$ for all $x_1, \ldots, x_{4p+1} \in R$, but R satisfies no polynomial identity.

(B) There is a prime ring R of characteristic 0 with derivation $d \neq 0$ such that $[d(x_1)\, d(x_2), d(x_3)\, d(x_4)]\, d(x_5)[d(x_6)\, d(x_7), d(x_8)\, d(x_9)] = 0$ for all $x_1, \ldots, x_9 \in R$, but R satisfies no polynomial identity.

In light of these examples we look at the following question. Suppose a prime ring R with derivation $d \neq 0$ is an algebra over a commutative ring A such that $d(R)$ is contained in a finitely generated submodule of R. Does R satisfy a polynomial identity? We show that the answer is yes. We will arrive at this as a simple consequence of a theorem about functions more general than derivations.

We begin by defining this more general collection of functions.

DEFINITION. $f: R \to R$ is a semi-derivation of R if there exists a function $g: R \to R$ such that

(1) $f(xy) = f(x)g(y) + xf(y) = f(x)y + g(x)f(y)$ for all $x, y \in R$ and

(2) $f(g(x)) = g(f(x))$ for all $x \in R$.

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An example of semi-derivations which are not derivations are functions of the form $f = g - 1$, where g is a homomorphism of R and 1 is the identity map on R. Since most of the work in this paper will be in the more general context of semi-derivations, we will also obtain results on homomorphism of prime rings.

We proceed by proving the key

LEMMA 1. *Let R be any ring and f a semi-derivation of R such that $f^{2n-1} \neq 0$. If R is an algebra over a commutative ring A such that $f^n(R)$ is contained in a finitely generated submodule of R, then R contains non-zero left and right ideals that are contained in finitely generated submodules of R.*

Proof. If $y \in R$ then since $f^n(R)$ is contained in a finitely generated submodule of R, so is

$$W = f^n(R) + \sum_{i=0}^{2n-1} f^n(R) f^i(g^{2n-1-i}(y)).$$

(Viewing f^0 and g^0 as identity maps.)

Let $x \in R$ and fix y as above, then by repeated use of laws (1) and (2) on semi-derivations we get

$$f^n(xy) = \sum_{i=0}^{n} \binom{n}{i} f^{n-i}(x) f^i(g^{n-i}(y)). \tag{$*$}$$

If in $(*)$ we replace x by $f^{n-1}(x)$ and y by $g^{n-1}(y)$ we see that $f^{n-1}(R) f^n(g^{n-1}(y)) \subset W$. Now if in $(*)$ we replace x by $f^{n-2}(x)$ and y by $f(g^{n-2}(y))$ we see that $f^{n-2}(R) f^{n+1}(g^{n-2}(y)) \subset W$. If we continue this procedure of replacing x by $f^{n-j}(x)$ and y by $f^{j-1}(g^{n-j}(y))$ for $j = 1, 2, \ldots, n$, we finally obtain that $Rf^{2n-1}(y) \subset W$. Since there is some $y \in R$ such that $f^{2n-1}(y) \neq 0$ we see that there exists a non-zero left ideal of R that is contained in a finitely generated submodule of R. Similarly we can show the existence of a non-zero right ideal with the same property.

In showing the existence of the left ideal in the proof of Lemma 1 we only used one of the two formulas in law (1) on semi-derivations. It can be noted that if g is surjective then Lemma 1 can be proven even if f satisfies only one of the two formulas in (1). Our main result will now follow as a special case of

THEOREM 1. *Let R be a prime ring with semi-derivation f. Suppose R is an algebra over a commutative ring A such that $f^n(R)$ is contained in a finitely generated submodule. Then if $f^{2n-1} \neq 0$, R is an order in a simple algebra finite dimensional over it center.*

Proof. By Lemma 1 there exist non-zero left and right ideals, L and T respectively, that are contained in finitely generated A modules. Since R is prime, there exists $r \in R$ such that $I = LrT$ is a non-zero ideal of R. However, I

is also contained in a finitely generated A module and it therefore satisfies a polynomial identity. In a prime ring if a non-zero ideal satisfies a polynomial identity, then the entire ring satisfies a polynomial identity. Therefore R also satisfies a polynomial identity. By a theorem of Posner, any prime ring satisfying a polynomial identity is an order in a simple algebra finite dimensional over its center.

Later we will show that even if f is a derivation, the conclusion of Theorem 1 need not hold if we weaken the condition $f^{2n-1} \neq 0$ to $f^{2n-2} \neq 0$. In Theorem 1 if we let f be a derivation and let $n = 1$ we obtain our main result which is

THEOREM 2. *Let R be a prime ring with derivation $d \neq 0$. Suppose R is an algebra over a commutative ring A such that $d(R)$ is contained in a finitely generated submodule. Then R is an order in a simple algebra finite dimensional over its center.*

If we consider the case where R is an algebra over a field we get Corollaries to Theorems 1 and 2.

COROLLARY 1. *Let R be a prime ring with semi-derivation f. Suppose R is an algebra over a field F such that $f^n(R)$ is contained in a finite dimensional subspace. Then if $f^{2n-1} \neq 0$, R is a simple algebra finite dimensional over its center.*

Proof. The ideal I obtained in the proof of Theorem 1 is in this case a finite dimensional subspace of R. Therefore I is a prime, artinian ring satisfying a polynomial identity; so it follows that I is a simple algebra finite dimensional over its center. However, I must contain a central idempotent of R, thus $I = R$.

As before we obtain a result on derivations as a special case.

COROLLARY 2. *Let R be a prime ring with derivation $d \neq 0$. Suppose R is an algebra over a field F such that $d(R)$ is contained in a finite dimensional subspace. Then R is a simple algebra finite dimensional over its center.*

We can easily strengthen Theorem 2 and Corollary 2 to give us Corollary 3 which we state without proof.

COROLLARY 3. *Let R be a prime ring with derivation $d \neq 0$ and I a non-zero ideal of R.*

(a) *If R is an algebra over a commutative ring A such that $d(I)$ is contained in a finitely generated submodule, then R is an order in a simple algebra finite dimensional over its center.*

(b) *If R is an algebra over a field F such that $d(I)$ is contained in a finite dimensional subspace, then R is a simple algebra finite dimensional over its center.*

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As previously stated, if g is a homomorphism of a ring then $f=g-1$ is a semi-derivation. Since Corollary 3 is really a result on semi-derivations, by viewing it in terms of homomorphisms we have:

COROLLARY 4. *Let R be a prime ring with homomorphism $g\neq 1$ and I a non-zero ideal of R.*

(a) *If R is an algebra over a commutative ring A such that $\{g(i)-i \mid i\in I\}$ is contained in a finitely generated submodule, then R is an order in a simple algebra finite dimensional over its center.*

(b) *If R is an algebra over a field F such that $\{g(i)-i \mid i\in I\}$ is contained in a finite dimensional subspace, then R is a simple algebra finite dimensional over its center.*

We conclude by showing that the condition $f^{2n-1}\neq 0$ in Theorem 1 cannot be weakened to $f^{2n-2}\neq 0$. In particular we will give an example of a prime ring R with derivation d such that R is an algebra over a field F and $d^n(R)$ is a finite dimensional subspace with $d^{2n-2}\neq 0$, but R satisfies no polynomial identity:

Let F be a field of characteristic 0 or $p\geq 2n-1$. Suppose R is the ring of countable matrices over F with only a finite number of non-zero entries; that is, $R=\bigcup_{m=1}^{\infty} F_m$ where F_m is the $m\times m$ matrices over F. Now let $a=e_{12}+e_{23}+\cdots+e_{n-1,n}$ so $a^n=0$ and $a^{n-1}\neq 0$. Define derivation d by $d(x)=ax-xa$ for all $x\in R$. Then for any positive integer k,

$$d^k(x)=\sum_{i=0}^{k}(-1)^i\binom{k}{i}a^{k-i}xa^i.$$

Therefore $d^{2n-1}=0$ and $d^{2n-2}\neq 0$. We also see that

$$d^n(R)\subset \sum_{\substack{i+j=n\\ i,j\geq 1}} a^iRa^j\subset F_n.$$

Therefore $d^n(R)$ is a finite dimensional subspace of R and $d^{2n-2}\neq 0$, but R satisfies no polynomial identity.

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