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A SUBSPACE THEOREM FOR ORDINARY LINEAR DIFFERENTIAL EQUATIONS

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Abstract

The study of the S-unit equation for algebraic numbers rests very heavily on Schmidt's Subspace Theorem. Here we prove an effective subspace theorem for the differential function field case, which should be valuable in the proof of results concerning the S-unit equation for function fields. Theorem 1 states that either

$$\mathrm{Ord}_a(L_1 L_2 \cdots L_n)$$

has a given upper bound where

$$L_1(\mathbf{P}),\, L_2(\mathbf{P}),\, \ldots,\, L_n(\mathbf{P})$$

are linearly independent linear forms in the polynomials

$$\mathbf{P} = (P_1(x),\, P_2(x),\, \ldots,\, P_n(x))$$

with coefficients that are formal power series solutions about $x = 0$ of non-zero differential equations and where Ord_a denotes the order of vanishing about a regular (finite) point of functions $f_{k,i} : (k = 1, n;\, i = 1, n)$ or

$$\mathbf{P} = (P_1(x),\, P_2(x),\, \ldots,\, P_n(x))$$

lies inside one of a finite number of proper subspaces of $(K(x))^n$. The proof of the theorem is based on the wroskian methods and graded sub-rings of Picard-Vessiot extensions developed by D. V. Chudnovsky and G. V. Chudnovsky in their function field analogues of the Roth and Schmidt theorems. A brief discussion concerning the possiblity of a subspace theorem for a product of valuations including the infinite one is also included.

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1. Introduction

For a long time there has been much interest concerning the accuracy with which algebraic numbers can be approximated by rational numbers. In 1955, K. F. Roth [6] showed that if α is real and algebraic of degree at least 2 then for each $\varepsilon \geq 0$ the inequality

$$(1) \qquad \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\varepsilon}}$$

has only many solutions in rationals p/q. This result is best possible of its type but non-effective in the sense that it does not yield an upper bound for the largest value of q. Later C. F. Osgood [4] proved an effective "Roth-type" result for the solutions of algebraic differential equations, and D. V. Chudnovsky and G. V. Chudnovsky [1] used wronskian methods and graded subrings of Picard-Vessiot extensions to prove an effective analogue of Roth's Theorem for the solutions of ordinary linear differential equations.

A powerful generalisation of (1) for a system of algebraic linear forms was given by Schmidt's Subspace Theorem [8] which was later extended by H. P. Schlickewei [7] to the p-adic case. Both results were ineffective but found to be crucial in the study of the S-unit equation

$$(2) \qquad \alpha_1 x_1 + \alpha_1 x_2 + \cdots + \alpha_n x_n = 0,$$

summarized by J. H. Evertse and K. Györy [2] in their results concerning (2) for $n > 2$.

The purpose of this paper is to present a proof of an effective subspace theorem for the differential function field case based on the methods used by D. V. Chudnovsky and G. V. Chudnovsky [1] in their function field analogue of Roth's Theorem.

The theorem in this paper gives a bound on a valuation of a product of linear forms $L_1, L_2, \dots, L_n$ where the coefficients of each linear form are functions of x. As such, it is similar to a result obtained by C. F. Osgood [5, Theorem XIII]. However, the methods used by Osgood to obtain the auxillary polynomial rely on Nevanlinna theory and are therefore different from those used in this paper. Also the exceptional subspaces, though implicit in Osgood's work (the auxiliary polynomial is required to be non-zero for the bound to hold), are neither considered in detail or utilised, as they are here, to present a function field analogue of Schmidt's Subspace Theorem.

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2. Definitions

(1) If $y = y(x)$ is a function of x then

$$y^{(j)} = \left(\frac{\partial}{\partial x}\right)^{j}, \qquad 0 \leq j \leq \infty,$$

is the jth derivative of y with respect to x.

(2) The notation

$$a_{ij}: (1 = 1, n; j = 1, m)$$

is used to denote the expression

$$a_{ij} \text{ for } i = 1 \text{ to } n \text{ and } j = 1 \text{ to } m.$$

(3) Let $\alpha_{k,i} = 0: (k = 1, n; i = 1, n)$ be a system of non-zero linear differential equations of the form

$$(3) \qquad \alpha_{k,i} = \sum_{j=0}^{N_{k,i}-1} a_{k,i,j+1} y^{(j)} = 0: (k = 1, n; i = 1, n)$$

having coefficients $a_{k,i,j+1}: (k = 1, n; i = 1, j = 0, N_{k,i} - 1)$ belonging to some differential field F with field of constants C. A differential field containing F is called a Picard-Vessiot extension of F for $\alpha_{k,i}: (k = 1, n; i = 1, n)$ if

(a) $M = F\langle y_{1,1,1}, y_{1,1,2}, \ldots, y_{n,n,N_{n,n}}\rangle$ where $y_{k,i,1}, y_{k,i,2}, \ldots, y_{k,i}, N_{n,n}$ are $N_{k,i}$ solutions of (3) for $(k = 1, n; i = 1, n)$ which are linearly independent over C, and

(b) M has the same field of constants as F, that is, M has the field of constants C.

(4) If F is a differential field of charactristic zero then a non-zero differential polynomial R over F is an expression of the form

$$R(y_i^{(j)}: (i = 1, n; j = 0, m))$$

where $y_1(x), y_2(x), \ldots, y_n(x)$ are differential indeterminates, n and m are non-negative integers and

$$R(y_i^{(j)}: (i = 1, n; j = 0, m))$$

is a polynomial in the $n(m+1)$ variables

$$y_i^{(j)}: (i = 1, n; j = 0, m)$$

having coefficients in F.

(5) Let $f = f(x)$ be a function of x. Then $\operatorname{Ord}_a f$ is the order of vanishing of f about a finite point $x = a$. If f is written as a power series about $x = a$ in the following way,

$$f = \sum_{-\infty}^{\infty} c_i (x-a)^i ,$$

then $\operatorname{Ord}_a f$ is the smallest i for which $c_i \neq 0$ if such an i exists. Otherwise we say that $\operatorname{Ord}_a f = -\infty$.

3. The theorem

THEOREM 1. *Let K denote an algebraically closed field of characteristic zero. Suppose*

$$L_1(\mathbf{P}), L_2(\mathbf{P}), \ldots, L_n(\mathbf{P})$$

are linearly independent linear forms in

$$\mathbf{P} = (P_1(x), P_2(x), \ldots, P_n(x))$$

with coefficients

$$f_{k,i} = f_{k,i}(x) \colon (k = 1, n; i = 1, n)$$

where

(1) *$P_i(x)$ is a polynomial in x with coefficients in K: $(i = 1, n)$ and*

(2) *$f_{k,i}(x)$ is a formal power series solution about $x = 0$ of a non-zero linear differential equation*

$$\alpha_{k,i} = \alpha_{k,i}((f_{k,i}), (f_{k,i})^{(1)}, (f_{k,i})^{(2)}, \ldots, (f_{k,i})^{(N_{k,i})}) = 0$$

with coefficients in $K(x)$: $(k = 1, n; i = 1, n)$. Let Ord_a denote the order of vanishing about a regular (finite) point of functions $f_{k,i}$: $(k = 1, n; i = 1, n)$. Then for each $\varepsilon > 0$ either

$$\operatorname{Ord}_a(L_1 L_2 \cdots L_n) \leq (1+\varepsilon) \sum_{i=1}^{n} \deg P_i + C$$

where C is an effectively computable constant (that is, it is independent of $\mathbf{P}$), or

$$\mathbf{P} = (P_1(x), P_2(x), \ldots, P_n(x))$$

lies inside one of a finite number of proper subspaces of $(K(x))^n$.

The proof of Theorem 1 relies upon the construction of a differential polynomial

$$D_N(P_1, P_2, \ldots, P_n)$$

and, by proving a series of lemmas, finding upper and lower bounds on the quantity

$$\mathrm{Ord}_a D_N(P_1, P_2, \dots, P_n).$$

Let M be the Picard-Vessiot extension of $K(x)$ corresponding to the differential equations $\alpha_{k,i} = 0$: $(k = 1, n; i = 1, n)$. that is M is the minimal differential extension of $K(x)$ containing $f_{k,i,r}$: $(k = 1, n; i = 1, n; r = 1, N_{k,i})$ with the field of constants K. For $N \geq 0$ we denote by M_N the vector space over K generated by monomials of the form

$$\prod_{k=1}^{n} \prod_{i=1}^{n} \prod_{r=1}^{N_{k,i}} f_{k,r,r}^{m_{k,i,r}} \quad \text{where} \quad \sum_{k=1}^{n} \sum_{i=1}^{n} \sum_{r=1}^{N_{k,i}} m_{k,i,r} = N.$$

According to Hilbert's Theorem, later generalised by Serre and henceforth referred to as the Serre-Hilbert Theorem [3], $\dim_k M_N = P_0(N)$ for $N \geq N_0$, where $P_0(N)$ is an integer valued polynomial. We define μ_N to be equal to $\dim_k M_N$ and we take functions

$$f_j^N : (j = 1, \mu_N)$$

as a basis of M_N, and introduce the following auxillary polynomial in the differential indeterminates $P_1, P_2, \dots, P_n$:

$$D_N(P_1, P_2, \dots, P_n) = \frac{W(P_i f_f^N : (i = 1, n; j = 1, \mu_N))}{(W(f_j^N : (j = 1, \mu_N))}$$

where

$$W(h_1, h_2, \dots, h_m) = \det\left(\left(\left(\frac{\partial}{\partial x}\right)^{l-1} h_s\right) : (l = 1, m; s = 1, m)\right).$$

THEOREM 2. (1) *$D_N(P_1, P_2, \dots, P_n)$ is a differential polynomial in P_1, $P_2, \dots, P_n$,*

(2) *the coefficients of the differential polynomial are invariant under the action of the differential Galois group of M over $K(x)$ and so the coefficients belong to $K(x)$, and*

(3) $\mathrm{Ord}_a D_N(P_1, P_2, \dots, P_n) \leq \mu_N \sum_{i=1}^{n} \deg P_i$.

Since the wronskian is formed by taking the sum of products consisting of elements taken from each row, (1) is clearly true. We prove (2) by using the following two lemmas.

LEMMA 1. *The coefficients of $D_N(P_1, P_2, \dots, P_n)$ consist of the sum of terms of the form*

$$\frac{D_1 \times D_2 \times \cdots \times D_n}{(\overline{W})_n},$$

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where D_i *is a determinant of the form*

$$\begin{vmatrix} (f_1^N)^{(v_{i,1}-1)} & (f_2^N)^{(v_{i,1}-1)} & \dots & (f_{\mu_N}^N)^{(v_{i,1}-1)} \\ (f_1^N)^{(v_{i,2}-1)} & (f_2^N)^{(v_{i,2}-1)} & \dots & (f_{\mu_N}^N)^{(v_{i,2}-1)} \\ (f_1^N)^{(v_{i,3}-1)} & (f_2^N)^{(v_{i,3}-1)} & \dots & (f_{\mu_N}^N)^{(v_{i,3}-1)} \\ \vdots & \vdots & & \vdots \\ (f_1^N)^{(v_{i,\mu_N}-1)} & (f_2^N)^{(v_{i,\mu_N}-1)} & \dots & (f_{\mu_N}^N)^{(v_{i,\mu_N}-1)} \end{vmatrix}$$

for $i = 1, n$ *and where*

$$\overline{W} = W(f_1^N : (j = 1, \mu_N)).$$

Lemma 1 follows from the application and manipulation of the definition of deteminant.

LEMMA 2. *Under the action of the differential Galois group of* M *over* $K(x)$, *any determinant of the form*

$$D = \begin{vmatrix} (f_1^N)^{(k_1)} & (f_2^N)^{(k_1)} & \dots & (f_{\mu_N}^N)^{(k_1)} \\ (f_1^N)^{(k_2)} & (f_2^N)^{(k_2)} & \dots & (f_{\nu_N}^N)^{(k_2)} \\ (f_1^N)^{(k_3)} & (f_2^N)^{(k_3)} & \dots & (f_{\mu_N}^N)^{(k_3)} \\ \vdots & \vdots & & \vdots \\ (f_1^N)^{(k_{\mu_N})} & (f_2^N)^{(k_{\mu_N})} & \dots & (f_{\mu_N}^N)^{(k_{\mu_N})} \end{vmatrix}$$

is sent to some constant times itself.

Under the action of the differential Galois group, G, of M over $K(x)$, each f_j^N is sent to a linear combination of $f_1^N, f_2^N, \dots, f_{\mu_N}^N$ in the following way: for any automorphism $\Gamma \in G$,

$$\Gamma : f_r^N \to \sum_{j=1}^{\mu_N} c_{r,j} f_j^N,$$

and

$$\Gamma : (f_r^N)^{(k_t)} \to \sum_{j=1}^{\mu_N} c_{r,j} (f_j^N)^{(k_t)} \quad \text{for } t = 1, \mu_N,$$

where $c_{r,j} \in K : (r = 1, \mu_N ; j + 1, \mu_N)$. Lemma 2 follows from the application of the above results to each element of D and from the algebraic transformation of determinants.

By Lemma 1, the coefficients of $D_N(P_1, P_2, \dots, P_n)$ consist of the sum of terms of the form

$$\frac{D_1 \times D_2 \times \cdots \times D_n}{(\overline{W})^n},$$

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which, applying Lemma 2, are clearly invariant under the action of the differential Galois group G and hence belong to $K(x)$. Hence $D_N(P_1, P_2, \ldots, P_n)$ is a polynomial in $P_1, P_1^{(1)}, \ldots, P_n^{(\mu_N - 1)}$ having terms of the form

$$A_{\bar{a}} \prod_{i=1}^{n} \prod_{j=1}^{\mu_N} P_i^{(a_{i,j})}$$

where

$$A_{\bar{a}} \in K(x) \quad \text{and} \quad \sum_{i=1}^{n} \sum_{j=1}^{\mu_N} a_{i,j} \le n\mu_{N-1}.$$

Therefore

$$\operatorname{Ord}_a D_N(P_1, P_2, \ldots, P_n) \le \mu_N \sum_{i=1}^{n} \deg P_i + C_1(N) \tag{4}$$

where $C_1(N)$ is a constant independent of $P_1, P_2, \ldots, P_n$.

LEMMA 3. *If f_l^{N-1} belongs to the basis of M_{N-1} and $f_{k,i}$ is a solution of one of the original differential equations $\alpha_{k,i}$, then*

$$f_l^{N-1} \sum_{i=1}^{n} P_i f_{k,i} = \sum_{i=1}^{n} P_i \sum_{j=1}^{\mu_N} C_l^{i,j} f_j^N,$$

where the $C_l^{i,j}$ are constants belonging to K and the vectors $C_l = C_l^{i,j}$: $(i = 1, n; j = 1, \mu_n)$ are linearly independent over K.

Clearly $M_1 M_{n-1} \subset M_N$ and so

$$f_{k,i} f_l^{N-1} \in M_N.$$

It follows that

$$f_{k,i} f_l^{N-1} = \sum_{j=1}^{\mu_N} C_l^{i,j} f_j^N$$

where $C_l^{i,j} \in K$: $(i = 1, n; j = 1, \mu_N)$ for $l = 1, \mu_{N-1}$. Multiplying by P_i and summing for $i = 1, \ldots, n$ gives the required result. The vectors $C_l = C_l^{i,j}$: $(i = 1, n; j = 1, \mu_N)$ are clearly linearly independent over K, for suppose on the contrary that there are constants $d_l \in K$ not all zero such that

$$\sum_{l=1}^{\mu_{N-1}} C_l d_l = 0.$$

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Then

$$\sum_{l=1}^{\mu_{N-1}} C_l^{i,j} d_l = 0 \quad \text{for } (i = 1, n; j = 1, \mu_N).$$

Multiplying by f_j^N and summing for $j = 1, \ldots, \mu_N$ we see that

$$\sum_{l=1}^{\mu_{N-1}} d_l \sum_{j=1}^{\mu_N} C_l^{i,j} f_j^N = 0.$$

Hence

$$\sum_{l=1}^{\mu_{N-1}} (d_l f_{k,i}) f_l^{N-1} = f_{k,i} \sum_{l=1}^{\mu_{N-1}} d_l f_l^{N-1} = 0,$$

implying that

$$\sum_{l=1}^{\mu_{N-1}} d_l f^{N-1}/l = 0,$$

which is a contradiction of the linear independence of the f_l^{N-1} over K. Hence the $C_l^{i,j}: (l = 1, \mu_{N-1})$ are linearly independent over K.

LEMMA 4. *We can make a non-singular transformation that reduces the determinant*

$$W = W(P_i f_j^N : (i = 1, n; j = 1, \mu_N))$$

to the form $\hat{W}$, *the determinant of the matrix* $\hat{A}$ *in which the first* μ_{N-1} *columns have the form*

$$\left(\left(\frac{\delta}{\delta x} \right)^{(s-1)} \left\{ \sum_{i=1}^{n} \sum_{j=1}^{\mu_N} P_i f_j^N C_l^{i,j} \right\} s = 1, n\mu_N \right)$$

for $l = 1, \mu_{N-1}$ *or*

$$\left(\left(\frac{\delta}{\delta x} \right)^{(s-1)} \left\{ f_l^{(N-1)} \sum_{i=l}^{n} P_i f_{k,i} \right\} s = 1, n\mu_N \right)$$

for $l = 1, \mu_{N-1}$.

Lemma 4 follows if we can make a non-singular transformation that reduces W to the form W_k, the determinant of the matrix A_k in which the first k columns have the above form, for $k = 1, \mu_{N-1}$. This is proved using induction on k.

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LEMMA 5. *The equations*

$$\prod_{\lambda=0}^{(n\mu_N-1)} \left(\frac{\partial}{\partial P_n^{(\lambda)}}\right)^{e_{k,\lambda+1}} D_N(P_1, P_2, \dots, P_n) = 0$$

are satisfied whenever

$$\sum_{\lambda=0}^{(n\mu_N-1)} e_{k,\lambda+1} < \mu_{N-1}$$

at

$$\sum_{i=1}^{n} P_i f_{k,i} = 0$$

for $k = 1, n$.

Clearly, under the transformation of Lemma 4, the above equations are all satisfied if and only if

$$\prod_{\lambda=0}^{(n\mu_N-1)} \left(\frac{\partial}{\partial P_n^{(\lambda)}}\right)^{e_{k,\lambda+1}} \hat{W} = 0$$

whenever

$$\sum_{\lambda=0}^{(n\mu_N-1)} e_{k,\lambda+1} < \mu_{N-1}$$

at

$$\sum_{i=l}^{n} P_i f_{k,i} = 0$$

for $k = 1, n$, which follows since each of the derivatives of $\hat{W}$ consists of the sum of determinants, each having at least one of their first μ_{N-1} columns identical to that of $\hat{W}$. Hence at

$$\sum_{i=1}^{n} P_i f_{k,i} = 0,$$

the expression

$$\prod_{\lambda=0}^{(n\mu_N-1)} \left(\frac{\partial}{\partial P_n^{(\lambda)}}\right)^{e_{k,\lambda+1}} \hat{W} \quad \left(\text{where } \sum_{\lambda=0}^{(n\mu_N-1)} e_{k,\lambda+1} < \mu_{N-1}\right)$$

consists of the sum of determinants, each of which has at least one column consisting entirely of zeros.

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LEMMA 6. (1) *$D_N(P_1, P_2, \dots, P_n)$ can be expressed as the sum of monomials each of which has the form*

$$A_{\bar{e}} \prod_{k=1}^{n} \prod_{\lambda=0}^{(n\mu_N - 1)} \left(\left\{ \frac{L_k}{f_{k,n}} \right\}^{(\lambda)} \right)^{e_{k,\lambda+1}}$$

where

$$\sum_{\lambda=0}^{(n\mu_N-1)} e_{k,\lambda+1} \geq \mu_{N-1} \quad \textit{for } k = 1, n,$$

where

$$\bar{e} = \{e_{1,1}, e_{1,2}, \dots, e_{n,n\mu_N}\}$$

and where for each set $\bar{e}$,

$$A_{\bar{e}} = A_{\bar{e}}(f_{k,i}^{(\lambda)}: (k = 1, n; i = 1, n; \lambda = 0, (n\mu_N - 1)))$$

is some polynomial in

$$f_{k,i}^{(\lambda)}: (k = 1, n; i = 1, R; \lambda = 0, n\mu_N - 1).$$

(2) *If $D_N(P_1, P_2, \dots, P_n)$ is non-zero, then*

$$\operatorname{Ord}_a D_N(P_1, P_2, \dots, P_n) \geq \mu_{N-1} \operatorname{Ord}_a L_1 L_2 \cdots L_n) - C_3(N) \tag{5}$$

where $C_3(N)$ is a constant depending only on N and $f_j^N: (j = 1, \mu_N)$.

Part (1) follows from the Taylor expansion of $D_N(P_1, P_2, \dots, P_n)$ about $L_1 = 0$, $L_2 = 0, \dots, L_n = 0$ and Lemma 5. Part (2) follows from (1) and the properties of Ord_a.

PROOF OF THEOREM 1. From (4) and (5) we see, if D_N is non-zero, that

$$\mu_N \sum_{i=1}^{n} \deg P_i + C_1(N) \geq \mu_{N-1} \operatorname{Ord}_a(L_1 L_2 \cdots L_n) - C_3(N)$$

and so

$$\operatorname{Ord}_a(L_1 L_2 \cdots L_n) \leq \frac{\mu_N}{\mu_{N-1}} \sum_{i=1}^{n} \deg P_i + C_4(N).$$

Now, according to Hilbert's Theorem [3],

$$\frac{\mu_N}{\mu_{N-1}} = \frac{\dim_k M_N}{\dim_k M_{N-1}} \to 1 \quad \text{as } N \to \infty.$$

Hence, if $D_N(P_1, P_2, \dots, P_n) \neq 0$, for $N \geq N_1(\varepsilon)$, we have

$$\operatorname{Ord}_a(L_1 L_2 \cdots L_n) \leq (1 + \varepsilon) \sum_{i=1}^{n} \deg P_i + C_4(N), \tag{6}$$

where $C_4(N)$ is a constant independent of $P_1, P_2, \ldots, P_n$. However, if $D_N(P_1, P_2, \ldots, P_n) = 0$ the functions $P_i f_j^N : (i = 1, n; j = 1, \mu_N)$ are linearly dependent over K. Hence

$$\sum_{i=1}^{n} P_i \left(\sum_{j=1}^{\mu_N} c_{i,j} f_j^N \right) = 0 \quad \text{for constants } c_{i,j} \in K \text{ not all zero.}$$

Choosing a basis of M_N over $K(x)$, $g_r^N : (r = 1, \alpha_N)$ say, having dimension α_N, each of the f_j^N can be expressed as a linear combination of the $g_r^N : (r = 1, \alpha_N)$ having coefficients in $K(x)$ with degrees bounded by some $m = m(N)$. That is,

$$f_j^N = \sum_{r=1}^{\alpha_N} a_{r,j}(x) g_r^N : (j = 1, \mu_N)$$

and so

$$\begin{aligned} \sum_{j=1}^{\mu_N} c_{i,j} f_j^N &= \sum_{j=1}^{\mu_N} \sum_{r=1}^{\alpha_N} c_{i,j} a_{r,j}(x) g_r^N \\ &= \sum_{r=1}^{\alpha_N} \left(\sum_{j=1}^{\mu_N} c_{i,j} a_{r,j}(x) \right) g_r^N \qquad (7) \\ &= \sum_{r=1}^{\alpha_N} b_{r,i}(x) g_r^N \end{aligned}$$

where the $b_{r,i}(x) \in K(x)$ with degrees bounded by m, not all zero. Now as we saw above,

$$\sum_{i=1}^{n} P_i \left(\sum_{j=1}^{\mu_N} c_{i,j} f_j^N \right) = 0.$$

Hence

$$\sum_{i=1}^{n} P_i \left(\sum_{r=1}^{\alpha_N} b_{r,i}(x) g_r^N \right) = 0$$

and so

$$\sum_{r=1}^{\alpha_N} g_r^N \left(\sum_{i=1}^{n} P_i b_{r,i}(x) \right) = 0.$$

At least one of the linear relations

$$\sum_{i=1}^{n} P_i b_{r,i}(x)$$

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is non-trivially zero, by the linear independence of the g_r^N over $K(x)$. That is, there exists at least one linear relation among the P_i of the form

$$B_1(x)P_1(x) + B_2(x)P_2(x) + \cdots + B_n(x)P_n(x) = 0,$$

where $B_i(x) \in K(x)$: $(i = 1, n)$ and where

$$\deg B_i(x) \leq m' = m'(N): (i = 1, n).$$

Because the degrees of the polynomials $B_i(x)$: $(i = 1, n)$ are bounded, only finitely many linear relations of the above form exist. Now the expressions

$$B_1(x)P_1(x) + B_2(x)P_2(x) + \cdots + B_n(x)P_n(x) = 0$$

imply that at most $(n-1)$ of $P_1, P_2, \ldots, P_n$ are linearly independent, and hence from (6) and the above, either

$$\mathrm{Ord}_a(L_1L_2\cdots L_n) \leq (1+\varepsilon)\sum_{i=1}^{n}\deg P_i + C_4(N)$$

or the polynomials $P_1, P_2, \ldots, P_n$ lie inside a finite number of rational subspaces of $(K(x))^n$.

In considering a possible extension of Theorem 1 to obtain an upper bound for

$$\prod_{d\in A}\mathrm{Ord}_d(L_1L_2\cdots L_n),$$

where $A = \{a_1, a_2, \ldots, a_r, \infty\}$ and where $a_1, a_2, \ldots, a_r$ are elements of K, it seems difficult to approach this type of result using the homogeneous form of the wronskian. It is the inclusion of the point $x = \infty$ in A that is responsible for this lack of conformity. One might also expect that one could put $\varepsilon = 0$ in Theorem 1.

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