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SOME ORBITAL INTEGRALS AND A TECHNIQUE FOR COUNTING REPRESENTATIONS OF $GL_2(F)$

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Introduction. Let F be a local field of characteristic zero, with q elements in its residue field, ring of integers $\mathcal{O}_F$, uniformizer ϖ_F and maximal ideal $\mathcal{G}_F$. Let $G_F = GL_2(F)$. We fix Haar measures dg and dz on G_F and Z_F, the centre of G_F, so that

$$\text{meas}(K) = \text{meas } Z(\mathcal{O}_F) = 1$$

where $K = GL_2(\mathcal{O}_F)$ is a maximal compact subgroup of G_F. If T is a torus in G_F we take dt to be the Haar measure on T such that

$$\text{meas}(T^M) = 1$$

where T^M denotes the maximal compact subgroup of T.

For any nonnegative integer c we define

$$K_c = \left\{ \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in K | \gamma \in \mathcal{G}^c \right\}.$$

Let ψ_c denote the characteristic function of K_c. In §1 we compute the following orbital integrals:

$$\Psi_c(T, t) = \int_{T\backslash G} \psi_c(g^{-1}tg)dg.$$

In [5, §3], Langlands computes these integrals for $c = 0$. He makes use of the Bruhat-Tits building of G_F, and we use the same tools. Perhaps the details contained in §1 will be helpful to those studying [5].

Let $n(c, F)$ denote the number of irreducible, unitary, admissible representations, π, of G_F such that

i) π is special or supercuspidal,

ii) $c(\pi) = c$,

iii) $\epsilon_\pi = 1$,

where $c(\pi)$ is the conductor of π (see [1]), and ϵ_π is the central character of π. In Section 2 we describe a method for computing $n(c, F)$ and we undertake explicit computations when c is odd.

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1. The orbital integrals. There are several conjugacy classes of tori in G_F. A split torus is one which is conjugate to

$$A = \left\{\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \middle| \alpha, \beta \in F^x\right\}.$$

The other conjugacy classes are in a one-to-one correspondence with quadratic extension fields of F. If T is a nonsplit torus in G_F, then the set of eigenvalues of elements of T is the multiplicative subgroup of a quadratic extension of F. Two tori in G_F are conjugate if and only if they are isomorphic (see [**3**, §7].

LEMMA 1. *Let H be an open subgroup of G_F and λ a function on G_F such that*

$$\lambda(h^{-1}gh) = \lambda(g)$$

for all $h \in H$. Then, for all $a \in A$,

$$\int_{A\backslash G_F} \lambda(g^{-1}ag)dg = \text{meas }(H) \int_{C\backslash G_F/ZH} \lambda(g^{-1}ag)dg$$

where

$$C = \left\{\begin{bmatrix} \varpi_F^{\,n} & 0 \\ 0 & 1 \end{bmatrix} \middle| n \in \mathbf{Z}\right\}.$$

If T is any nonsplit torus in G_F and $t \in T$, then

$$\int_{T\backslash G_F} \lambda(g^{-1}tg)dg = [T: T^M Z]^{-1}\text{meas }(H) \int_{G_F/HZ} \lambda(g^{-1}hg)dg.$$

In particular, these formulae hold for H equal to K_c and λ equal to ψ_c.

Proof. Let T be a nonsplit torus and $t \in T$.

$$\int_{T\backslash G_F} \lambda(g^{-1}tg)dg = \int_{T\backslash G_F/H} \lambda(g^{-1}tg)\frac{\text{meas }(gHg^{-1})}{\text{meas }(T \cap gHg^{-1})}dg$$

$$= \frac{\text{meas }(H)}{\text{meas }(T^M)} \int_{T\backslash G_F/H} \lambda(g^{-1}tg)\frac{\text{meas }(T^M)}{\text{meas }(T \cap gHg^{-1})}dg$$

$$= \text{meas }(H)[T: T^M Z]^{-1} \int_{G_F/HZ} \lambda(g^{-1}tg)dg.$$

In the split case we proceed in the same way but use $A = C \times A^M Z$.

Let V be some 2-dimensional vector space over F and let X_F be the set of classes of $\mathcal{O}_F$-lattices in V. We say L_1 and L_2 are in the *same class* if L_1 is an F-multiple of L_2. Consider X_F as the set of points of a graph. We join two points M_1 and M_2 by a line of unit length if there are lattices $L_1 \in M_1$ and $L_2 \in M_2$ such that L_1 has index q as a subgroup of L_2. X_F is called the *Bruhat-Tits building* of G_F. Bruhat and Tits have defined the building of much more general groups, but we need only this simple case (see [**6**]). If $m \in X_F$, there are exactly $q + 1$ points joined to m by a line of unit length. It is not hard to verify that any two points in X_F are joined by some path and that there is a unique path of minimal length. This makes the building of G_F particularly easy to work with, but it does not hold true in general; in fact, even for GL_3 the situation is much more complicated (see [**4**]). If $m_1, m_2 \in X_F$, we define $\operatorname{dist}_{X_F}(m_1, m_2)$ in the obvious way.

We identify G_F with $GL(V)$. Thus G_F (and G_F/Z) acts on X_F. There is a unique point $p_0 \in X_F$ which is fixed by K. Let C be as in Lemma 1. Then the line $\mathfrak{A}$ formed by the orbit of p_0 under the action of C is called the *standard apartment* of X_F with respect to A. The connection with A is that $\mathfrak{A}$ is the set of points of X_F fixed by A^M. In fact, if T is any split torus on G_F, then the standard apartment of X_F with respect to T is the line of points fixed by T^M. For $c \in \mathbf{Z}$ we let

$$p_c = \begin{bmatrix} \varpi_F{}^c & 0 \\ 0 & 1 \end{bmatrix} p_0.$$

If $m_1, m_2 \in X_F$, we let (m_1, m_2) denote the minimal path from m_1 to m_2; we distinguish between (m_1, m_2) and (m_2, m_1). We say g *fixes* (m_1, m_2) if g fixes each point of (m_1, m_2) or, what is the same thing, if g fixes m_1 and m_2.

LEMMA 2. *The subgroup of G_F which fixes (p_0, p_{-c}) is equal to K_cZ.*

Proof. The group which fixes p_0 is KZ, and so the group which fixes p_{-c} is

$$\begin{bmatrix} \bar{\omega}^{-c} & 0 \\ 0 & 1 \end{bmatrix} KZ \begin{bmatrix} \bar{\omega}^{c} & 0 \\ 0 & 1 \end{bmatrix} = \left\{ \begin{bmatrix} \alpha & \beta\bar{\omega}^{-n} \\ \gamma\bar{\omega}^{n} & \delta \end{bmatrix} \,\middle|\, \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in K \right\} \cdot Z.$$

The intersection of the two groups is K_cZ.

Let g be a diagonalizable element of G_F; i.e., g is conjugate, in G_F, to a diagonal matrix. We define

$$r_F(g) = \nu_F\left(\frac{\alpha - \beta}{\alpha\beta}\right)$$

where α, β are the eigenvalues of g and ν_F is defined by

$$|\gamma|_F = q^{\nu_F(\gamma)}$$

for all $\gamma \in F$ with $|\ \ |_F$ the usual absolute value on F.

LEMMA 3. *Let g be a diagonalizable element of G_F contained in some split torus D. Let $\mathscr{D}$ be the standard apartment with respect to D. If $p \in X_F$, then $gp = p$ if and only if*

$$\operatorname{dist}_{X_F}(p, \mathscr{D}) \leqq r_F(g).$$

Proof. This is Lemma 3.2 of [**5**].

For r and c nonnegative integers, we define $\mathscr{L}(c, r)$ to be the number of lines (m_1, m_2) of length c such that

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(i) $\mathrm{dist}_{X_F}(m_i, \mathfrak{A}) \leqq r$ for $i = 1, 2$,
(ii) $\mathrm{dist}_{X_F}(m_1, \mathfrak{A}) = \mathrm{dist}_{X_F}(m_1, p_0)$.

LEMMA 4. *For $c \neq 0$, c even and $r \geqq c/2$,*

$$\mathscr{L}(c, r) = (q + 1)q^{c/2+r-1}.$$

For c odd and $r \geqq [c/2]$,

$$\mathscr{L}(c, r) = 2q^{r+[c/2]}.$$

For $c \neq 0$ and $r < [c/2]$,

$$\mathscr{L}(c, r) = 2q^{2r}.$$

For all $r \geqq 0$,

$$\mathscr{L}(0, r) = q^r.$$

Proof. We must count the lines of length c in Figure 1 which start at a point

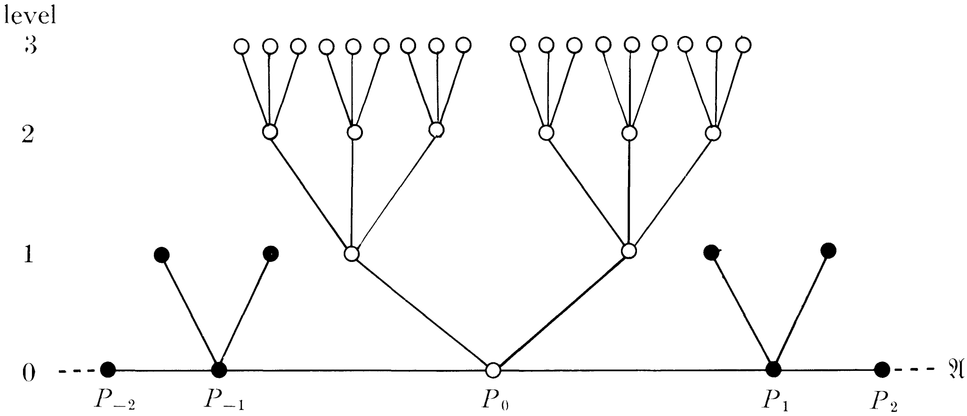

FIGURE 1

marked by an open dot (Figure 1 is the diagram for $\mathscr{L}(2, 3)$ with $q = 3$). The base line is the standard apartment $\mathfrak{A}$. There are $(q - 1)q^{n-1}$ starting points at level n for $0 < n \leqq r$; i.e., there are $(q - 1)q^{n-1}$ points, p, such that

$$\mathrm{dist}_{X_F}(p, \mathfrak{A}) = \mathrm{dist}(p, p_0) = n.$$

Suppose that c is even and $r \geqq c$. A line starting at level r must proceed towards $\mathfrak{A}$ for $c/2$ steps and then any one of q directions may be taken at each of the next $c/2$ steps. Thus from each point at level r originate $q^{c/2}$ lines of the required type. At the opposite end, a line starting from the level 0 point has $(q + 1)$ initial directions and then $c - 1$ choices of q directions. In this manner we obtain the following table:

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Level	*No. of Points*	*Lines per Point*	*Total*
0	1	$(q+1)q^{c-1}$	$q^c + q^{c-1}$
1	$(q-1)$	$(q+1)q^{c-1}$	$q^{c+1} + q^{c-1}$
2	$(q-1)q$	$(q+1)q^{c-1}$	$q^{c+2} + q^c$
.	.	.	.
.	.	.	.
.	.	.	.
$r-c-1$	$(q-1)q^{r-c-2}$	$(q+1)q^{c-1}$	$q^{r-1} + q^{r-3}$
$r-c$	.	$(q+1)q^{c-1}$	$q^r + q^{r-2}$
$r-c+1$	.	q^{c-1}	$q^r + q^{r-1}$
$r-c+2$	.	q^{c-1}	$q^{r+1} + q^r$
$r-c+3$	.	q^{c-2}	$q^{r+1} + q^r$
$r-c+4$	$(q-1)q^{r-c+3}$	q^{c-2}	$q^{r+2} + q^{r+1}$
.	.	.	.
.	.	.	.
$r-3$	$(q-1)q^{r-4}$	$q^{c/2+1}$	$q^{r+c/2-2} + q^{r+c/2-3}$
$r-2$	.	$q^{c/2+1}$	$q^{r+c/2-1} + q^{r+c/2-2}$
$r-1$	.	$q^{c/2}$	$q^{r+c/2-1} + q^{r+c/2-2}$
r	$(q-1)q^{r-1}$	$q^{c/2}$	$q^{r+c/2} + q^{r+c/2-1}$

If we add all this together, we get the promised result. All the other cases are just as simple—and just as tedious!

THEOREM 1. *Let D be a split torus in G_F and let a be a regular element of D^M. Then*

$$\Psi_c(D, a) = \begin{cases} \operatorname{meas}(K_c)\mathscr{L}(c, r(a)), & \text{if } |\det a|_F = 1 \\ 0 & \text{if } |\det a|_F \neq 1. \end{cases}$$

Proof. Without loss of generality we may take D to be equal to A. Let S denote the set of lines in X_F of length c. From Lemma 1 it follows that

$$G_F/K_cZ \to S$$
$$gK_cZ \mapsto (gp_0, gp_{-c})$$

is a bijection. Let S_0 be the subset of lines (m_1, m_2) in S such that

$$\operatorname{dist}_{X_F}(m_1, \mathfrak{A}) = \operatorname{dist}_{X_F}(m_1, p_0).$$

If $L \in S$, then CL, the orbit of L under the action of C, has exactly one element in S_0. Therefore, Lemma 1 yields

$$\Psi_c(A, a) = \operatorname{meas}(K_c) \int_{C\backslash G_F/K_cZ} \psi_c(g^{-1}ag)dg$$
$$= \sum_{(gp_0, gp_{-c}) \in S_0} \psi_c(g^{-1}ag).$$

It is clear that $\psi_c(g^{-1}ag)$ is zero for all g if $|\det a|_F \neq 1$, and so we assume

$|\det a|_F = 1$. We have, from Lemma 2, that

$$\psi_c(g^{-1}ag) = 1$$

$\Leftrightarrow g^{-1}ag \in K_c \Leftrightarrow g^{-1}ag$ fixes $(p_0, p_{-c}) \Leftrightarrow a$ fixes (gp_0, gp_c). Lemma 3 finishes the proof.

In order to compute $\Psi_c(T, t)$ for nonsplit tori we must work in a field large enough so that the matrices of T are diagonalizable, and then embed X_F in the larger building. Let E be a quadratic extension of F. We view E as a 2-dimensional vector space over F. If $\alpha \in E^\times$, then

$$\begin{aligned} \alpha : E &\rightarrow E \\ x &\rightarrow \alpha x \end{aligned}$$

defines an element of $GL_F(E)$. We identify $GL_F(E)$ with G_F. The subgroup, T_F, corresponding to $E^\times$ is a nonsplit torus. The lattices of E are $\mathcal{O}_F$-sub-modules of E and two lattices are in the same class if one is an $F^\times$-multiple of the other. Thus we have a model of X_F, consisting of classes of lattices in E, on which T_F acts in a natural way. If L is a lattice in E, we let $[L]$ denote the class containing L.

LEMMA 5. *If E/F is unramified and $T_F \cong E^\times$, then $[\mathcal{O}_E]$ is the only point in X_F fixed by $T_F{}^M$. If E/F is ramified, then $[\mathcal{O}_E]$ and $[\varpi_E \mathcal{O}_E]$ are the only points in X_F fixed by $T_F{}^M$.*

Proof. If $[L]$ is fixed by $T_F{}^M$, then it is easy to check that L must be a fractional ideal in $\mathcal{O}_E$. The rest is straightforward.

We now want to embed X_F in X_E. Let X_E consist of classes of $\mathcal{O}_E$-lattices of $E \times E$. It is not hard to show that $E \otimes_F E \cong E \times E$ as E vector spaces under the map

$$\text{(1)} \quad \begin{aligned} E \otimes_F E &\rightarrow E \times X \\ a \otimes b &\mapsto (ab, a\bar{b}) \end{aligned}$$

where $\bar{b}$ is the conjugate of b with respect to E/F. If $[L]$ is a point in X_F, then we obtain a point in X_E by

$$\text{(2)} \quad [L] \rightarrow [L \otimes_{\mathcal{O}_F} \mathcal{O}_E].$$

Actually, the right-hand side is a lattice class in $E \otimes_F E$, but (1) allows us to consider it as a point in X_E. In this way, we view X_F as a subset of X_E. If E/F is unramified, then each point of X_E has $q^2 + 1$ points distance one from it, and the embedding preserves distances. If E/F is ramified, then there are still $q + 1$ points distance one from each point in X_F, but between any two points of X_F there is a point of X_E, not in X_F. In this case distance is not preserved; in fact, if $m_1, m_2 \in X_F$,

$$\operatorname{dist}_{X_E}(m_1, m_2) = 2 \operatorname{dist}_{X_F}(m_1, m_2).$$

Next, we define the action of G_E on X_E to be compatible with (2), the injection of G_F into G_E and the previously defined action of G_F on X_F. In other words, we identify G_E with $GL(E \times E)$ so that the following diagram commutes:

$$\begin{array}{ccc} G_F & \hookrightarrow & G_E \\ \downarrow & & \downarrow \\ GL_F(E) & \rightarrow & GL_E(E \times E). \end{array}$$

The top map is injection, the left side has been defined above and the bottom line is defined by (2); i.e.,

$$[gL \otimes_{\mathcal{O}_F} \mathcal{O}_E] = g[L \otimes_{\mathcal{O}_F} \mathcal{O}_E].$$

In particular, if $\alpha \in E^x$, then α corresponds to a diagonalizable element of $GL_E(E \times E)$ with eigenvalues α and $\bar{\alpha}$. If $p \in X_F$ and $g \in G_F$, then p is fixed by g acting on X_E if and only if it is fixed by g acting on X_F. The torus $T_F \subseteq G_F$ is mapped into a split torus T_E in G_E. The apartment $\mathfrak{A}_E$ is the line fixed by T_E^M.

Lemma 6. *Let* $m_1 = [\mathcal{O}_E \otimes_{\mathcal{O}_F} \mathcal{O}_E]$ *and* $m_2 = [\varpi_E\mathcal{O}_E \otimes_{\mathcal{O}_F} \mathcal{O}_E]$. *If* E/F *is unramified, then* $m_1 = m_2$ *and* $\mathfrak{A}_E \cap X_F = \{m_1\}$. *If* E/F *is ramified, then* $m_1 \neq m_2$ *and*

(1) $\mathfrak{A}_E \cap X_F$ *is empty;*

(2) $\operatorname{dist}_{X_F}(X_F, \mathfrak{A}_E) = \operatorname{dist}_{X_E}(m_i, \mathfrak{A}_E) = \delta(E/F)$,

where $\delta(E/F) = \nu_E$ *(different of* E/F*) and* $i = 1, 2$. *Furthermore, if* m *is any other point of* X_F, *then* $\operatorname{dist}_{X_E}(m, \mathfrak{A}_E) > \delta(E/F)$.

Proof. Suppose that E/F is unramified. Then

$$(3) \quad T_E^M = T_F^M \cdot Z(\mathcal{O}_E).$$

Therefore, a point of X_E is fixed by T_E^M if and only if it is fixed by T_F^M. Lemma 5 says that the only point in X_F which is fixed by T_F^M is $[\mathcal{O}_E]$, and so the only point of X_F in X_E fixed by T_E^M is m_1.

Suppose that E/F is ramified. We no longer have (3), but any point on $\mathfrak{A}_E$ must still be fixed by T_F^M. Thus, Lemma 5 implies

$$\mathfrak{A}_E \cap X_F \subseteq \{m_1, m_2\}.$$

Multiplication by ϖ_E interchanges $[\mathcal{O}_E]$ and $[\varpi_E\mathcal{O}_E]$. Therefore, the diagonalizable element of $GL_E(E \times E)$ corresponding to ϖ_E interchanges m_1 and m_2. This transformation has eigenvalues ϖ_E and $\bar{\varpi}_E$. Since $\operatorname{dist}_{X_F}(m_1, m_2) = 1$ there must be a point, m, of X_E, not in X_F, which is between m_1 and m_2 and which is fixed by ϖ_E. From Lemma 3, it follows that

$$\operatorname{dist}_{X_E}(m, \mathfrak{A}_E) \leqq \nu_E \left(1 - \frac{\bar{\varpi}_E}{\varpi_E}\right)$$

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and

$$\operatorname{dist}_{X_E}(m_1, \mathfrak{A}_E) > \nu_E\left(1 - \frac{\bar{\omega}_E}{\omega_E}\right).$$

Since $\operatorname{dist}_{X_E}(m, m_1) = 1$, we have

$$\begin{aligned}\operatorname{dist}_{X_E}(m_1, \mathfrak{A}_E) &= 1 + \nu_E\left(1 - \frac{\bar{\omega}_E}{\omega_E}\right) \\ &= \nu_E(\omega_E - \bar{\omega}_E) \\ &= \delta(E/F).\end{aligned}$$

For nonnegative integers c and r, let $\mathscr{M}(c, r)$ denote the number of lines (n_1, n_2) in X_F of length c such that

$$\operatorname{dist}_{X_F}(n_1, n) \leqq r, \quad \text{and} \quad \operatorname{dist}_{X_F}(n_2, n) \leqq r.$$

where n is a fixed but arbitrary point in X_F.

LEMMA 7. *$c \neq 0$, c even and $r \geqq c/2$,*

$$\mathscr{M}(c, r) = \frac{q^{c-1}(q+1)}{(q-1)}\left(q^{r-c/2+1} + q^{r-c/2} - 2\right).$$

For c odd and $r \geqq [c/2] + 1$,

$$\mathscr{M}(c, r) = \frac{2q^{c-1}(q+1)}{q-1}\left(q^{r-[c/2]} - 1\right).$$

For $r < c/2$,

$$\mathscr{M}(c, r) = 0.$$

For all $r > 0$,

$$\mathscr{M}(0, r) = (q+1)q^{r-1}$$

and

$$\mathscr{M}(0, 0) = 1.$$

Proof. For $q = 3$, $\mathscr{M}(c, 3)$ is the number of lines of length c in Figure 2. For

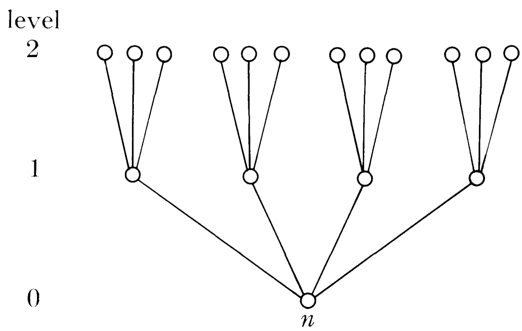

FIGURE 2

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c odd and $[c/2] < c < r$ we can produce, as in the proof of Lemma 4, the following table:

Level	*No. of Points*	*Lines per Point*	*Total*
0	1	0	0
1	$q+1$	0	0
2	$(q+1)q$	0	0
.	.	.	.
.	.	.	.
.	.	.	.
$c-r-1$	$(q+1)q^{c-r-2}$	0	0
$c-r$	$(q+1)q^{c-r-1}$	q^r	$q^c + q^{c-1}$
$c-r+1$	.	q^{r-1}	$q^c + q^{c-1}$
$c-r+2$	.	q^{r-1}	$q^{c+1} + q^{c-1}$
$c-r+3$	.	q^{r-2}	$q^{c+1} + q^{c-1}$
.	.	.	.
.	.	.	.
.	.	.	.
$r-2$	.	$q^{[c/2]+1}$	$q^{r+[c/2]-1} + q^{r+[c/2]-2}$
$r-1$	.	$q^{[c/2]+1}$	$q^{r+[c/2]} + q^{r+[c/2]-1}$
r	$(q+1)q^{r-1}$	$q^{[c/2]}$	$q^{r+[c/2]} + q^{r+[c/2]-1}$.

If we add all this up, and fiddle about for a bit, then we get the advertised result. The other cases are just more of the same sort of thing.

The next two lemmas are standard results.

LEMMA 8. *If T is a torus in G_F, isomorphic to $E^\times$ where E is a quadratic extension of F, then*

$$[T : T^M Z] = e(E/F)$$

where $e(E/F)$ is the ramification index of E over F.

LEMMA 9. *If c is any positive integer, then*

$$[K : K_c] = q^{c-1}(q+1).$$

THEOREM 2. *If B is a nonsplit torus in G_F, isomorphic to an unramified quadratic extension of F, and $b \in \hat{B}$, then*

$$\Psi_c(B, b) = \begin{cases} \text{meas}(K_c)(c\mathscr{M}, r_E(b)) & \text{if } |\det b|_F = 1 \\ 0 & \text{if } |\det b|_F \neq 1. \end{cases}$$

Proof. It is clear that the integral is zero if $|\det b|_F \neq 1$, and so we assume $|\det b|_F = 1$. Let S be the set of lines in X_F of length c. Then there is a $1-1$

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correspondence with G/K_cZ given by

$$G/K_cZ \to S$$
$$gK_cZ \mapsto (gp_0, gp_{-c}).$$

Therefore, from Lemmas 1 and 8,

$$\psi_c(B, b) = \operatorname{meas}(K_c) \int_{G/K_cZ} \psi_c(g^{-1}bg)dg$$
$$= \sum_{(gp_0, gp_{-c}) \in S} \psi_c(g^{-1}bg).$$

We embed X_F in X_E, and use Lemma 3, to obtain

$$\psi_c(g^{-1}bg) = 1$$

if and only if

$$\operatorname{dist}_{X_E}(gp_0, \mathfrak{A}_E) \leqq r_E(b) \quad \text{and} \quad \operatorname{dist}_{X_E}(gp_{-c}, \mathfrak{A}_E) \leqq r_E(b).$$

But there is a unique point, m, in $X_F \cap \mathfrak{A}_E$. Therefore, for $p \in X_F$,

$$\operatorname{dist}_{X_E}(p, \mathfrak{A}_E) = \operatorname{dist}_{X_E}(p, m).$$

For nonnegative integers c and r, let $\mathcal{N}(c, r)$ be the number of lines, (n_1, n_2), of length c in X_F such that

$$\operatorname{dist}_{X_F}(n_i, l_1) \leqq r \quad \text{or} \quad \operatorname{dist}_{X_F}(n_i, l_2) \leqq r$$

for $i = 1, 2$, where l_1 and l_2 are two arbitrary but fixed points in X_F such that $\operatorname{dist}_{X_F}(l_1, l_2) = 1$. For $q = 3$, $\mathcal{N}(c, 2)$ is the number of lines of length c in Figure 3.

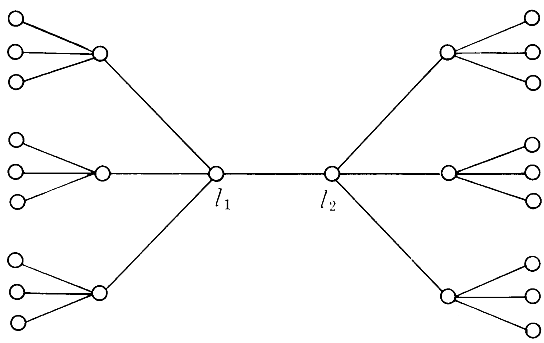

FIGURE 3

LEMMA 10. *For $c \neq 0$, c even and $r \geqq c/2$,*

$$\mathcal{N}(c, r) = \frac{2q^{c-1}(q+1)}{(q-1)} (q^{r-c/2+1} - 1).$$

For c odd and $r \geqq [c/2]$,

$$\mathcal{N}(c, r) = \frac{2q^{c-1}}{q-1}(2q^{r-[c/2]+1} - q - 1).$$

For all $r > 0$,

$$\mathcal{N}(0, r) = \frac{2(q^{r+1} - 1)}{q - 1}.$$

For $r < [c/2]$,

$$\mathcal{N}(c, r) = 0.$$

Proof. We could prove this lemma in the same manner as Lemmas 4 and 7, but there is a simpler way. If we bend Figure 3 at the point l_1, then we get Figure 4 (ignore the points marked by closed dots). Thus $\mathcal{N}(c, r)$ is equal to $\mathcal{M}(c, r)$ plus the number of lines with an end point at one of the q^r points on level $r + 1$. If c is odd, then a line of length c can have at most one end point at level $r + 1$ and so, for $r \geqq [c/2]$,

$$\mathcal{N}(c, r) = \mathcal{M}(c, r) + 2q^{r+[c/2]}.$$

If c is even, we must be careful not to count twice the lines with both end points on level $r + 1$. For c even and $r \geqq c/2$, we get

$$\mathcal{N}(c, r) = \mathcal{M}(c, r) + q^{r+c/2} + q^{r+c/2-1}.$$

Combining this with Lemma 7, we obtain the required formulae.

THEOREM 3. *If* T *is a nonsplit torus in* G_F, *isomorphic to a ramified quadratic extension* E *of* F, *and* $t \in \hat{T}$, *then*

$$\Psi_c(T, t) = \begin{cases} \frac{1}{2}\operatorname{meas}(K_c)\mathcal{N}\left(c, \left[\dfrac{r_E(t) - \delta(E/F)}{2}\right]\right), & \text{if } |\det t|_F = 1 \\ 0 & , \text{ if } |\det t|_F \neq 1. \end{cases}$$

Proof. We assume $|\det t|_F = 1$. If S is the set of lines of length c in X_F, then

$$G_F/K_cZ \rightarrow S$$
$$gK_cZ \rightarrow (gp_0, gp_{-c})$$

is a bijection. Therefore, from Lemmas 1 and 8

$$\Psi_c(T, t) = 2\operatorname{meas}(K_c) \sum_{(gp_0, gp_{-c}) \in S} \psi_c(g^{-1}tg).$$

We embed X_F in X_E. Let m_1 and m_2 be as defined in Lemma 6 and let m be the point of X_E between m_1 and m_2 (see the proof of Lemma 6). For $p \in X_F$, we have

$$\begin{aligned} \operatorname{dist}_{X_E}(p, \mathfrak{A}_E) &= \delta(E/F) - 1 + \operatorname{dist}_E(p, m) \\ &= \delta(E/F) + \min\{2\operatorname{dist}_{X_F}(p, m_1), 2\operatorname{dist}_{X_F}(p, m_2)\}. \end{aligned}$$

The situation is illustrated by Figure 4. The open dots are points of X_F and the closed dots are points in X_E not in X_F. The base line is $\mathfrak{A}_E$. Figure 3 must be bent at m_1 to relate it to Figure 4.

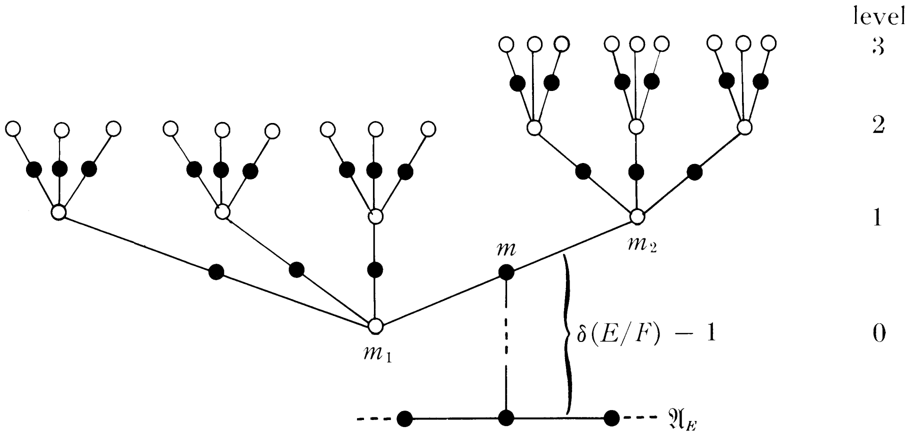

FIGURE 4

From Lemmas 2 and 3 we have

$$\psi_c(g^{-1}tg) = 1$$

if and only if

$$t \text{ fixes } gp_0 \text{ and } gp_{-c}$$

if and only if

$$\mathrm{dist}_{X_E}(gp_0, \mathfrak{A}_E) \leqq r_E(t) \quad \text{and} \quad \mathrm{dist}_{X_E}(gp_{-c}, \mathfrak{A}_E) \leqq r_E(t).$$

The theorem follows easily.

If we combine Theorem 1, 2 and 3 with Lemmas 4, 7, 9 and 10 we can establish the following table of values of $\psi_c(T, t)$.

For A a split torus, $a \in \hat{A}^M$ and $r = r_F(a)$

$$\psi_c(A, a) = \begin{cases} 2q^{2r-c+1}(q+1)^{-1} & \text{for } r < [c/2] \\ q^{r-c/2} & \text{for } r \geqq c/2,\ c \text{ even, } c \neq 0 \\ 2q^{r-[c/2]}(q+1)^{-1} & \text{for } r \geqq [c/2],\ c \text{ odd.} \end{cases}$$

For $B \cong E^\times$ where E/F is an unramified quadratic extension, $b \in \hat{B}^M$ and $r = r_E(b)$.

$$\Psi_c(B, b) = \begin{cases} 0 & \text{for } r < c/2 \\ (q-1)^{-1}(q^{r-c/2+1} + q^{r-c/2} - 2) & \text{for } r \geqq c/2,\ c \text{ even} \\ 2(q-1)^{-1}(q^{r-[c/2]} - 1) & \text{for } r \geqq [c/2] + 1,\ c \text{ odd.} \end{cases}$$

For $T \cong E^\times$ where E/F is a ramified quadratic extension, $t \in \hat{T}^M$ and

$l = [(r_E(t) - \delta(E/F))/2]$,

$$\psi_c(T, t) = \begin{cases} 0 & \text{for } l < [c/2] \\ (q-1)^{-1}(q^{l-c/2+1} - 1) & \text{for } l \geqq c/2,\ c \text{ even} \\ (q^2-1)^{-1}(2q^{l-[c/2]+1} - q - 1) & \text{for } l \geqq [c/2],\ c \text{ odd.} \end{cases}$$

2. The computation of $n(c, F)$. We shall produce a function (Lemma 13), f_c, on $Z\backslash G_F$, which is locally constant and has compact support such that

$$\text{(4)} \qquad \operatorname{tr}(\pi(f_c)) = \begin{cases} 1 & \text{if } c(\pi) = c \text{ and } \epsilon_\pi = 1 \\ 0 & \text{otherwise} \end{cases}$$

where π is any irreducible unitary admissible representation of G_F. There exists a locally integrable class function X_π, for each π, such that (see [3, §7])

$$\operatorname{tr}(\pi(f_c)) = \int_{Z\backslash G_F} \chi_\pi(g) f_c(g) dg.$$

If we apply equation 7.2.2 of [3], we obtain

$$\int_{Z\backslash G_F} \chi_\pi(g) f_c(g) dg = \tfrac{1}{2} \sum_{T\in S} \int_{Z\backslash T} \Delta(t) \int_{T\backslash G} \chi_\pi(g) f_c(g^{-1}tg) dg dt$$
$$= \tfrac{1}{2} \sum_{T\in S} \int_{Z\backslash T} \Delta(t) \chi_\pi(t) F_c(T, t) dt$$

where S is a complete set of nonconjugate tori in G_F, containing A,

$$\Delta(g) = \left| \frac{(\alpha_1 - \alpha_2)^2}{\alpha_1\alpha_2} \right|_F$$

with α_1 and α_2 the eigenvalues of g, and

$$F_c(T, t) = \int_{T\backslash G} f_c(g^{-1}tg) dg.$$

Let $S' = S - \{A\}$. We define a function $\mathbf{F}_c$ on $\bigcup_{T\in S} Z\backslash T = \mathscr{U}$ by

$$\mathbf{F}_c(a) = F_c(A, a), \quad a \in Z\backslash A$$

and

$$\mathbf{F}_c(t) = \operatorname{meas}(Z\backslash T) F_c(T, t) \quad \text{for } T \in S' \text{ and } t \in T.$$

Then

$$\operatorname{tr}(\pi(f_c)) = \tfrac{1}{2} \sum_{T\in S} \frac{1}{\operatorname{meas}(Z\backslash T)} \int_{Z\backslash T} \Delta(t) \chi_\pi(t) \mathbf{F}_c(t) dt.$$

If $T \in S$, then $T - \hat{T}$ ($\hat{T}$ is the set of regular elements of T) has measure zero, and so

$$\operatorname{tr}(\pi(f_c)) = \tfrac{1}{2} \sum_{T\in S} \frac{1}{\operatorname{meas}(Z\backslash T)} \int_{Z\backslash \hat{T}} \Delta(t) \chi_\pi(t) \mathbf{F}_c(t) dt.$$

As in [3, p. 480], we define a measure on $\mathscr{U}' = \bigcup_{T\in S'} T$, by

$$\int_{\mathscr{U}'} f(s)ds = \tfrac{1}{2}\sum_{T\in S'} \frac{1}{\operatorname{meas}(Z\backslash T)}\int_{Z\backslash \hat{T}} \Delta(t)f(t)dt.$$

This defines an inner product which we denote by $\langle\ ,\ \rangle$ and we let $L^2(\mathscr{U}')$ denote the corresponding space of functions on $\mathscr{U}'$. From [3, Chapters 15, 16] we known that the characters χ_π are an orthonormal basis of $L^2(\mathscr{U}')$ where π runs over D, the set of special and supercuspidal representations of G_F.

We now suppose that

(5) $\quad F_c(A, a) = 0$

for all $a \in A$. This will be shown to be true in the case c is odd. Then

$$\operatorname{tr}(\pi(f_c)) = \langle \mathbf{F}_c, \bar{\chi}_\pi\rangle.$$

Therefore,

$$\mathbf{F}_c = \sum_{\pi\in D} a_\pi \chi_\pi$$

where

$$\begin{aligned} a_\pi &= \langle \mathbf{F}_c, \chi_\pi\rangle \\ &= \operatorname{tr}(\pi(f_c)) \\ &= \begin{cases} 1 & \text{if } c(\pi) = c \text{ and } \epsilon_\pi = 1 \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Hence

$$\langle \mathbf{F}_c, \mathbf{F}_c\rangle = \sum_{\pi\in D} a_\pi^2 = \sum_{\pi\in D} a_\pi = n(c, F).$$

We have shown

THEOREM 4. *If $F_c(A, a) = 0$, for all $a \in A$, then*

$$n(c, F) = \langle \mathbf{F}_c, \mathbf{F}_c\rangle = \tfrac{1}{2}\sum_{T\in S'} \operatorname{meas}(Z\backslash T)\int_{Z\backslash\hat{T}} \Delta(t)F_c(T, t)^2 dt.$$

The next lemma presents (5) in a more transparent way.

LEMMA 11. *Suppose that f is a locally constant function on G_F with compact support. Then* $\operatorname{tr}(\pi(f)) = 0$ *for all principal series representations π if and only if $F(A, a) = 0$ for all $a \in \hat{A}$ where*

$$F(A, a) = \int_{A\backslash G} f(g^{-1}ag)dg.$$

Proof. From [**3**, Proposition 7.6] it follows that $\operatorname{tr}(\pi(f)) = 0$ for all principal series representations π if and only if all the Fourier coefficients of $F(A,\ )$ are zero.

LEMMA 12. *If c is odd, then $\mathbf{F}_c$ is identically zero on $\hat{A}$.*

Proof. Let π denote a principal series representation of G_F. If $\epsilon_\pi = 1$, then $c(\pi)$ is even, and so, by Lemma 11, $\operatorname{tr}(\pi(F_c)) = 0$.

It is now time to construct the functions f_c. We observe that $\operatorname{tr}(\pi((\operatorname{meas} K_c)^{-1}\psi_c))$ is equal to the number of times that π contains the trivial representation when restricted to K_c. Therefore, if ϵ_π is not identically 1, then $\operatorname{tr}(\pi(f_c)) = 0$ for all c. Suppose that $\epsilon_\pi \equiv 1$. Then (see [**1**] or [**2**])

$$(6) \quad (\operatorname{meas}(K_c))^{-1}\operatorname{tr}(\pi(\psi_c)) = \begin{cases} 0 & \text{if } c < c(\pi) \\ c - c(\pi) + 1 & \text{if } 0 \leqq c(\pi) \leqq c. \end{cases}$$

LEMMA 13. *Let*

$$f_0 = \psi_0,$$
$$f_1 = -2\psi_0 + (\operatorname{meas} K_1)^{-1}\psi_1,$$

and for $c \geqq 2$,

$$f_c = (\operatorname{meas} K_{c-2})^{-1}\psi_{c-2} - 2(\operatorname{meas} K_{c-1})^{-1}\psi_{c-1} + (\operatorname{meas} K_c)^{-1}\psi_c.$$

Then f_c satisfies (4).

Proof. It is a simple exercise to verify the lemma by means of (6).

One could now verify Lemma 12 directly because $\mathbf{F}_c$ is a linear combination of the integrals computed in §1.

Let $T \in S'$ and let E be the corresponding quadratic extension of F. We shall write δ_T to mean $\delta(E/F)$. Since the eigenvalues of any element, t, of T must be conjugate with respect to E/F, $|\det t|_F = 1$ forces the eigenvalues of t to be units. Therefore, the set of $t \in T$ such that $|\det t|_F = 1$ corresponds to $\mathcal{O}_E^\times$. We define, for n a nonnegative integer,

$$H_T(n) = \{t \in T | r_E(t) = n \text{ and } |\det t|_F = 1\}.$$

For fixed $T \in S$, the function $F_c(T, t)$ depends only on $r_E(t)$ and so we shall write $F_c(T, t) = F_c(T, r_E(t))$.

LEMMA 14. *If E/F is unramified and $n \geqq 0$, then*

$$\operatorname{meas}(H_T(n)) = \begin{cases} (q-1)(q+1)^{-1}q^{-n}, & n \geqq 1 \\ q(q+1)^{-1} & , \quad n = 0. \end{cases}$$

If E/F is ramified, then $H_T(n)$ is empty if n is not of the form $2m + \delta_T$ for some nonnegative integer m, and

$$\operatorname{meas}(H_T(2m + \delta_T)) = q^{-(m+1)}(q-1).$$

Proof. We have

$$\operatorname{meas}(H_T(n)) = \operatorname{meas}\{\alpha \in \mathcal{O}_E^\times | \nu_E(\alpha - \bar{\alpha}) = n\}.$$

Suppose that E/F is ramified. Let J be a complete set of representatives of

$\mathscr{O}_E^\times/\bar{\omega}_E\mathscr{O}_E^\times$ such that $0 \in J$ and $J - \{0\} \subseteq \mathscr{O}_F^\times$. If $\alpha \in \mathscr{O}_E^\times$, we can write

$$\alpha = \sum_{m=0}^{\infty} (\epsilon_{2m} + \epsilon_{2m+1}\bar{\omega}_E)\bar{\omega}_F^{\,m}$$

where $\epsilon_m \in J$ and $\epsilon_0 \neq 0$. Thus

$$\alpha - \bar{\alpha} = \sum_{m=0}^{\infty} \epsilon_{2m+1}(\bar{\omega}_E - \bar{\bar{\omega}}_E)\bar{\omega}_F^{\,m}.$$

If ϵ_{2m+1} is the first odd numbered coefficient which is not zero, then

$$\begin{aligned} \nu_E(\alpha - \bar{\alpha}) &= \nu_E((\bar{\omega}_E - \bar{\bar{\omega}}_E)\bar{\omega}_F^{\,m}) \\ &= \delta(E/F) + 2m. \end{aligned}$$

The rest is straightforward and we omit it. The unramified case is similar.

The function $\Delta(t)$ has the following explicit values:

$$\Delta(t) = \begin{cases} q^{-2r_E(t)} & \text{if } E/F \text{ is unramified} \\ q^{-r_E(t)} & \text{if } E/F \text{ is ramified.} \end{cases}$$

Let B denote the unique unramified torus in S', and let $S'' = S - \{A, B\}$. We can put the last few facts into the formula in Theorem 4 and, for odd c, obtain

$$\begin{aligned} \langle \mathbf{F}_c, \mathbf{F}_c \rangle &= \tfrac{1}{2} \sum_{T \in S'} \operatorname{meas}(Z\backslash T) \int_{Z\backslash \hat{T}} \Delta(t) F_c(T, t)^2 dt \\ &= \tfrac{1}{2} \sum_{n=0}^{\infty} \operatorname{meas} H_B(n) q^{-2n} F_c(T_0, n)^2 \\ &\quad + \tfrac{1}{2} \sum_{T \in S''} 2 \sum_{m=0}^{\infty} \operatorname{meas} H_T(2m + \delta_T) q^{-2m-\delta_T} F_c(T, 2m + \delta_T)^2 \\ &= \tfrac{1}{2} \sum_{n=0}^{\infty} (q-1)(q+1)^{-1} q^{-3n} F_c(T_0, n)^2 \\ &\quad + \sum_{T \in S''} q^{-\delta_T} \sum_{m=0}^{\infty} q^{-3m-1}(q-1)^{-1}{}_cF(T, 2m + \delta_T)^2. \end{aligned} \tag{7}$$

We shall need

LEMMA 15.

$$\sum_{T \in S' - \{T_0\}} q^{-\delta_T} = 2q^{-1}.$$

Proof. This lemma is a special case of a more general theorem due to Serre. It can be proved in several ways. It can be proved by the same techniques we have been using. One has only to use the Weyl integration formula to obtain

$$1 = \int_G \psi_0(g) dg = \tfrac{1}{2} \sum_{T \in S} \int_T \Delta(t) \Psi_0(T, t) dt.$$

If we put explicit values of $\psi_0(T, t)$ into the above, then the identity drops out.

THEOREM 5. *If c is an odd integer, then*

$$n(c, F) = \begin{cases} 2q^{(c-3)/2}(q-1) & \textit{for } c \geqq 2 \\ 2 & \textit{for } c = 1. \end{cases}$$

Proof. Suppose that π is a supercuspidal representation of G_F. Then $c(\pi) \geqq 2$. Therefore, for $c = 1$, we need only count the special representations with trivial central character. It is not hard to show that there are exactly 2 of these.

We now take c to be at least 2. To make the computation we have to put explicit values for $F_c(T, t)$ into (7). We start with the unramified torus B. We shall use the results of §1 without giving specific references.

$$f_c = (\mathrm{meas}(K_{c-2}))^{-1}\psi_{c-2} - 2(\mathrm{meas}(K_{c-1}))^{-1}\psi_{c-1} + (\mathrm{meas}(K_c))^{-1}\psi_c.$$

Therefore, if $r = r(b)$

$$F_c(B, b) = \mathscr{M}(c-2, r) - 2\mathscr{M}(c-1, r) + \mathscr{M}(c, r).$$

A bit of computation shows that

$$F_c(B, b) = \begin{cases} -2(q-1)^2 q^{c-3} & \text{if } r \geqq (c-1)/2 \\ 0 & \text{if } r \leqq (c-1)/2. \end{cases}$$

Hence,

$$\begin{aligned} &\tfrac{1}{2}\int_{Z\backslash \hat{T}_0} (F_c(B, b))^2 \Delta(t)\,db \\ &= \tfrac{1}{2}\sum_{n=(c-1)/2}^{\infty} (q-1)(q+1)^{-1}q^{-3n}(-2(q-1)q^{c-3})^2 \\ &= \frac{2q^{(c-3)/2}(q-1)(q^2-1)}{q^2+q+1}. \end{aligned}$$

Now suppose that $T \in S''$ and let $m = (r_E(t) - \delta_T)/2$ (in view of Lemma 14, m is a positive integer). Then

$$F_c(T, t) = \tfrac{1}{2}\{\mathscr{N}(c-2, m) - 2\mathscr{N}(c-1, m) + \mathscr{N}(c, m)\}.$$

It follows that

$$F_c(T, t) = \begin{cases} -q^{c-3}(q^2-1) & \text{for } l > (c-1)/2 \\ q^{c-3} & \text{for } l = (c-1)/2 \\ 0 & \text{for } l < (c-1)/2. \end{cases}$$

Therefore,

$$\begin{aligned} &\int_{Z\backslash T} \Delta(t)(F_c(T, t))^2 dt \\ &= (q-1)q^{(c-5)/2-\delta_T} + q^{-\delta_T}\sum_{m=(c-1)/2}^{\infty} q^{-3m-1}(q-1)^{-1}(-q^{c-3}(q^2-1))^2 \\ &= \frac{q^{(c-1)/2-\delta_T}(q-1)(q+2)}{q^2+q+1}. \end{aligned}$$

Applying Lemma 15, we get

$$\sum_{T\in S''}\int_{Z\backslash\hat{T}}\Delta(t)(F_c(T,t))^2dt$$
$$=\sum_{TES''}q^{-\delta_T}\left(\frac{q^{(c-1)/2}(q-1)(q+2)}{q^2+q+1}\right)$$
$$=\frac{2q^{(c-3)/2}(q-1)(q+2)}{q^2+q+1}.$$

Adding this to the unramified term produces the advertised values of $n(c, F)$.

To make this method work for c even it is necessary to construct functions which behave like f_c but whose orbital integrals vanish on the split torus. While this can probably be done, it is not clear what form these functions should take. Recently J. Tunnell [**7**] has found a completely different method which imposes no restrictions on c or ϵ_π.

References

1. W. Casselman, *On some results of Atkin and Lehner*, Math. Ann. *201* (1973), 301–314.

2. P. Deligne, *Formes modulaires et représentations de* GL_2, Modular functions of one variable (Springer, London-New York, *349* (1973), 55–106.

3. H. Jaquet and R. Langlands, *Automorphic forms on* $GL(2)$ (Springer, London-New York, *114*, 1970).

4. R. Kottwitz, Thesis, Harvard University, to appear.

5. R. Langlands, *Base change for* $GL(2)$, notes for lectures at the Institute for Advanced Study, Autumn, 1975.

6. I. Macdonald, *Spherical functions on a group of p-adic type*, Ramanujan Institute Publications No. 2, University of Madras, 1971.

7. J. Tunnell, Harvard University, to appear.

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