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A MODULAR BISIMULATION CHARACTERISATION FOR FRAGMENTS OF HYBRID LOGIC

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Abstract. There are known characterisations of several fragments of hybrid logic by means of invariance under bisimulations of some kind. The fragments include $\{\downarrow, @\}$ with or without nominals (Areces, Blackburn, Marx), @ with or without nominals (ten Cate), and $\downarrow$ without nominals (Hodkinson, Tahiri). Some pairs of these characterisations, however, are incompatible with one another. For other fragments of hybrid logic no such characterisations were known so far. We prove a generic bisimulation characterisation theorem for all standard fragments of hybrid logic, in particular for the case with $\downarrow$ and nominals, left open by Hodkinson and Tahiri. Our characterisation is built on a common base and for each feature extension adds a specific condition, so it is modular in an engineering sense.

§1. Introduction. Hybrid logic dates back to Arthur Prior's works from 1960, but the story is somewhat convoluted so instead of giving direct references we direct the reader to Blackburn [6] for details. Hybrid logics can be described as a version of modal logics with an additional machinery to refer to individual evaluation points. The ability to refer to specific states has several advantages from the point of view of logic and of formal specification. For example, hybrid logics allow a more uniform proof theory [10, 14] and model theory [15, 16] than non-hybrid modal logics. From a computer science perspective, hybrid logics are eminently applicable to describing behavioural dynamics: *if at this state something holds, then at some state accessible from it something else holds*. This view has been particularly leveraged to the specification and modelling of reactive and event/data-based systems [22, 17, 26]. Other applications include reasoning on semi-structured data [12] and description logics beyond terminological boxes [9].

For definitions and an introductory treatment of modal and hybrid logics we refer the reader to Blackburn *et al.* [7]; our terminology and notation comes largely from there, in particular, we use $\downarrow$ and @ as the current-state binder and the state-relativisation operator. In general, we assume familiarity with modal and hybrid logics at the level of [7], anything beyond that we will define.

Just as in modal logic, there is a *standard translation* of hybrid languages to suitable first-order languages which gives an interpretation of the former in the latter. A celebrated characterisation theorem due to van Benthem (see [28]) states that a first-order formula is equivalent to a translation of a modal formula if and only if it is invariant under bisimulation. Similar characterisations exist for certain fragments of hybrid logic: Areces *et al.* [2] give one for the fragment with @ and $\downarrow$, the fragment with @ was characterised by ten Cate [27], and Hodkinson and Tahiri in [20] characterised the fragment with $\downarrow$ but, importantly, without nominals. As we will see, these characterisations are rather disparate and some of them do not extend to richer fragments. In particular, the method used in [20] does not cover the language with nominals, and the authors pose the problem of finding a characterisation for this fragment. They also tabulate existing results, and ask, more generally, which fragments can be characterised by some form of bisimulations. Here is a version of this table:

Hybrid features	Invariance under
$\emptyset$	Bisimulations
$\{\downarrow\}$ w/o nominals	Quasi-injective bisimulations
$\{\downarrow\}$ with nominals	*unknown*
$\{@\}$	$\mathcal{H}(@)$-bisimulations
$\{@, \downarrow\}$	ω-bisimulations
$\{\exists\}$	*unknown*
$\{@, \exists\}$	equivalent to FOL
full feature set	equivalent to FOL

In this article we answer these questions by providing a characterisation for all (sensible) fragments of the hybrid language by means of a fine-tuned version of ω-bisimulation, a notion introduced in Areces *et al.* [2], which provides one crucial ingredient for our results. The other crucial ingredient is Thm. 3 in §5, which shows that first-order sentences cannot distinguish between ω-bisimulation and the hybrid counterpart of elementary equivalence. Its proof is obviously inspired by Lindström's celebrated characterisation of first-order logic, but more specifically by an earlier use of the same technique by Badia [4]. Let us remark that although Areces *et al.* [2] do discuss the possibility of applying their notion to other fragments of the language, they do not state any results in this direction. Indeed, the proof of their characterisation theorem for the language with $\downarrow$, even without @, uses the condition for @ essentially, to deal with nominals. Our results are completely modular, in the engineering sense that you can choose a subset $\mathcal{F}$ of the features of a hybrid language, and we give a characterisation theorem for the language fragment for $\mathcal{F}$ using conditions pertaining to $\mathcal{F}$.

§2. Terminology and notation. As we already mentioned, we follow the terminology of Blackburn *et al.* [7] for modal and hybrid logic. For general model theory, we follow Hodges [18], at least in spirit. We write $\mathfrak{A} = (A, R^{\mathfrak{A}}, f^{\mathfrak{A}}, c^{\mathfrak{A}}, \dots)$ for structures, with A being the universe, and $R^{\mathfrak{A}}, f^{\mathfrak{A}}, c^{\mathfrak{A}}$ interpretations of a relation R, a function f, and a constant c of the appropriate signature Σ. We use the same notation $\mathfrak{M} = (M, R^{\mathfrak{M}}, V^{\mathfrak{M}})$ for Kripke structures, where $R^{\mathfrak{M}}$ is a binary accessibility relation and $V^{\mathfrak{M}}$ a valuation assigning subsets of M to propositions. We also write $m' \in R^{\mathfrak{M}}(m)$ or $m \; R^{\mathfrak{M}} \; m'$ for $(m, m') \in R^{\mathfrak{M}}$, and do so for any binary relation.

Often we will consider models with a distinguished element; we will call them *pointed* and write $(\mathfrak{M}, m)$ for a model $\mathfrak{M}$ with a distinguished element $m \in M$. The class of all pointed models of a formula $\phi(x)$ with a free variable x will be denoted by $\mathrm{Mod}_*(\phi)$. As usual, we dispense with superscripts unless we have a good reason to use them; we also use a typical shorthand notation for sequences, namely, $\overline{m} = (m_1, \dots, m_n)$ together with $\overline{m}(i) = m_i$. If convenient, especially for comparing sequences of different lengths, we can also view sequences of elements of M as finite words over the alphabet M.

We deal exclusively with hybrid propositional logic, so the language is built out of the atomic propositions in PROP $\cup$ NOM $\cup$ WVAR, where PROP is the set of propositional

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variables, NOM the set of nominals, and WVAR the countable set of world variables (the sets PROP and NOM could be of any cardinality). Formulas are defined by the grammar

$$\phi, \psi ::= \bot \mid p \mid s \mid x \mid \neg\phi \mid \phi \vee \psi \mid \Diamond\phi \mid {\downarrow}x \cdot \phi \mid @_w \phi \mid \exists x \cdot \phi$$

where $p \in$ PROP, $s \in$ NOM, $x \in$ WVAR, and $w \in$ WVAR $\cup$ NOM. Even without $\exists$, hybrid logic is more expressive than modal logic, since for example ${\downarrow}x \cdot \Diamond x$ expresses reflexivity of the current state. With $\exists$ it is even stronger, for example $\neg\exists x \cdot (x \wedge \Diamond x)$ expresses irreflexivity of the accessibility relation. In presence of $\exists$ the operation $\downarrow$ is redundant as ${\downarrow}x \cdot \phi$ is equivalent to $\exists x \cdot (x \wedge \phi)$. In presence of $\exists$ and $@$ the expressivity of hybrid logic is the same as that of full first-order logic.

Extending the familiar concept from modal logic, we define the *degree* $\mathsf{dg}(\phi)$ of a hybrid formula ϕ in the usual recursive way:

- $\mathsf{dg}(\phi) = 0$ for $\phi \in$ PROP $\cup$ NOM $\cup$ WVAR,
- $\mathsf{dg}(-\phi) = \mathsf{dg}(\phi)$ for $- \in \{\neg, @_w\}$,
- $\mathsf{dg}(\phi \vee \psi) = \max\{\mathsf{dg}(\phi), \mathsf{dg}(\psi)\}$,
- $\mathsf{dg}(-\phi) = \mathsf{dg}(\phi) + 1$ for $- \in \{\Diamond, {\downarrow}x\,\cdot, \exists x\,\cdot\}$.

Note that the degree increases only on formulas whose *standard translations* (to be recalled later) involve quantification.

2.1. Semantics of hybrid logic. A model for the hybrid language is a structure $\mathfrak{M} = \big(M, R^{\mathfrak{M}}, (s^{\mathfrak{M}})_{s \in \text{NOM}}, V^{\mathfrak{M}}\big)$, where $V^{\mathfrak{M}}\colon \text{PROP} \to \wp(M)$ is a valuation fixing which propositions hold in which worlds. An *assignment* is any map $\nu\colon \text{WVAR} \to M$. For an assignment ν, a world variable x, and an element $m \in M$ we define the x-variant ν^x_m of ν to be the map

$$\nu^x_m(y) = \begin{cases} m & \text{if } y = x, \\ \nu(y) & \text{if } y \neq x. \end{cases}$$

Given a model $\mathfrak{M}$, an assignment $\nu\colon \text{WVAR} \to M$, and an element $m \in M$, we define inductively

- $\mathfrak{M}, \nu, m \Vdash \bot$ never holds
- $\mathfrak{M}, \nu, m \Vdash p$ if $p \in$ PROP and $m \in V^{\mathfrak{M}}(p)$
- $\mathfrak{M}, \nu, m \Vdash s$ if $s \in$ NOM and $m = s^{\mathfrak{M}}$
- $\mathfrak{M}, \nu, m \Vdash x$ if $x \in$ WVAR and $m = \nu(x)$
- $\mathfrak{M}, \nu, m \Vdash \phi \vee \psi$, if $\mathfrak{M}, \nu, m \Vdash \phi$ or $\mathfrak{M}, \nu, m \Vdash \psi$
- $\mathfrak{M}, \nu, m \Vdash \neg\phi$ if $\mathfrak{M}, \nu, m \nVdash \phi$
- $\mathfrak{M}, \nu, m \Vdash \Diamond\phi$ if $\mathfrak{M}, \nu, m' \Vdash \phi$ for some $m' \in R^{\mathfrak{M}}(m)$
- $\mathfrak{M}, \nu, m \Vdash {\downarrow}x \cdot \phi$ if $x \in$ WVAR and $\mathfrak{M}, \nu^x_m, m \Vdash \phi$
- $\mathfrak{M}, \nu, m \Vdash @_s \phi$ if $s \in$ NOM and $\mathfrak{M}, \nu, s^{\mathfrak{M}} \Vdash \phi$
- $\mathfrak{M}, \nu, m \Vdash @_x \phi$ if $x \in$ WVAR and $\mathfrak{M}, \nu, \nu(x) \Vdash \phi$
- $\mathfrak{M}, \nu, m \Vdash \exists x \cdot \phi$ if $x \in$ WVAR and $\mathfrak{M}, \nu^x_{m'}, m \Vdash \phi$ for some $m' \in M$

The operators ${\downarrow}x\,\cdot$ and $\exists x\,\cdot$ *bind* the world variable x, whereas $@_x$ does not (to see this, note that evaluating ${\downarrow}x \cdot \phi$ at m is independent of the assignment, whereas $@_x$ is not). An occurrence of a world variable x in a formula ϕ is *free* if it is not in the scope of an operator binding x, a world variable x is free in ϕ if it has at least one free occurrence in ϕ; we write $\phi(x_1, \ldots, x_k)$ for a hybrid formula whose free world variables occur in the sequence $x_1, \ldots, x_k$. A *hybrid sentence* is a hybrid formula with no free world variables. For a sentence ϕ the assignment ν is irrelevant, so we write $\mathfrak{M}, m \Vdash \phi$.

2.2. Translations and fragments. To each hybrid language there corresponds a first-order language consisting of a binary predicate R, a set NOM of constants, and the set PROP$^\circ$ of predicates P, one for each $p \in$ PROP. For a hybrid signature $\Sigma = (\text{PROP}, \text{NOM})$ of propositions and nominals, let $\Sigma^\circ = (R, \text{PROP}^\circ, \text{NOM})$ be the first-order signature corresponding to Σ, and let $\mathcal{L}_{\Sigma^\circ}$ denote the corresponding first-order language (with $\approx$ denoting equality). For each first-order model $\mathfrak{M} = (M, R^{\mathfrak{M}}, (s^{\mathfrak{M}})_{s \in \text{NOM}}, (P^{\mathfrak{M}})_{P \in \text{PROP}^\circ})$ one can define a hybrid model $(M, R^{\mathfrak{M}}, (s^{\mathfrak{M}})_{s \in \text{NOM}}, V^{\mathfrak{M}})$ over Σ such that $V^{\mathfrak{M}}(p) = P^{\mathfrak{M}}$ for all $p \in$ PROP. Assume that WVAR $= \{z_i : i \in \omega\}$. We pick two new variables $\boldsymbol{x}$ and $\boldsymbol{y}$, and define *standard translations* $ST_{\boldsymbol{x}}(\phi)$ and $ST_{\boldsymbol{y}}(\phi)$ of a hybrid formula ϕ by simultaneous recursion as follows, with $p \in$ PROP, $w \in$ NOM $\cup$ WVAR, and $z \in$ WVAR:

$$
\begin{aligned}
ST_{\boldsymbol{x}}(\bot) &= \bot & ST_{\boldsymbol{y}}(\bot) &= \bot \\
ST_{\boldsymbol{x}}(p) &= P(\boldsymbol{x}) & ST_{\boldsymbol{y}}(p) &= P(\boldsymbol{y}) \\
ST_{\boldsymbol{x}}(w) &= w \approx \boldsymbol{x} & ST_{\boldsymbol{y}}(w) &= w \approx \boldsymbol{y} \\
ST_{\boldsymbol{x}}(\neg\phi) &= \neg ST_{\boldsymbol{x}}(\phi) & ST_{\boldsymbol{y}}(\neg\phi) &= \neg ST_{\boldsymbol{y}}(\phi) \\
ST_{\boldsymbol{x}}(\phi \vee \psi) &= ST_{\boldsymbol{x}}(\phi) \vee ST_{\boldsymbol{x}}(\psi) & ST_{\boldsymbol{y}}(\phi \vee \psi) &= ST_{\boldsymbol{y}}(\phi) \vee ST_{\boldsymbol{y}}(\psi) \\
ST_{\boldsymbol{x}}(\Diamond\phi) &= \exists \boldsymbol{y} \cdot (\boldsymbol{x}\; R\; \boldsymbol{y} \wedge ST_{\boldsymbol{y}}(\phi)) & ST_{\boldsymbol{y}}(\Diamond\phi) &= \exists \boldsymbol{x} \cdot (\boldsymbol{y}\; R\; \boldsymbol{x} \wedge ST_{\boldsymbol{x}}(\phi)) \\
ST_{\boldsymbol{x}}(\downarrow z \cdot \phi) &= \exists z \cdot (z \approx \boldsymbol{x} \wedge ST_{\boldsymbol{x}}(\phi)) & ST_{\boldsymbol{y}}(\downarrow z \cdot \phi) &= \exists z \cdot (z \approx \boldsymbol{y} \wedge ST_{\boldsymbol{y}}(\phi)) \\
ST_{\boldsymbol{x}}(@_w\, \phi) &= \boldsymbol{x} \approx w \wedge ST_{\boldsymbol{x}}(\phi) & ST_{\boldsymbol{y}}(@_w\, \phi) &= \boldsymbol{y} \approx w \wedge ST_{\boldsymbol{y}}(\phi) \\
ST_{\boldsymbol{x}}(\exists z \cdot \phi) &= \exists z \cdot ST_{\boldsymbol{x}}(\phi) & ST_{\boldsymbol{y}}(\exists z \cdot \phi) &= \exists z \cdot ST_{\boldsymbol{y}}(\phi)
\end{aligned}
$$

The way we define translations is slightly different from Areces *et al.* [2], but our definitions make explicit the transitional aspect of $\Diamond$, the quantificational aspect of $\downarrow$ and $\exists$, and the identity-like aspect of $@$. The resulting concept is essentially the same as all other particular forms of standard translations. Namely, we have that for each hybrid sentence ϕ (no free world variables), the formula $ST_{\boldsymbol{x}}(\phi)$ has precisely one variable, $\boldsymbol{x}$ or $\boldsymbol{y}$, free, and moreover

(SE) $\mathfrak{M}, m \Vdash \phi \iff \mathfrak{M} \vDash ST_{\boldsymbol{x}}(\phi)[m]$

where the right-hand side is the usual first-order satisfaction of $ST_{\boldsymbol{x}}(\phi)$ on evaluating $\boldsymbol{x}$ to m; the same holds for $ST_{\boldsymbol{y}}(\phi)$ of course. Alternatively, the right-hand side can be expressed in a signature expanded by a single constant intepreted as m, by $(\mathfrak{M}, \boldsymbol{m}) \vDash ST_{\boldsymbol{m}}(\phi)$. This would have the advantage of translating sentences to sentences.

The full hybrid language over Σ is exactly as expressive as the first-order language over Σ°. To say more about expressivity, we recall the *standard back translation* introduced in Hodkinson and Tahiri [20]. Let F be the function from $\mathcal{L}_{\Sigma^\circ}$ to the hybrid language over Σ defined inductively by putting

$$
\begin{aligned}
F(\bot) &= \bot \\
F(P(w)) &= @_{F(w)}\, p \\
F(x) &= x \\
F(s) &= s \\
F(\neg\phi) &= \neg F(\phi) \\
F(\phi \vee \psi) &= F(\phi) \vee F(\psi)
\end{aligned}
$$

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$$F(\exists x \cdot \phi) = \exists x \cdot F(\phi)$$
$$F(w\ R\ w') = @_{F(w)} \Diamond F(w')$$
$$F(w \approx w') = @_{F(w)} F(w')$$

The standard back translation of a formula $\phi(x)$, with x its only free variable, is the hybrid sentence $SBT(\phi[x]) = \exists x \cdot (x \wedge F(\phi(x)))$. It is easy to show that

$$\mathfrak{M} \vDash \phi[m] \iff \mathfrak{M}, m \Vdash SBT(\phi(x))$$

for any model $\mathfrak{M}$ and any $m \in M$. It is clear from the definitions that in presence of $\exists$ and $@$ the hybrid language captures the whole first-order one. Letting $\mathcal{F} = \{@, \downarrow, \exists, \textsc{nom}\}$ be the hybrid language features beyond modal logic, and recalling that $\downarrow$ is expressible in presence of $\exists$, we obtain five interesting fragments of the language over $\mathcal{F}$, namely, $\{@\}$, $\{\downarrow\}$, $\{\exists\}$, $\{@, \downarrow\}$ and $\{@, \exists\}$, each with and without nominals. For each of these there is a natural question about a characterisation of its standard translation, that is, a characterisation of the set of precisely those first-order formulas that are equivalent to translations of hybrid sentences from the given fragment.

An appropriate notion for such characterisations proved to be *bisimulation*, which we will now recall. Let $\mathfrak{M} = (M, R^{\mathfrak{M}}, V^{\mathfrak{M}})$ and $\mathfrak{N} = (N, R^{\mathfrak{N}}, V^{\mathfrak{N}})$ be models. A relation $B \subseteq M \times N$ is a bisimulation from $\mathfrak{M}$ to $\mathfrak{N}$ if for all $m\ B\ n$ the following conditions hold:

(prop) $m \in V^{\mathfrak{M}}(p)$ iff $n \in V^{\mathfrak{N}}(p)$ for all $p \in \textsc{prop}$;
(forth) for all $m' \in R^{\mathfrak{M}}(m)$ there is an $n' \in R^{\mathfrak{N}}(n)$ with $m'\ B\ n'$;
(back) for all $n' \in R^{\mathfrak{N}}(n)$ there is an $m' \in R^{\mathfrak{M}}(m)$ with $m'\ B\ n'$.

So defined, bisimulation is a similarity relation between models, hence by a terminological quirk one says that two models related by a bisimulation are *bisimilar* (a consistent terminology would have the relation called *bisimilarity*, but while logics are typically consistent, logicians are not). Two pointed models $(\mathfrak{M}, m)$ and $(\mathfrak{N}, n)$ are called bisimilar if there is a bisimulation $B \subseteq M \times N$ such that $m\ B\ n$. In this paper, we do not require the bisimulation to be non-empty; instead, we work with bisimilar pointed models, which implicitly ensures non-emptiness.

For a purely modal language, van Benthem [28] proved that a first-order formula is equivalent to the standard translation of a modal sentence if, and only if, it is invariant under bisimulations, where a sentence ϕ is *invariant under bisimulation* if for all bisimilar pointed models $(\mathfrak{M}, m)$ and $(\mathfrak{N}, n)$ we have $\mathfrak{M}, m \Vdash \phi$ iff $\mathfrak{N}, n \Vdash \phi$. Adding nominals to the language breaks this correspondence. For consider the structures $\mathfrak{M}$ and $\mathfrak{N}$ in Fig. 1, with $B = \{(\mathrm{m}_0, \mathrm{n}_0), (\mathrm{m}_1, \mathrm{n}_1), (\mathrm{m}_2, \mathrm{n}_1)\}$ indicated by dashed lines. Taking $V^{\mathfrak{M}}(p) = M$ and $V^{\mathfrak{N}}(p) = N$ on all $p \in \textsc{prop}$, we immediately have that B is a bisimulation. Now, assuming that for distinct nominals s and t we have $\mathrm{s}^{\mathfrak{M}} = \mathrm{m}_0$, $\mathrm{t}^{\mathfrak{M}} = \mathrm{m}_1$, $\mathrm{s}^{\mathfrak{N}} = \mathrm{n}_0$, $\mathrm{t}^{\mathfrak{N}} = \mathrm{n}_1$ we see that B does not preserve nominals, since $\mathfrak{N}, \mathrm{n}_1 \Vdash \mathrm{t}$ but $\mathfrak{M}, \mathrm{m}_2 \nVdash \mathrm{t}$. The problem is easily repaired by strengthening the notion of bisimulation. It suffices to add the condition

(nom) $m = s^{\mathfrak{M}}$ iff $n = s^{\mathfrak{N}}$ for every $s \in \textsc{nom}$,

to recover the characterisation. Note that (nom) forces bisimulations to be bijections between $\textsc{nom}^{\mathfrak{M}}$ and $\textsc{nom}^{\mathfrak{N}}$.

A similar problem arises if we add $\downarrow$ and world variables (but not nominals) to the purely modal language. For consider the structures $\mathfrak{M}$ and $\mathfrak{N}$ in Fig. 2, with B given

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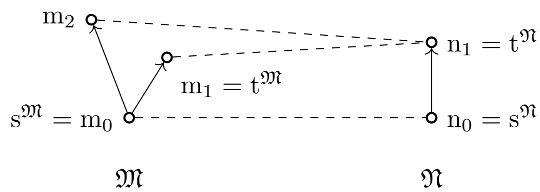

FIGURE 1. Bisimilar but not preserving nominals

by the dashed lines. Let $V^{\mathfrak{M}}(p) = M$ and $V^{\mathfrak{N}}(p) = N$ for all $p \in$ PROP; moreover, let $\nu^{\mathfrak{M}}(\mathrm{z}_i) = \mathrm{m}_i$ and $\nu^{\mathfrak{N}}(\mathrm{z}_i) = \mathrm{n}_{i \bmod 2}$ for all $\mathrm{z}_i \in$ WVAR. Then B is a bisimulation

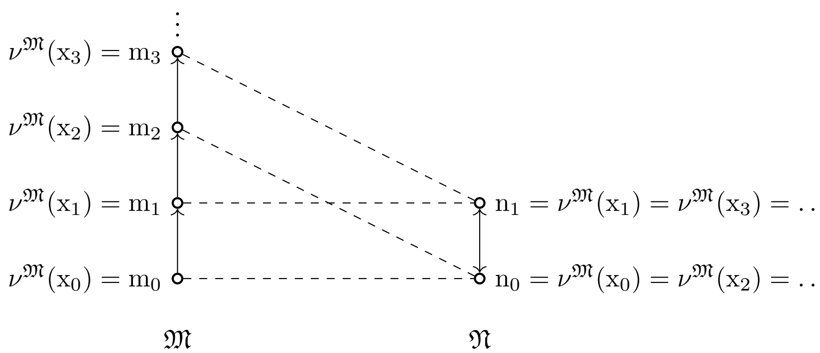

FIGURE 2. Bisimilar but not preserving $\downarrow$

not preserving $\downarrow$, as for example $\mathfrak{N}, \mathrm{n}_0 \Vdash {\downarrow}\mathrm{x} \cdot \Diamond\Diamond\mathrm{x}$, but $\mathfrak{M}, \mathrm{m}_0 \nVdash {\downarrow}\mathrm{x} \cdot \Diamond\Diamond\mathrm{x}$. Similar examples can easily be constructed with $\mathfrak{N}$ being a cycle of length ℓ (in particular, a loop). As an aside, note that it would not be reasonable to require preservation of open formulas, as nothing short of a bijection can preserve world variables.

§3. Quasi-injective bisimulations. To deal with $\downarrow$, Blackburn and Seligman [8] introduce *quasi-injective bisimulations*, that is, relations satisfying the usual bisimulation conditions together with the requirement that distinct bisimulation images of the same state are mutually inaccessible. To be precise, let $(R^{\mathfrak{M}})^*$ be the reflexive, transitive closure of $R^{\mathfrak{M}}$ and let us say that m' is *reachable* in $\mathfrak{M}$ from m if $m' \in (R^{\mathfrak{M}})^*(m)$. A bisimulation $B \subseteq M \times N$ is *quasi-injective* if $m\ B\ n$, $m\ B\ n'$, and $n \neq n'$ imply that $n' \notin (R^{\mathfrak{N}})^*(n)$ and $n \notin (R^{\mathfrak{N}})^*(n')$, and the analogous symmetric condition also holds. They prove that formulas of the language with $\downarrow$ but without nominals are preserved under quasi-injective bisimulations, hence any formula equivalent to a standard translation of a sentence of that language is invariant under quasi-injective bisimulations. Hodkinson and Tahiri [20] prove the converse, thus obtaining a characterisation theorem.

THEOREM 1. *For a hybrid language over a signature Σ without nominals and only involving $\downarrow$, the following are equivalent for a first-order formula ϕ over Σ°:*

1. *ϕ is equivalent to a translation of a hybrid sentence.*
2. *ϕ is invariant under quasi-injective bisimulations.*

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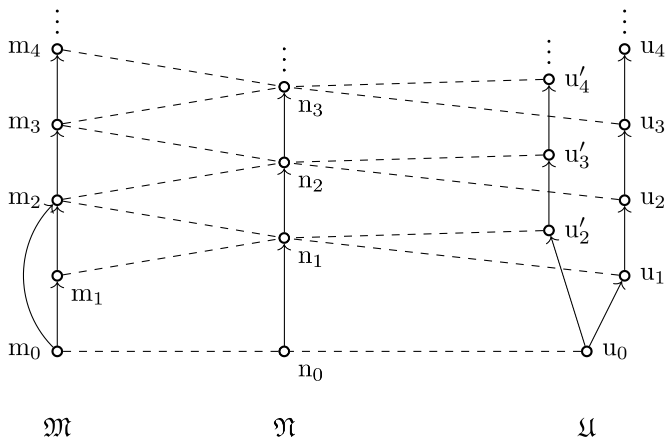

FIGURE 3. $\mathfrak{N}$ and $\mathfrak{M}$ are $\downarrow$-bisimilar but not quasi-injectively bisimilar; $\mathfrak{N}$ and $\mathfrak{U}$ are quasi-injectively bisimilar.

Quasi-injective bisimulation faces an obvious problem with nominals: even mutually inaccessible states cannot be bisimilar to a single state whenever that single state has a name. Aware of this, Hodkinson and Tahiri ask whether a characterisation can be obtained for the language including nominals. The example below shows that $\downarrow$-preserving bisimulations need not be quasi-injective, suggesting that quasi-injectivity is too strong; see also Ex. 3 in §4.

EXAMPLE 1. Consider structures $\mathfrak{M}$ and $\mathfrak{N}$ of Fig. 3. The relation B indicated by dashed arrows is clearly a bisimulation; also clearly it is not quasi-injective. Yet, B preserves $\downarrow$-formulas, as we will now show. To do so, we make use of *$\downarrow$-unravelling*, defined in Hodkinson and Tahiri [20, Defs. 3.7 and 3.8]. For our purposes it suffices to observe that for acyclic graphs $\downarrow$-unravelling coincides with the usual unravelling. Hodkinson and Tahiri show (see [20, Prop. 3.10]) that a model and its $\downarrow$-unravelling are $\downarrow$-bisimilar, hence $\downarrow$-sentences are invariant under $\downarrow$-unravelling. Therefore, $\downarrow$-sentences are invariant under unravelling acyclic digraphs. Now consider structures $\mathfrak{U}$ and $\mathfrak{N}$ of Fig. 3, and observe that (i) $\mathfrak{U}$ is an unravelling of $\mathfrak{M}$, (ii) $\mathfrak{M}$ and $\mathfrak{N}$ are acyclic digraphs, (iii) $\mathfrak{U}$ and $\mathfrak{N}$ are quasi-injectively bisimilar. It follows that $\downarrow$-sentences are invariant under $B \subseteq M \times N$. But B is not quasi-injective.

§4. Bisimulations with memory. Our approach to bisimulations is motivated by Areces *et al.* [2] where in order to arrive at a notion of bisimulation appropriate for hybrid languages the bisimilarity relation is endowed with memory: pairs of states are not bisimilar *per se* but in connection to their histories—we compare strings over the alphabet of states rather than single states. In [2, Sect. 3.3], this idea is fleshed out in the form of k- and ω-bisimulations. We recall the definitions below, with inessential modifications in presentation and terminology.

DEFINITION 1. Let $\mathfrak{M}$ and $\mathfrak{N}$ be Kripke structures. A relation $B_k \subseteq (M^k \times M) \times (N^k \times N)$ is a *k-bisimulation* from $\mathfrak{M}$ to $\mathfrak{N}$ if for all $(\overline{m}, m)\ B_k\ (\overline{n}, n)$ the following hold:

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(prop) $m \in V^{\mathfrak{M}}(p)$ iff $n \in V^{\mathfrak{N}}(p)$ for all $p \in$ PROP;
(nom) $m = s^{\mathfrak{M}}$ iff $n = s^{\mathfrak{N}}$ for every $s \in$ NOM;
(wvar) $\overline{m}(j) = m$ iff $\overline{n}(j) = n$ for all $1 \leq j \leq k$;
(forth) for all $m' \in R^{\mathfrak{M}}(m)$ there is an $n' \in R^{\mathfrak{N}}(n)$ with $(\overline{m}, m')\ B_k\ (\overline{n}, n')$;
(back) for all $n' \in R^{\mathfrak{N}}(n)$ there is an $m' \in R^{\mathfrak{M}}(m)$ with $(\overline{m}, m')\ B_k\ (\overline{n}, n')$;
(atv) $(\overline{m}, \overline{m}(j))\ B_k\ (\overline{n}, \overline{n}(j))$ for all $1 \leq j \leq k$;
(atn) $(\overline{m}, s^{\mathfrak{M}})\ B_k\ (\overline{n}, s^{\mathfrak{N}})$ for all $s \in$ NOM;

An ω-*bisimulation* from $\mathfrak{M}$ to $\mathfrak{N}$ is a family of k-bisimulations $(B_k)_{k \in \omega}$ from $\mathfrak{M}$ to $\mathfrak{N}$ such that for all natural numbers $k \in \omega$ and all tuples $(\overline{m}, m) \in M^k \times M$ and $(\overline{n}, n) \in N^k \times N$ the following condition is satisfied:

(st) if $(\overline{m}, m)\ B_k\ (\overline{n}, n)$, then $(\overline{m} \frown m, m)\ B_{k+1}\ (\overline{n} \frown n, n)$,

where $\frown$ stands for concatenation. (Note that technically an ω-bisimulation is a relation between M^* and N^*.)

Definition 1 is obtained from the definition of ω-bisimulation from [2, Sect. 3.3] by removing the condition (bind), which already is covered by rule (st). Areces *et al.* call (st) the storage rule as the operator $\downarrow$ is often informally called *store*, one intuitive reading of $\downarrow x \cdot \phi$ being "store the current state under label x in ϕ". As in [2], @ is handled by the two clauses (atv) and (atn) for variables and nominals separately. For handling the existential quantification we need the following condition:

(ex) If $(\overline{m}, m)\ B_k\ (\overline{n}, n)$, then:
 (ex-f) for all $m' \in M$ there is an $n' \in N$ with $(\overline{m} \frown m', m)\ B_{k+1}\ (\overline{n} \frown n', n)$,
 (ex-b) for all $n' \in N$ there is an $m' \in M$ with $(\overline{m} \frown m', m)\ B_{k+1}\ (\overline{n} \frown n', n)$.

The (ex)-rule is introduced in this paper. We included quantification because our results are parameterised by a set of hybrid language features $\mathcal{F}$, and it is valuable to study any fragment of the hybrid language.

Definition 2 ($\mathcal{F}$-ω-bisimulation). Let $\mathcal{F} \subseteq \{@, \downarrow, \exists, \text{NOM}\}$ be a set of hybrid language features. Let $\mathsf{cond}(\mathcal{F})$ be the smallest subset of the set of conditions given above, which includes {(prop), (forth), (back)} and satisfies the following requirements:

- if NOM $\in \mathcal{F}$, then {(nom), (wvar)} $\subseteq \mathsf{cond}(\mathcal{F})$, where NOM $\in \mathcal{F}$ means that any nominal is a sentence and every world variable is a formula;
- if $\downarrow \in \mathcal{F}$, then (st) $\in \mathsf{cond}(\mathcal{F})$;
- if $\exists \in \mathcal{F}$, then (ex) $\in \mathsf{cond}(\mathcal{F})$,
- if $@ \in \mathcal{F}$, then {(atv), (atn)} $\in \mathsf{cond}(\mathcal{F})$;
- if $\{\downarrow, \text{NOM}\} \subseteq \mathcal{F}$, then {(st), (nom), (wvar)} $\subseteq \mathsf{cond}(\mathcal{F})$;
- if $\{@, \text{NOM}\} \subseteq \mathcal{F}$, then {(atv), (atn), (nom), (wvar)} $\subseteq \mathsf{cond}(\mathcal{F})$;
- if $\{\downarrow, @\} \subseteq \mathcal{F}$, then {(st), (atv), (atn)} $\subseteq \mathsf{cond}(\mathcal{F})$;
- if $\{\exists, @\} \subseteq \mathcal{F}$, then {(ex), (atv), (atn)} $\subseteq \mathsf{cond}(\mathcal{F})$;
- if $\{\exists, \text{NOM}\} \subseteq \mathcal{F}$, then {(ex), (st), (nom), (wvar)} $\subseteq \mathsf{cond}(\mathcal{F})$;[1]
- if $\{\downarrow, @, \text{NOM}\} \subseteq \mathcal{F}$, then {(st), (atv), (atn), (nom), (wvar)} $\subseteq \mathsf{cond}(\mathcal{F})$;
- if $\{\exists, @, \text{NOM}\} \subseteq \mathcal{F}$, then $\mathsf{cond}(\mathcal{F})$ contains all the conditions.

An ω-bisimulation $(B_k)_{k \in \omega}$ from $\mathfrak{M}$ and $\mathfrak{N}$ in the hybrid language of $\mathcal{F}$ will be called an $\mathcal{F}$-ω-*bisimulation* from $\mathfrak{M}$ to $\mathfrak{N}$ if the conditions $\mathsf{cond}(\mathcal{F})$ hold. Two pointed models

[1]Since $\downarrow x \cdot \phi$ is semantically equivalent to $\exists x \cdot (x \wedge \phi)$, this fragment is semantically closed under store; hence, (st) $\in \mathsf{cond}(\mathcal{F})$.

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$(\mathfrak{M}, m)$ and $(\mathfrak{N}, n)$ are *$\mathcal{F}$-ω-bisimilar* if there exists an *$\mathcal{F}$-ω-bisimulation* from $\mathfrak{M}$ to $\mathfrak{N}$ such that $m\ B_0\ n$.

Note that the possibilities above are exhaustive, as $\downarrow$ is redundant in presence of $\exists$. In practice we will avoid using precise names such as $\{\downarrow, \text{NOM}\}$-$\omega$-bisimulation, and rely on context and circumlocutions to clarify what kind of an ω-bisimulation we need. The remainder of this paper should be read with this principle in mind.

EXAMPLE 2. Consider again the structures $\mathfrak{M}$ and $\mathfrak{N}$ of Ex. 1 and the bisimulation relation B from $\mathfrak{M}$ to $\mathfrak{N}$. For each of the pairs $(\mathrm{m}_0, \mathrm{n}_0)$, $(\mathrm{m}_i, \mathrm{n}_i)$, and $(\mathrm{m}_{i+1}, \mathrm{n}_i)$ in B with $i \geq 1$, writing λ for the empty sequence, define inductively for $k \geq 0$

$$\begin{aligned}
S_0^{0,0} &= S_0^{i,i} = S_0^{i+1,i} = \{(\lambda, \lambda)\}\\
S_{k+1}^{0,0} &= \{(\overline{m} \frown \mathrm{m}_0, \overline{n} \frown \mathrm{n}_0) : (\overline{m}, \overline{n}) \in S_k^{0,0}\}\\
S_{k+1}^{i,i} &= \{(\overline{m} \frown \mathrm{m}_0, \overline{n} \frown \mathrm{n}_0) : (\overline{m}, \overline{n}) \in S_k^{i,i}\}\\
&\quad \cup \{(\overline{m} \frown \mathrm{m}_j, \overline{n} \frown \mathrm{n}_j) : 1 \leq j \leq i,\ (\overline{m}, \overline{n}) \in S_k^{i,i}\}\\
S_{k+1}^{i+1,i} &= \{(\overline{m} \frown \mathrm{m}_0, \overline{n} \frown \mathrm{n}_0) : (\overline{m}, \overline{n}) \in S_k^{i+1,i}\}\\
&\quad \cup \{(\overline{m} \frown \mathrm{m}_{j+1}, \overline{n} \frown \mathrm{n}_j) : 1 \leq j \leq i,\ (\overline{m}, \overline{n}) \in S_k^{i+1,i}\}
\end{aligned}$$

and finally put

$$B_k = \{(\overline{m} \frown \mathrm{m}_i, \overline{n} \frown \mathrm{n}_j) : \mathrm{m}_i\ B\ \mathrm{n}_j,\ (\overline{m}, \overline{n}) \in S_k^{i,j}\}.$$

In particular, $B_0 = \{(\lambda \frown m, \lambda \frown n) : m\ B\ n\}$. Indeed, for each $k \geq 0$, the relation B_k satisfies all the requirements of a k-bismulation between $\mathfrak{M}$ and $\mathfrak{N}$, except (atv). The construction of B_k is a good example of what we are aiming at, so let us dwell on it for a while. One can view the sets $S_k^{i,j}$ as recording k pairs of states visited sequentially in a "run" starting at $(\mathrm{m}_0, \mathrm{n}_0)$ and visiting pairs accessible from $(\mathrm{m}_0, \mathrm{n}_0)$. In each run, only single pairs of accessible states are recorded: for instance, if $(\mathrm{m}_1, \mathrm{n}_1)$ is visited, then $(\mathrm{m}_2, \mathrm{n}_1)$ is not. An analogy to think of is a bisimulation between nondeterministic automata. Less metaphorically, the construction is such that the sequence of recorded pairs of states $S_k^{i,j}$ can only contain pairs of states $(\mathrm{m}_k, \mathrm{n}_\ell)$ such that $k \leq i$ and $\ell \leq j$, where $(\mathrm{m}_i, \mathrm{n}_j)$ is the current state. Crucially for (wvar), the construction separates between runs including and excluding m_1; if $(\mathrm{m}_1, \mathrm{n}_1)$ has been visited, then only the pairs $(\mathrm{m}_i, \mathrm{n}_i)$ are visited; if not, then only the pairs $(\mathrm{m}_{i+1}, \mathrm{n}_i)$. This property is preserved by transitions in $\mathfrak{M}$ and $\mathfrak{N}$, respectively, so (forth) and (back) hold. Finally, it is easy to verify that the family $(B_k)_{0 \leq k}$ also satisfies (st).

LEMMA 1. *Let Σ be a signature without nominals. Let $\mathfrak{M}$ and $\mathfrak{N}$ be Σ-structures related by a quasi-injective bisimulation. Then there is a $\{\text{NOM}, \downarrow\}$-$\omega$-bisimulation between $\mathfrak{M}$ and $\mathfrak{N}$.*

PROOF. Let B be a quasi-injective bisimulation between $\mathfrak{M}$ and $\mathfrak{N}$. Define $B_k \subseteq (M^k \times M) \times (N^k \times N)$ for all $k \geq 0$ as follows

$$\begin{aligned}
B_k = \Big\{ \big((\overline{m}, m), (\overline{n}, n)\big) : m\ B\ n \wedge&\\
\forall j \in \{1, \dots, k\} \cdot \overline{m}(j)\ B\ \overline{n}(j) \wedge&\\
m \in (R^{\mathfrak{M}})^*(\overline{m}(j)) \wedge n \in (R^{\mathfrak{N}})^*(\overline{n}(j)) &\Big\}
\end{aligned}$$

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Properties (back), (forth), and (st) are satisfied immediately. For (wvar), let $(\overline{m}, m)\ B_k\ (\overline{n}, n)$. First assume $\overline{m}(j) = m$ for some $1 \leq j \leq k$. Then $\overline{m}(j)\ B\ \overline{n}(j)$, $m\ B\ n$, and $n \in (R^{\mathfrak{N}})^*(\overline{n}(j))$ by definition of B_k. But then $\overline{n}(j) = n$ as otherwise B would not be quasi-injective. For the other direction just interchange n and m. ⊣

As demonstrated by Ex. 1, the converse of Lem. 1 does not hold. Indeed, the next example shows that quasi-injective bisimulations tend to preserve non-first-order properties such as having uncountable cardinality. It would be interesting to find out the precise strength of quasi-injective bisimulations, but it is beyond the scope of the present article, and may prove elusive.

Example 3. Let $\mathfrak{R} = (\mathbb{R}, \leq)$ and $\mathfrak{Q} = (\mathbb{Q}, \leq)$ be the usual reals and rationals with their usual orders. The player $\exists$, say, Eloïse, has a winning strategy in the Ehrenfeucht-Fraïssé game $\mathrm{EF}_\omega(\mathfrak{R}, \mathfrak{Q})$. Let the relation $B_k \subseteq (\mathbb{R}^k \times \mathbb{R}) \times (\mathbb{Q}^k \times \mathbb{Q})$ be given by putting $(\overline{r}, r)\ B_k\ (\overline{q}, q)$ if and only if (i) $(\overline{r}, \overline{q})$ is the pair of sequences arising from the first k rounds in some play of $\mathrm{EF}_\omega(\mathfrak{R}, \mathfrak{Q})$ in which Eloïse follows her winning strategy, and (ii) (r, q) is the pair consisting of Abelard's move and Eloïse's response in the $k+1$ round. It is not difficult to see that the family $(B_k)_{k\in\omega}$ is an ω-bisimulation between $\mathfrak{R}$ and $\mathfrak{Q}$. However, there is no non-empty quasi-injective bisimulation between $\mathfrak{R}$ and $\mathfrak{Q}$. For suppose there is one, say S, and suppose $r\ S\ q$. Then, by quasi-injectivity, S must be a bijective map between $\{u \in \mathbb{R} : r \leq u\}$ and $\{w \in \mathbb{Q} : q \leq w\}$. This is clearly impossible because of the cardinalities of these sets.

The following example shows that the "memory" (understood as assigning values to world variables) is irrelevant in the absence of store $\downarrow$ or existential quantification $\exists$.

Example 4. Let Σ be a hybrid signature with no nominals and one propositional symbol p. Let $\mathfrak{M}$ and $\mathfrak{N}$ be the Σ-models shown to the left and right, respectively, in Fig. 4. In term rewriting terminology [3], both models are abstract rewriting systems

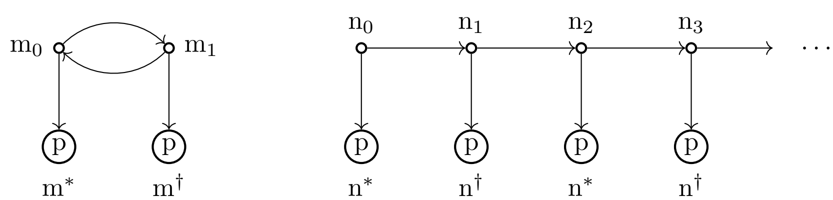

Figure 4. Bisimilarity without store

that are locally confluent but not confluent; p stands for the normal form property. Let $B_0 = \{(\mathrm{m}_0, \mathrm{n}_{2i}) : i \in \omega\} \cup \{(\mathrm{m}_1, \mathrm{n}_{2i+1}) : i \in \omega)\} \cup \{(\mathrm{m}^*, \mathrm{n}^*), (\mathrm{m}^\dagger, \mathrm{n}^\dagger)\}$ and $B_k = \emptyset$ for all $k > 0$. Then $(B_k)_{k\in\omega}$ is an $\mathcal{F}$-ω-bisimulation from $\mathfrak{M}$ to $\mathfrak{N}$, where $\mathcal{F} \subseteq \{\textsc{nom}, @\}$.

§5. Characterisation theorems. Our proof approach of bisimulation invariance characterisations for hybrid logic is modelled after Badia [4], where such a characterisation was given for bi-intuitionistic logic. That in turn was motivated by the proof of the celebrated Lindström characterisation of first-order logic in [21]. The main technicality in our approach is a finite approximation of the notion of k-bisimulation in the sense of Areces *et al.* [2]. An illustration is the family $S_k^{i,j}$ of relations, given in Ex. 2. In general, this allows us to manoeuvre around first-order undefinability of ω-bisimulations.

Definition 3 (Basic (k,ℓ)-bisimulation). Let $\mathfrak{M} = (M, R^{\mathfrak{M}}, V^{\mathfrak{M}})$ and $\mathfrak{N} = (N, R^{\mathfrak{N}}, V^{\mathfrak{N}})$ be models of the purely modal language. A system of relations $(Z_i^k)_{i\leq\ell}$ where $Z_i^k \subseteq (M^k \times M) \times (N^k \times N)$ will be called a *basic* (k,ℓ)*-bisimulation* from $\mathfrak{M}$ to $\mathfrak{N}$ if the following holds for all $i \leq \ell$:

(prop) if $(\overline{m}, m)\ Z_i^k\ (\overline{n}, n)$ and $p \in \text{PROP}$, then $m \in V^{\mathfrak{M}}(p)$ iff $n \in V^{\mathfrak{N}}(p)$,
(ω-forth) if $(\overline{m}, m)\ Z_{i-1}^k\ (\overline{n}, n)$ and $m' \in R^{\mathfrak{M}}(m)$, then $(\overline{m}, m')\ Z_i^k\ (\overline{n}, n')$ for some $n' \in R^{\mathfrak{N}}(n)$,
(ω-back) if $(\overline{m}, m)\ Z_{i-1}^k\ (\overline{n}, n)$ and $n' \in R^{\mathfrak{N}}(n)$, then $(\overline{m}, m')\ Z_i^k\ (\overline{n}, n')$ for some $m' \in R^{\mathfrak{M}}(m)$.

For pointed models $(\mathfrak{M}, m)$ and $(\mathfrak{N}, n)$, we add the condition $m\ Z_0^0\ n$.

Notice that (ω-forth) and (ω-back) from Def. 3 are applicable at most ℓ times while (forth) and (back) from Def. 1 is applicable any number of times.

Definition 4 (Extended (k,ℓ)-bisimulation). Let Σ be a hybrid signature and let $\mathfrak{M} = (M, R^{\mathfrak{M}}, (s^{\mathfrak{M}})_{s\in\text{NOM}}, V^{\mathfrak{M}})$ and $\mathfrak{N} = (N, R^{\mathfrak{N}}, (s^{\mathfrak{N}})_{s\in\text{NOM}}, V^{\mathfrak{N}})$ be Kripke structures over Σ. Let $(Z_i^k)_{i\leq\ell}$ be a basic (k,ℓ)-bisimulation from $\mathfrak{M}$ to $\mathfrak{N}$. Consider the following conditions:

(nom) If $(\overline{m}, m)\ Z_i^k\ (\overline{n}, n)$ and $s \in \text{NOM}$, then $m = s^{\mathfrak{M}}$ iff $n = s^{\mathfrak{N}}$.
(wvar) If $(\overline{m}, m)\ Z_i^k\ (\overline{n}, n)$ and $1 \leq j \leq k$, then $\overline{m}(j) = m$ iff $\overline{n}(j) = n$.
(atv) If $(\overline{m}, m)\ Z_i^k\ (\overline{n}, n)$ and $1 \leq j \leq k$, then $(\overline{m}, \overline{m}(j))\ Z_i^k\ (\overline{n}, \overline{n}(j))$.
(atn) If $(\overline{m}, m)\ Z_i^k\ (\overline{n}, n)$ and $s \in \text{NOM}$, then $(\overline{m}, s^{\mathfrak{M}})\ Z_i^k\ (\overline{n}, s^{\mathfrak{N}})$.

For a set $\mathcal{F}$ of hybrid language features, the basic (k,ℓ)-bisimulation $(Z_i^k)_{i\leq\ell}$ will be called an $\mathcal{F}$-(k,ℓ)*-bisimulation* from $\mathfrak{M}$ to $\mathfrak{N}$ if the conditions $\mathsf{cond}(\mathcal{F})$ from Def. 2 hold. For pointed models $(\mathfrak{M}, m)$ and $(\mathfrak{N}, n)$, we add the condition $m\ Z_0^0\ n$, as before.

Note that $\emptyset$-$(0,\ell)$-bisimulation is the bisimulation for the purely modal language, while $\{\text{NOM}, @\}$-$(0,\ell)$-bisimulation is the bisimulation for the hybrid language obtained by adding nominals and retrieve to the modal language. As we mentioned already, a k-bisimulation can be viewed as relation between words of length $k+1$. Now we extend the notion to words of countably infinite arbitrary lengths.

Definition 5 ($\mathcal{F}$-(ω,ℓ)-bisimulation). Let Σ be a hybrid signature and let $\mathcal{F}$ be a set of hybrid language features. Let $\mathfrak{M} = (M, R^{\mathfrak{M}}, (s^{\mathfrak{M}})_{s\in\text{NOM}}, V^{\mathfrak{M}})$ and $\mathfrak{N} = (N, R^{\mathfrak{N}}, (s^{\mathfrak{N}})_{s\in\text{NOM}}, V^{\mathfrak{N}})$ be models over Σ. Let $(Z_i^k)_{\substack{i\leq\ell\\k\in\omega}}$ be a system of relations such that $(Z_i^k)_{i\leq\ell}$ is an $\mathcal{F}$-(k,ℓ)-bisimulation from $\mathfrak{M}$ to $\mathfrak{N}$ for each $k \in \omega$. Consider the following *extensibility* conditions:

(st) If $(\overline{m}, m)\ Z_{i-1}^k\ (\overline{n}, n)$, then $(\overline{m} \frown m, m)\ Z_i^{k+1}\ (\overline{n} \frown n, n)$.
(ex) If $(\overline{m}, m)\ Z_{i-1}^k\ (\overline{n}, n)$, then:
  (ex-f) for all $m' \in M$, $(\overline{m} \frown m', m)\ Z_i^{k+1}\ (\overline{n} \frown n', n)$ for some $n' \in N$;
  (ex-b) for all $n' \in N$, $(\overline{m} \frown m', m)\ Z_i^{k+1}\ (\overline{n} \frown n', n)$ for some $m' \in M$.

For a set $\mathcal{F}$ of hybrid language features, $(Z_i^k)_{\substack{i\leq\ell\\k\in\omega}}$ is an $\mathcal{F}$-(ω,ℓ)*-bisimulation* from $\mathfrak{M}$ to $\mathfrak{N}$ if the conditions $\mathsf{cond}(\mathcal{F})$ from Def. 2 hold.

The following result shows that two pointed models are indistinguishable by formulas of degree at most ℓ if and only if there is $\mathcal{F}$-(ω, ℓ)-bisimulation between them.

THEOREM 2 ($\mathcal{F}$-(ω, ℓ)-bisimulation). *Consider a hybrid language defined over a finite signature with features from $\mathcal{F}$, and two models $\mathfrak{M}$ and $\mathfrak{N}$ for this hybrid language. Let $\ell, j \in \omega$ be two natural numbers. Let $(\overline{a}, a) \in M^j \times M$ and $(\overline{b}, b) \in N^j \times N$ be two tuples of elements. The following are equivalent:*

(i) there is an $\mathcal{F}$-(ω, ℓ)-bisimulation $(Z_i^k)_{\substack{i \leq \ell \\ k \in \omega}}$ from $\mathfrak{M}$ to $\mathfrak{N}$ such that $(\overline{a}, a)\ Z_0^j\ (\overline{b}, b)$;

(ii) for each hybrid formula $\phi(\overline{x})$ of degree at most ℓ with $\overline{x}$ a sequence of world variables of length j, we have $\mathfrak{M}, \overline{a}, a \Vdash \phi[\overline{x}]$ iff $\mathfrak{N}, \overline{b}, b \Vdash \phi[\overline{x}]$.[2]

PROOF. For (i) $\Longrightarrow$ (ii), we first show that the following property holds for all $i \leq \ell$, all $k \in \omega$, all formulas $\phi(\overline{x})$ with $\overline{x} = x_1, \ldots, x_k$ of degree $\mathsf{dg}(\phi) \leq i$, all $(\overline{m}, m) \in M^k \times M$, and all $(\overline{n}, n) \in N^k \times N$:

$(*)$ if $(\overline{m}, m)\ Z_{\ell-i}^k\ (\overline{n}, n)$, then $\mathfrak{M}, \overline{m}, m \Vdash \phi[\overline{x}]$ iff $\mathfrak{N}, \overline{n}, n \Vdash \phi[\overline{x}]$.

We proceed by induction on complexity of ϕ.

Case $p \in$ PROP: We assume $(\overline{m}, m)\ Z_{\ell-i}^k\ (\overline{n}, n)$. Then we have

$$\mathfrak{M}, \overline{m}, m \Vdash p \iff m \in V^{\mathfrak{M}}(p) \overset{\text{(prop)}}{\iff} n \in V^{\mathfrak{N}}(p) \iff \mathfrak{N}, \overline{n}, n \Vdash p$$

Case $s \in$ NOM: As previously, assume $(\overline{m}, m)\ Z_{\ell-i}^k\ (\overline{n}, n)$; then

$$\mathfrak{M}, \overline{m}, m \Vdash s \iff m = s^{\mathfrak{M}} \overset{\text{(nom)}}{\iff} n = s^{\mathfrak{N}} \iff \mathfrak{N}, \overline{n}, n \Vdash s$$

Case x_e: Similarly, using (wvar) we obtain

$$\mathfrak{M}, \overline{m}, m \Vdash x_e[\overline{x}] \iff \overline{m}(e) = m \overset{\text{(wvar)}}{\iff} \overline{n}(e) = n \iff \mathfrak{N}, \overline{n}, n \Vdash x_e[\overline{x}]$$

The *cases* of $\neg$, $\wedge$, and $\vee$ are straightforward consequences of the induction hypothesis.

Case $\phi = \Diamond\psi$: We have $\mathsf{dg}(\phi) \leq i \leq \ell$ and $\mathsf{dg}(\psi) \leq i - 1$. Assume that $(\overline{m}, m)\ Z_{\ell-i}^k$ $(\overline{n}, n)$. Hence

$$\begin{aligned}
& \mathfrak{M}, \overline{m}, m \Vdash \Diamond\psi[\overline{x}] \\
\iff & \mathfrak{M}, \overline{m}, m' \Vdash \psi[\overline{x}] \text{ for some } m' \in M \text{ with } m\ R^{\mathfrak{M}}\ m' \\
\overset{\text{I. H.}}{\Longrightarrow} & \mathfrak{N}, \overline{n}, n' \Vdash \psi[\overline{x}] \text{ for some } n' \in N \text{ with } n'\ R^{\mathfrak{N}}\ n' \\
\iff & \mathfrak{N}, \overline{n}, n \Vdash \Diamond\psi[\overline{x}]
\end{aligned}$$

where I. H. is applicable since $(\overline{m}, m)\ Z_{\ell-i}^k\ (\overline{n}, n)$ implies $(\overline{m}, m')\ Z_{\ell-(i-1)}^k\ (\overline{n}, n')$ by (ω-forth); the backward direction follows similarly using (ω-back).

Case $\phi = {\downarrow} x \cdot \psi$: Again, $\mathsf{dg}(\phi) \leq i \leq \ell$ and $\mathsf{dg}(\psi) \leq i - 1$.

$$\begin{aligned}
& \mathfrak{M}, \overline{m}, m \Vdash {\downarrow} x \cdot \psi[\overline{x}] \\
\iff & \mathfrak{M}, \overline{m} \frown m, m \Vdash \psi[\overline{x} \frown x] \\
\overset{\text{I. H.}}{\iff} & \mathfrak{N}, \overline{n} \frown n, n \Vdash \psi[\overline{x} \frown x] \\
\iff & \mathfrak{N}, \overline{n}, n \Vdash {\downarrow} x \cdot \psi[\overline{x}]
\end{aligned}$$

[2] $\mathfrak{M}, \overline{m}, m \Vdash \phi[\overline{x}]$ means that $\mathfrak{M}, \nu, m \Vdash \phi$ for some assignment ν which maps $\overline{x}(i)$ to $\overline{m}(i)$ for all $i \in \{1, \ldots, j\}$.

where I. H. is applicable since $(\overline{m}, m)\ Z^k_{\ell-i}\ (\overline{n}, n)$ implies $(\overline{m} \frown m, m)\ Z^{k+1}_{\ell-(i-1)}\ (\overline{n} \frown n, n)$ by (st).

Case $\phi = @_{x_e} \psi$: Then $e \in \{1, \dots, k\}$.

$$\mathfrak{M}, \overline{m}, m \Vdash @_{x_e} \psi[\overline{x}] \iff \mathfrak{M}, \overline{m}, \overline{m}(e) \Vdash \psi[\overline{x}]$$
$$\overset{\text{I. H.}}{\iff} \mathfrak{N}, \overline{n}, \overline{n}(e) \Vdash \psi[\overline{x}] \iff \mathfrak{N}, \overline{n}, n \Vdash @_{x_e} \psi[\overline{x}]$$

where I. H. is applicable since $(\overline{m}, m)\ Z^k_{\ell-i}\ (\overline{n}, n)$ implies $(\overline{m}, \overline{m}(e))\ Z^k_{\ell-i}\ (\overline{n}, \overline{n}(e))$ by (atv).

Case $\phi = @_s \psi$:

$$\mathfrak{M}, \overline{m}, m \Vdash @_s \psi[\overline{x}] \iff \mathfrak{M}, \overline{m}, s^{\mathfrak{M}} \Vdash \psi[\overline{x}]$$
$$\overset{\text{I. H.}}{\iff} \mathfrak{N}, \overline{n}, s^{\mathfrak{N}} \Vdash \psi[\overline{x}] \iff \mathfrak{N}, \overline{n}, n \Vdash @_s \psi[\overline{x}]$$

where I. H. is applicable since $(\overline{m}, m)\ Z^k_{\ell-i}\ (\overline{n}, n)$ implies $(\overline{n}, s^{\mathfrak{M}})\ Z^k_{\ell-i}\ (\overline{n}, s^{\mathfrak{N}})$ by (atn).

Case $\phi = \exists x \cdot \psi$: We have $\mathsf{dg}(\phi) \leq i \leq \ell$ and $\mathsf{dg}(\psi) \leq i - 1$.

$$\begin{aligned} & \mathfrak{M}, \overline{m}, m \Vdash \exists x \cdot \psi[\overline{x}] \\ \iff & \mathfrak{M}, \overline{m} \frown m', m \Vdash \psi[\overline{x} \frown x] \text{ for some } m' \in M \\ \overset{\text{I. H.}}{\Longrightarrow} & \mathfrak{N}, \overline{n} \frown n', n \Vdash \psi[\overline{x} \frown x] \text{ for some } n' \in N \\ \iff & \mathfrak{N}, \overline{n}, n \Vdash \exists x \cdot \psi[\overline{x}] \end{aligned}$$

where I. H. is applicable since $(\overline{m}, m)\ Z^k_{\ell-i}\ (\overline{n}, n)$ implies $(\overline{m} \frown m', m)\ Z^{k+1}_{\ell-(i-1)}\ (\overline{n} \frown n', n)$ for some $n' \in N$ by (ex-f); the converse direction follows similarly using (ex-b). This ends the proof of statement $(*)$.

We apply statement $(*)$ for $i = \ell$ to obtain: $(\overline{a}, a)\ Z^j_0\ (\overline{b}, b)$ implies $\mathfrak{M}, \overline{a}, a \Vdash \phi[\overline{x}]$ iff $\mathfrak{N}, \overline{b}, b \Vdash \phi[\overline{x}]$ for all formulas $\phi(\overline{x})$ of degree at most ℓ. This completes the proof of the implication from (i) to (ii).

For (ii) $\Longrightarrow$ (i), first consider for any model $\mathfrak{O}$, any $i, k \in \omega$, and any $(\overline{o}, o) \in O^k \times O$ from $\mathfrak{O}$, the bounded hybrid type of $\mathfrak{O}, \overline{o}, o$ as the set

$$\mathsf{htp}^{\leq i}_{\mathfrak{O}}(\overline{o}, o) = \{\phi(\overline{x}) : \mathsf{dg}(\phi) \leq i,\ \mathfrak{O}, \overline{o}, o \Vdash \phi[\overline{x}]\}$$

of hybrid formulas of degree at most i that are satisfied in $\mathfrak{O}$ at $(\overline{o}, o)$. Now define a family of relations $(Z^k_i)_{\substack{i \leq \ell \\ k \in \omega}}$ with

$$(\overline{m}, m)\ Z^k_i\ (\overline{n}, n) \text{ iff } \mathsf{htp}^{\leq \ell - i}_{\mathfrak{M}}(\overline{m}, m) = \mathsf{htp}^{\leq \ell - i}_{\mathfrak{N}}(\overline{n}, n)$$

for all $i, k \in \omega$ and all $(\overline{m}, m) \in M^k \times M$ and $(\overline{n}, n) \in N^k \times N$.

Immediately, by definition, we have that $(\overline{a}, a)\ Z^j_0\ (\overline{b}, b)$. We show that $(Z^k_i)_{\substack{i \leq \ell \\ k \in \omega}}$ is a $\mathcal{F}$-(ω, ℓ)-bisimulation.

Condition (prop): Assume that $(\overline{m}, m)\ Z^k_i\ (\overline{n}, n)$ and let $p \in \textsc{prop}$.

$$\begin{aligned} m \in V^{\mathfrak{M}}(p) & \iff \mathfrak{M}, \overline{m}, m \Vdash p \qquad (\text{since } \mathsf{htp}^{\leq \ell - i}_{\mathfrak{M}}(\overline{m}, m) = \mathsf{htp}^{\leq \ell - i}_{\mathfrak{N}}(\overline{n}, n)) \\ & \iff \mathfrak{N}, \overline{n}, n \Vdash p \\ & \iff n \in V^{\mathfrak{N}}(p) \end{aligned}$$

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Condition (ω-forth): Assume that $(\overline{m}, m)\ Z^k_{i-1}\ (\overline{n}, n)$ and let $m' \in R^{\mathfrak{M}}(m)$. As the set $\mathsf{htp}^{\leq \ell - i}_{\mathfrak{M}}(\overline{m}, m')$ contains only finitely many formulas up to logical equivalence (by the standard translation into first-order logic and the fact that for a bounded quantifier rank, and finite signature, there are only finitely many formulas of at most that rank up to logical equivalence [23, Lemma 2.4.8]), we can write a formula logically equivalent to $\bigwedge \mathsf{htp}^{\leq \ell - i}_{\mathfrak{M}}(\overline{m}, m')$. We have that:

$$\begin{aligned} \Diamond \bigwedge \mathsf{htp}^{\leq \ell - i}_{\mathfrak{M}}(\overline{m}, m') \in \mathsf{htp}^{\leq \ell - (i-1)}_{\mathfrak{M}}(\overline{m}, m) = \mathsf{htp}^{\leq \ell - (i-1)}_{\mathfrak{N}}(\overline{n}, n) \Longrightarrow \\ \mathfrak{N}, \overline{n}, n' \Vdash \bigwedge \mathsf{htp}^{\leq \ell - i}_{\mathfrak{M}}(\overline{m}, m') \text{ for some } n' \in R^{\mathfrak{N}}(n) \Longrightarrow \\ \mathsf{htp}^{\leq \ell - i}_{\mathfrak{M}}(\overline{m}, m') \subseteq \mathsf{htp}^{\leq \ell - i}_{\mathfrak{N}}(\overline{n}, n'). \end{aligned}$$

Suppose towards a contradiction that there is a $\phi(\overline{x}) \in \mathsf{htp}^{\leq \ell - i}_{\mathfrak{N}}(\overline{n}, n') \setminus \mathsf{htp}^{\leq \ell - i}_{\mathfrak{M}}(\overline{m}, m')$:

we have $\mathfrak{N}, \overline{n}, n' \Vdash \phi[\overline{x}]$ and $\mathfrak{M}, \overline{m}, m' \nVdash \phi[\overline{x}]$, that is, $\neg\phi(\overline{x}) \in \mathsf{htp}^{\leq \ell - i}_{\mathfrak{M}}(\overline{m}, m')$; since $\mathfrak{N}, \overline{n}, n' \Vdash \mathsf{htp}^{\leq \ell - i}_{\mathfrak{M}}(\overline{m}, m')$, we get $\mathfrak{N}, \overline{n}, n' \Vdash \neg\phi[\overline{x}]$, which is a contradiction.

Hence, $\mathsf{htp}^{\leq \ell - i}_{\mathfrak{M}}(\overline{m}, m') = \mathsf{htp}^{\leq \ell - i}_{\mathfrak{N}}(\overline{n}, n')$, which means $(\overline{m}, m')\ Z^k_i\ (\overline{n}, n')$.

Condition (ω-back): Analogous to (ω-forth).

Condition (nom): Assume that $(\overline{m}, m)\ Z^k_i\ (\overline{n}, n)$ and let $s \in \text{NOM}$.

$$\begin{aligned} s^{\mathfrak{M}} = m &\iff \mathfrak{M}, \overline{m}, m \Vdash s \qquad (\text{since } \mathsf{htp}^{\leq \ell - i}_{\mathfrak{M}}(\overline{m}, m) = \mathsf{htp}^{\leq \ell - i}_{\mathfrak{N}}(\overline{n}, n)) \\ &\iff \mathfrak{N}, \overline{n}, n \Vdash s \\ &\iff s^{\mathfrak{N}} = n \end{aligned}$$

Condition (wvar): Assume that $(\overline{m}, m)\ Z^k_i\ (\overline{n}, n)$ and let $\overline{x} = x_1, \ldots, x_k$. For any $1 \leq e \leq k$ we have

$$\overline{m}(e) = m \iff \mathfrak{M}, \overline{m}, m \Vdash x_e[\overline{x}] \iff \mathfrak{N}, \overline{n}, n \Vdash x_e[\overline{x}] \iff \overline{n}(e) = n$$

Condition (atv): Again, suppose $(\overline{m}, m)\ Z^k_i\ (\overline{n}, n)$ and $\overline{x} = x_1, \ldots, x_k$; let $1 \leq e \leq k$.

$$\begin{aligned} \phi(\overline{x}) \in \mathsf{htp}^{\leq \ell - i}_{\mathfrak{M}}(\overline{m}, \overline{m}(e)) &\iff \mathfrak{M}, \overline{m}, \overline{m}(e) \Vdash \phi[\overline{x}] \\ &\iff \mathfrak{M}, \overline{m}, m \Vdash @_{x_e} \phi[\overline{x}] \iff @_{x_e} \phi[\overline{x}] \in \mathsf{htp}^{\leq \ell - i}_{\mathfrak{M}}(\overline{m}, m) \\ &\iff @_{x_e} \phi[\overline{x}] \in \mathsf{htp}^{\leq \ell - i}_{\mathfrak{N}}(\overline{n}, n) \iff \phi(\overline{x}) \in \mathsf{htp}^{\leq \ell - i}_{\mathfrak{N}}(\overline{n}, \overline{n}(e)) \end{aligned}$$

showing that $\mathsf{htp}^{\leq \ell - i}_{\mathfrak{M}}(\overline{m}, \overline{m}(e)) = \mathsf{htp}^{\leq \ell - i}_{\mathfrak{N}}(\overline{n}, \overline{n}(e))$ and thus $(\overline{m}, \overline{m}(e))\ Z^k_i\ (\overline{n}, \overline{n}(e))$.

Condition (atn): Analogous to (atv).

Condition (st): Assume that $(\overline{m}, m)\ Z^k_{i-1}\ (\overline{n}, n)$, that is, $\mathsf{htp}^{\leq \ell - (i-1)}_{\mathfrak{M}}(\overline{m}, m) = \mathsf{htp}^{\leq \ell - (i-1)}_{\mathfrak{N}}(\overline{n}, n)$. We show $(\overline{m} \frown m, m)\ Z^{k+1}_i\ (\overline{n} \frown n, n)$:

$$\begin{aligned} \phi(\overline{x} \frown x) \in \mathsf{htp}^{\leq \ell - i}_{\mathfrak{M}}(\overline{m} \frown m, m) &\iff \mathfrak{M}, \overline{m} \frown m, m \Vdash \phi[\overline{x} \frown x] \\ &\iff \mathfrak{M}, \overline{m}, m \Vdash ({\downarrow} x \cdot \phi)[\overline{x}] \iff ({\downarrow} x \cdot \phi)(\overline{x}) \in \mathsf{htp}^{\leq \ell - (i-1)}_{\mathfrak{M}}(\overline{m}, m) \\ &\iff ({\downarrow} x \cdot \phi)(\overline{x}) \in \mathsf{htp}^{\leq \ell - (i-1)}_{\mathfrak{N}}(\overline{n}, n) \iff \mathfrak{N}, \overline{n}, n \Vdash ({\downarrow} x \cdot \phi)[\overline{x}] \\ &\iff \mathfrak{N}, \overline{n} \frown n, n \Vdash \phi[\overline{x} \frown x] \iff \phi(\overline{x} \frown x) \in \mathsf{htp}^{\leq \ell - i}_{\mathfrak{N}}(\overline{n} \frown n, n) \end{aligned}$$

Condition (ex): Analogous to (st). ⊣

The notion of (ω, ℓ)-bisimulation focuses on formulas of degree at most ℓ. In order to obtain an ω-bisimulation for countable models in terms of Areces *et al.* [2], however, it suffices to take the union of these approximating relations.

Lemma 2 ($\mathcal{F}$-ω-bisimulation). *Let Σ be a hybrid signature and let $\mathcal{F}$ be a set of hybrid language features. Let $\mathfrak{M}$ and $\mathfrak{N}$ be models over Σ. Let $(Z_i^k)_{\substack{i\in\omega \\ k\in\omega}}$ be a system of relations from $\mathfrak{M}$ to $\mathfrak{N}$ such that $(Z_i^k)_{\substack{i\leq\ell \\ k\in\omega}}$ is an $\mathcal{F}$-(ω, ℓ)-bisimulation for all $\ell \in \omega$. Then $\mathfrak{M}$ and $\mathfrak{N}$ are $\mathcal{F}$-ω-bisimilar.*

Proof. For any $k \in \omega$, we define $B_k = \bigcup_{\ell\in\omega} Z_\ell^k$. We show that $(B_k)_{k\in\omega}$ is an $\mathcal{F}$-ω-bisimulation, that is, all conditions from Def. 2 are satisfied. Assume that $(\overline{m}, m)\ B_k\ (\overline{n}, n)$. By definition, $(\overline{m}, m)\ Z_\ell^k\ (\overline{n}, n)$ for some $\ell \in \omega$.

Condition (prop): By (prop) for Z_ℓ^k, we have $m \in V^{\mathfrak{M}}(p)$ iff $n \in V^{\mathfrak{N}}(p)$ for all propositional symbols $p \in$ PROP.

Condition (nom): By (nom) for Z_ℓ^k, we have $m = s^{\mathfrak{M}}$ iff $n = s^{\mathfrak{M}'}$ for all nominals $s \in$ NOM.

Condition (wvar): By (wvar) for Z_ℓ^k, we have $m = \overline{m}(j)$ iff $n = \overline{n}(j)$ for all $1 \leq j \leq k$.

Condition (forth): Let $m' \in R^{\mathfrak{M}}(m)$. By ($\omega$-forth) for Z_ℓ^k, we have that $(\overline{m}, m')\ Z_{\ell+1}^k\ (\overline{n}, n')$ for some $n' \in R^{\mathfrak{N}}(n)$. Hence, $(\overline{m}, m')\ B_k\ (\overline{n}, n')$ for some $n' \in R^{\mathfrak{N}}(n)$.

Condition (back): Symmetric to (forth).

Condition (atv): By (atv) for Z_ℓ^k, we have $(\overline{m}, \overline{m}(j))\ Z_\ell^k\ (\overline{n}, \overline{n}(j))$ for all $1 \leq j \leq k$. Hence, $(\overline{m}, \overline{m}(j))\ B_k\ (\overline{n}, \overline{n}(j))$ for all $1 \leq j \leq k$.

Condition (atn): By (atn) for Z_ℓ^k, we have $(\overline{m}, s^{\mathfrak{M}})\ Z_\ell^k\ (\overline{n}, s^{\mathfrak{N}})$ for all $s \in$ NOM. Hence, $(\overline{m}, s^{\mathfrak{M}})\ B_k\ (\overline{n}, s^{\mathfrak{N}})$ for all $s \in$ NOM.

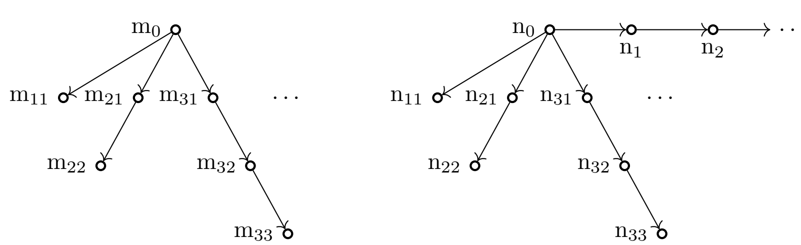
Condition (st): By (st) for Z_ℓ^k, we have $(\overline{m} \frown m, m)\ Z_{\ell+1}^{k+1}\ (\overline{n} \frown n, n)$. Hence, $(\overline{m} \frown m, m)\ B_{k+1}\ (\overline{n} \frown n, n)$.

Condition (ex-f): Let $m' \in M$ be a possible world. By (ex-f) for Z_ℓ^k, we have that $(\overline{m} \frown m', m)\ Z_{\ell+1}^{k+1}\ (\overline{n} \frown n', n)$ for some $n' \in N$. It follows that $(\overline{m} \frown m', m)\ B_{k+1}\ (\overline{n} \frown n', n)$.

Condition (ex-b): Analogous to (ex-f). ⊣

By Thm. 2 and Lem. 2, an $\mathcal{F}$-ω-bisimulation is at least as strong as $\mathcal{F}$-elementary equivalence. The following example shows that an $\mathcal{F}$-ω-bisimulation is strictly stronger than $\mathcal{F}$-elementary equivalence.

Example 5 ([1, Ex. 5.13]). Let Σ be the empty signature. Let $\mathfrak{M}$ be the model depicted to the left of the following diagram, which is a countably infinitely branched tree with the root m_0. Let $\mathfrak{N}$ be the model depicted to the right of the following diagram, which is obtained from $\mathfrak{M}$ by adding a new branch of countably infinite length.



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In any quantifier-free fragment of hybrid proposition logic, $(\mathfrak{M}, \mathrm{m}_0)$ and $(\mathfrak{N}, \mathrm{n}_0)$ from Ex. 5 are elementarily equivalent [5]. However, there is no bisimulation between $(\mathfrak{M}, \mathrm{m}_0)$ and $(\mathfrak{N}, \mathrm{n}_0)$, even in the language of propositional modal logic [13, Example 37]. In fact, by Thm. 2, there is a $\mathcal{F}$-(k, ℓ)-bisimulation $(Z_i^{k,\ell})_{\substack{i \leq \ell \\ k \in \omega}}$ for each $\ell \in \omega$, but this system does not satisfy $Z_i^{k,\ell} = Z_i^{k,\ell'}$ for all $i \in \omega$ and $\ell, \ell' \geq i$ that is required by Lem. 2.

It will now be useful to define an analogue of the usual notion of a type but only for formulas that happen to be equivalent to standard translations.

Definition 6 (Standard translation type & elementary equivalence). Consider a hybrid language defined over a signature Σ with features from $\mathcal{F}$. Let $(\mathfrak{M}, m)$ be a pointed model over Σ. Then $\mathsf{http}_{\mathfrak{M}}(m)$ denotes the set of translations of hybrid sentences satisfied by $(\mathfrak{M}, m)$, that is:

$$\mathsf{http}_{\mathfrak{M}}(m) = \{ST_{\boldsymbol{x}}(\phi) : \mathfrak{M}, m \Vdash \phi\} .$$

Two pointed Σ-models $(\mathfrak{M}, m)$ and $(\mathfrak{N}, n)$ are *$\mathcal{F}$-elementarily equivalent*, in symbols, $(\mathfrak{M}, m) \equiv_{\mathcal{F}} (\mathfrak{N}, n)$ if they cannot be distinguished by any hybrid sentence, i.e.,

$$\mathfrak{M}, m \Vdash \phi \text{ iff } \mathfrak{N}, n \Vdash \phi \text{ for all hybrid sentences } \phi \text{ over } \mathcal{F} .$$

The following result shows that first-order sentences cannot distinguish between $\mathcal{F}$-ω-bisimulation and $\mathcal{F}$-elementary equivalence.

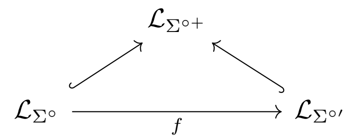
Theorem 3. *Consider a hybrid language defined over a finite signature Σ with features from $\mathcal{F}$. If a first-order formula $\phi(x)$ over Σ° can distinguish between two $\mathcal{F}$-elementary equivalent pointed models $(\mathfrak{M}_1, m_1)$ and $(\mathfrak{M}_2, m_2)$ over Σ, that is, $\mathfrak{M}_1 \vDash \phi[m_1]$ and $\mathfrak{M}_2 \vDash \neg\phi[m_2]$ and $(\mathfrak{M}_1, m_1) \equiv_{\mathcal{F}} (\mathfrak{M}_2, m_2)$, then $\phi(x)$ can distinguish $\mathcal{F}$-ω-bisimilar pointed models, that is, there exist pointed models $(\mathfrak{N}_1, n_1)$ and $(\mathfrak{N}_2, n_2)$ such that (a) $\mathfrak{N}_1 \vDash \phi[n_1]$ and $\mathfrak{N}_2 \vDash \neg\phi[n_2]$, and (b) $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are $\mathcal{F}$-ω-bisimilar.*

Proof. Let $(\mathfrak{M}_1, m_1)$ and $(\mathfrak{M}_2, m_2)$ be two $\mathcal{F}$-elementary equivalent pointed models. Let $\phi(x)$ be a first-order formula such that $\mathfrak{M}_1 \vDash \phi[m_1]$ and $\mathfrak{M}_2 \vDash \neg\phi[m_2]$. Since $\Sigma = (\textsc{prop}, \textsc{nom})$ is finite, Thm. 2 applies. Thus, for each $\ell < \omega$, there is an (ω, ℓ)-bisimulation $(Z_i^k)_{\substack{i \leq \ell \\ k \in \omega}}$ from $(\mathfrak{M}_1, m_1)$ to $(\mathfrak{M}_2, m_2)$.

In the following, we define, in first-order logic, an object-level characterisation of $\mathcal{F}$-(ω, ℓ)-bisimulations. Expand the signature $(R, \textsc{prop}^\circ, \textsc{nom})$ by adding:

- two unary predicates U_i $(i \in \{1, 2\})$,
- countably many predicates I_ℓ^k $(\ell, k \in \omega)$ each of arity $(k+1) \times (k+1)$,
- copies s' of each nominal $s \in \textsc{nom}$, and
- copies P' of each predicate $P \in \textsc{prop}^\circ$.

Call the expanded signature $\Sigma^{\circ+}$. Further, let $\Sigma^{\circ\prime}$ be the first-order signature obtained from Σ° by replacing each nominal $s \in \textsc{nom}$ with its copy s' and each predicate $P \in \textsc{prop}^\circ$ with its copy P'. We obtain a bijective mapping $f : \mathcal{L}_{\Sigma^\circ} \to \mathcal{L}_{\Sigma^{\circ\prime}}$, and then we define $\phi' = f(\phi)$.



To make the notation more transparent, we will write $I_\ell^k\ (\overline{x} \frown x, \overline{y} \frown y)$ in the form $I_\ell^k\binom{\overline{x}, x}{\overline{y}, y}$, where $\overline{x}$ and $\overline{y}$ are sequences of variables of length k. For each I_ℓ^k we let $J_\ell^k\binom{\overline{x}, x}{\overline{y}, y}$ stand for the formula

$$\bigwedge_{j=1}^{k} U_1(\overline{x}(j)) \wedge U_1(x) \wedge \bigwedge_{j=1}^{k} U_2(\overline{y}(j)) \wedge U_2(y) \wedge I_\ell^k\binom{\overline{x}, x}{\overline{y}, y}.$$

We define the following sentences over $\Sigma^{\circ +}$:

(init) $\exists x, y \cdot J_0^0\binom{x}{y} \wedge \phi[x] \wedge \neg\phi'[y]$

(Ψ_{prop}) $\forall \overline{x}, x, \overline{y}, y \cdot J_\ell^k\binom{\overline{x}, x}{\overline{y}, y} \rightarrow \bigwedge_{P \in \text{PROP}^\circ} (P(x) \leftrightarrow P'(y))$

(Ψ_{forth}) $\forall \overline{x}, x, \overline{y}, y, z_1 \cdot J_\ell^k\binom{\overline{x}, x}{\overline{y}, y} \wedge U_1(z_1) \wedge x\ R\ z_1 \rightarrow \exists z_2 \cdot J_{\ell+1}^k\binom{\overline{x}, z_1}{\overline{y}, z_2} \wedge y\ R\ z_2$

(Ψ_{back}) $\forall \overline{x}, x, \overline{y}, y, z_2 \cdot J_\ell^k\binom{\overline{x}, x}{\overline{y}, y} \wedge U_2(z_2) \wedge y\ R\ z_2 \rightarrow \exists z_1 \cdot J_{\ell+1}^k\binom{\overline{x}, z_1}{\overline{y}, z_2} \wedge x\ R\ z_1$

(Ψ_{wvar}) $\forall \overline{x}, x, \overline{y}, y \cdot J_\ell^k\binom{\overline{x}, x}{\overline{y}, y} \rightarrow \bigwedge_{j=1}^{k} (\overline{x}(j) = x \leftrightarrow \overline{y}(j) = y)$

(Ψ_{nom}) $\forall \overline{x}, x, \overline{y}, y \cdot J_\ell^k\binom{\overline{x}, x}{\overline{y}, y} \rightarrow \bigwedge_{s \in \text{NOM}} (x = s \leftrightarrow y = s')$

(Ψ_{atv}) $\forall \overline{x}, x, \overline{y}, y \cdot J_\ell^k\binom{\overline{x}, x}{\overline{y}, y} \rightarrow \bigwedge_{j=1}^{k} J_\ell^k\binom{\overline{x}, \overline{x}(j)}{\overline{y}, \overline{y}(j)}$

(Ψ_{atn}) $\forall \overline{x}, x, \overline{y}, y \cdot J_\ell^k\binom{\overline{x}, x}{\overline{y}, y} \rightarrow \bigwedge_{s \in \text{NOM}} J_\ell^k\binom{\overline{x}, s}{\overline{y}, s'}$

(Ψ_{st}) $\forall \overline{x}, x, \overline{y}, y \cdot J_{\ell-1}^k\binom{\overline{x}, x}{\overline{x}, x} \rightarrow J_\ell^{k+1}\binom{\overline{x} \frown x, x}{\overline{y} \frown y, y}$

$(\Psi_{\text{ex-f}})$ $\forall \overline{x}, x, \overline{y}, y, z_1 \cdot J_{\ell-1}^k\binom{\overline{x}, x}{\overline{y}, y} \wedge U_1(z_1) \rightarrow \exists z_2 \cdot J_\ell^{k+1}\binom{\overline{x} \frown z_1, x}{\overline{y} \frown z_2, y}$

$(\Psi_{\text{ex-b}})$ $\forall \overline{x}, x, \overline{y}, y, z_2 \cdot J_\ell^k\binom{\overline{x}, x}{\overline{y}, y} \wedge U_2(z_2) \rightarrow \exists z_1 \cdot J_\ell^{k+1}\binom{\overline{x} \frown z_1, x}{\overline{y} \frown z_2, y}$

Notice that each formula Ψ_S defined above is indexed by a condition S from Def. 3 and Def. 4 and is a restatement of S in first-order logic. Let $\Phi = \text{(init)} \cup (\bigcup_{S \in \mathsf{cond}(\mathcal{F})} \Psi_S)$. We claim that every finite subset of Φ is consistent. To see this let Φ_0 be one such finite subset and let $k, \ell \in \omega$ be such that J_i^j occurs in Φ_0 only if $j \leq k$ and $i \leq \ell$. Recall that there is an (ω, ℓ)-bisimulation $(Z_i^k)_{\substack{i \leq \ell \\ k \in \omega}}$ between $(\mathfrak{M}_1, m_1)$ and $(\mathfrak{M}_2, m_2)$. We can

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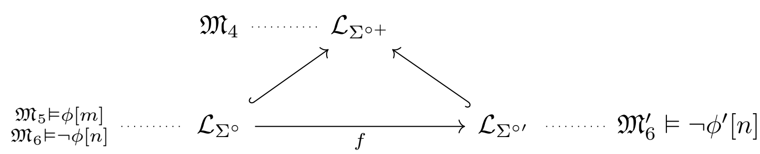
suppose without loss of generality that $M_1 \cap M_2 = \emptyset$ (if this is not the case, take disjoint isomorphic copies of M_1 and M_2). We construct $\mathfrak{M}_3$ by putting:

- $M_3 = M_1 \cup M_2$
- $R^{\mathfrak{M}_3} = R^{\mathfrak{M}_1} \cup R^{\mathfrak{M}_2}$
- $U_i^{\mathfrak{M}_3} = M_i$ for all $i \in \{1,2\}$
- $P^{\mathfrak{M}_3} = P^{\mathfrak{M}_1}$ and ${P'}^{\mathfrak{M}_3} = P^{\mathfrak{M}_2}$ for all predicates $P \in$ PROP$^\circ$
- $s^{\mathfrak{M}_3} = s^{\mathfrak{M}_1}$ and ${s'}^{\mathfrak{M}_3} = s^{\mathfrak{M}_2}$ for all nominals $s \in$ NOM
- $(I_i^j)^{\mathfrak{M}_3} = Z_i^j$ for all $j \le k$ and $i \le \ell$

This construction produces a model, since we have no non-constant functions in the signature. It follows that $\mathfrak{M}_3 \vDash \Phi_0$, as each formula ψ_S, where $S \in \mathsf{cond}(\mathcal{F})$, used to define Φ are simply restatements in first-order logic of conditions appearing in Def. 3 and Def. 4.

Since every finite subset of Φ is consistent, by compactness we get that Φ has a model, say, $\mathfrak{M}_4$. Thus, there are $m, n \in M_4$ such that $\mathfrak{M}_4 \vDash \phi[m]$, and $\mathfrak{M}_4 \vDash \neg\phi'[n]$.



Taking the reduct of $\mathfrak{M}_4$ to the signature Σ° we obtain a model $\mathfrak{M}_5 \vDash \phi[m]$. Now, taking the reduct of $\mathfrak{M}_4$ to the signature $\Sigma^{\circ\prime}$ we obtain a model $\mathfrak{M}_6' \vDash \neg\phi'[n]$. Then reducing $\mathfrak{M}_6'$ across the bijection f, we obtain a model $\mathfrak{M}_6 \vDash \neg\phi[n]$. The system of relations $(I_i^k)^{\mathfrak{M}_4}_{\substack{i \le \ell \\ k \in \omega}}$ is an $\mathcal{F}$-(ω,ℓ)-bisimulation for all $\ell \in \omega$. By Lem. 2, the structures $\mathfrak{M}_5$ and $\mathfrak{M}_6$ are $\mathcal{F}$-ω-bisimilar. ⊣

THEOREM 4. *Consider a hybrid language defined over a finite signature Σ with features from $\mathcal{F}$. A first-order formula $\phi(x)$ is equivalent to the standard translation of a hybrid sentence if, and only if, it is invariant under $\mathcal{F}$-ω-bisimulations of pointed models.*

PROOF. The forward direction follows immediately by Thm. 2 and property (SE). For the backward direction, first, we show that

(i) $\quad \mathrm{Mod}_*(\phi) = \bigcup_{(\mathfrak{M},m)\in \mathrm{Mod}_*(\phi)} \mathrm{Mod}_*(\mathsf{http}_{\mathfrak{M}}(m))$.

Obviously, $\mathrm{Mod}_*(\phi) \subseteq \bigcup_{(\mathfrak{M},m)\in \mathrm{Mod}_*(\phi)} \mathrm{Mod}_*(\mathsf{http}_{\mathfrak{M}}(m))$. Assume that $(\mathfrak{M}_2, m_2) \in \bigcup_{(\mathfrak{M},m)\in \mathrm{Mod}_*(\phi)} \mathrm{Mod}_*(\mathsf{http}_{\mathfrak{M}}(m))$. It follows that $\mathfrak{M}_2 \vDash \mathsf{http}_{\mathfrak{M}_1}(m_1)[m_2]$ for some pointed model $(\mathfrak{M}_1, m_1)$ such that $\mathfrak{M}_1 \vDash \phi[m_1]$. Note that for any hybrid sentence ψ, we have $\mathfrak{M}_1, m_1 \Vdash \psi$ if and only if $\mathfrak{M}_2, m_2 \vDash \psi$. Therefore, $(\mathfrak{M}_1, m_1)$ and $(\mathfrak{M}_2, m_2)$ are $\mathcal{F}$-elementary equivalent in their hybrid language, in symbols, $(\mathfrak{M}_1, m_1) \equiv_{\mathcal{F}} (\mathfrak{M}_2, m_2)$. Assume for reductio that $\mathfrak{M}_2 \nvDash \phi[m_2]$, so $\mathfrak{M}_2 \vDash \neg\phi[m_2]$. By Thm. 3, there exist $(\mathfrak{N}_1, n_1)$ and $(\mathfrak{N}_2, n_2)$ $\mathcal{F}$-ω-bisimilar such that $\mathfrak{N}_1 \vDash \phi[n_1]$ and $\mathfrak{N}_2 \vDash \neg\phi[n_2]$, which is a contradiction with the invariance of ϕ under $\mathcal{F}$-ω-bisimulation. Hence, $\mathrm{Mod}_*(\phi) \supseteq \bigcup_{(\mathfrak{M},m)\in \mathrm{Mod}_*(\phi)} \mathrm{Mod}_*(\mathsf{http}_{\mathfrak{M}}(m))$.

Secondly, take an arbitrary $(\mathfrak{M}, m) \in \mathrm{Mod}_*(\phi)$. Note that (i) implies that every model of $\mathsf{http}_{\mathfrak{M}}(m)$ must be a model of ϕ. But then $\mathsf{http}_{\mathfrak{M}}(m) \cup \{\neg\phi\}$ is unsatisfiable. By compactness of first-order logic, $\Psi_{(\mathfrak{M},m)} \cup \{\neg\phi\}$ is unsatisfiable for some finite $\Psi_{(\mathfrak{M},m)} \subseteq \mathsf{http}_{\mathfrak{M}}(m)$ (and we can pick a unique one for each $(\mathfrak{M}, m)$, if necessary using the axiom of choice). Hence, $\bigwedge \Psi_{(\mathfrak{M},m)}$ logically implies ϕ. Then using (i) we obtain,

(ii) $\quad \mathrm{Mod}_*(\phi) = \bigcup_{(\mathfrak{M},m)\in\mathrm{Mod}_*(\phi)} \mathrm{Mod}_*(\bigwedge \Psi_{(\mathfrak{M},m)})$.

However, this means that the set $\{\phi\} \cup \big\{\neg \bigwedge \Psi_{(\mathfrak{M},m)} : (\mathfrak{M}, m) \in \mathrm{Mod}_*(\phi)\big\}$ is unsatisfiable, and by compactness again, for some finite $\Theta \subseteq \{\neg \bigwedge \Psi_{(\mathfrak{M},m)} : (\mathfrak{M}, m) \in \mathrm{Mod}_*(\phi)\}$, the set $\{\phi\} \cup \Theta$ is unsatisfiable. Consequently, $\mathrm{Mod}_*(\phi) \subseteq \mathrm{Mod}_*(\neg \bigwedge \Theta)$. By its definition Θ is logically equivalent to

$$\neg \bigwedge \Psi_{(\mathfrak{M}_1,m_1)} \wedge \neg \bigwedge \Psi_{(\mathfrak{M}_2,m_2)} \wedge \cdots \wedge \neg \bigwedge \Psi_{(\mathfrak{M}_k,m_k)}$$

for some k, and therefore we have that $\neg \bigwedge \Theta$ is equivalent to

$$\bigwedge \Psi_{(\mathfrak{M}_1,m_1)} \vee \bigwedge \Psi_{(\mathfrak{M}_2,m_2)} \vee \cdots \vee \bigwedge \Psi_{(\mathfrak{M}_k,m_k)} .$$

Hence

$$\mathrm{Mod}_*(\neg \bigwedge \Theta) \subseteq \bigcup_{(\mathfrak{M},m)\in\mathrm{Mod}_*(\phi)} \mathrm{Mod}_*(\bigwedge \Psi_{(\mathfrak{M},m)})$$

so, using (ii) we get,

(iii) $\quad \mathrm{Mod}_*(\phi) = \mathrm{Mod}_*(\neg \bigwedge \Theta)$.

But now note that $\neg \bigwedge \Theta$ is a translation of a hybrid sentence, since the translations are closed under finite conjunctions and disjunctions. ⊣

The next result follows easily by the methods we have developed.

Theorem 5. *Consider a hybrid language defined over a finite signature Σ with features from $\mathcal{F}$. Let K be a class of pointed models over Σ. Then, K is axiomatisable by a hybrid sentence ϕ over Σ if and only if for some some ℓ, the class K is closed under $\mathcal{F}$-(ω, ℓ)-bisimulations.*

Proof. Let ϕ be a hybrid sentence axiomatising K, and let ℓ be the degree of ϕ. Then the left-to-right direction follows by Thm. 2. For the converse direction, assume K is closed under $\mathcal{F}$-(ω, ℓ)-bisimulations. Since Σ is finite, there are only finitely many formulas of degree at most ℓ up to logical equivalence. Recall that $\mathsf{htp}^{\leq \ell}_{\mathfrak{M}}(m)$ denotes the set of sentences of degree at most ℓ satisfied by $(\mathfrak{M}, m)$. Then for any pointed model $(\mathfrak{M}, m) \in K$, there exists a hybrid sentence $\phi_{(\mathfrak{M},m)}$ semantically equivalent to $\bigwedge \mathsf{htp}^{\leq \ell}_{\mathfrak{M}}(m)$. Further, there exists a hybrid sentence ϕ semantically equivalent to $\bigvee_{(\mathfrak{M},m)\in K} \bigwedge \phi_{(\mathfrak{M},m)}$. Then for any pointed model $(\mathfrak{M}, m)$ such that $\mathfrak{M}, m \Vdash \phi$, we have that $\mathfrak{M}, m \Vdash \phi_{(\mathfrak{N},n)}$ for some $(\mathfrak{N}, n) \in K$. It follows that $(\mathfrak{M}, m)$ and $(\mathfrak{N}, n)$ satisfy the same hybrid sentences of degree at most ℓ. By Thm. 2 again, $(\mathfrak{M}, m)$ are $(\mathfrak{N}, n)$ are $\mathcal{F}$-(ω, ℓ)-bisimilar. Since K is closed under $\mathcal{F}$-(ω, ℓ)-bisimulations, $(\mathfrak{M}, m) \in K$. ⊣

§6. Undecidability of invariance. In van Benthem [29] it is shown that invariance under standard bisimulations is undecidable for first-order formulas. Hodkinson and Tahiri [20], using the same proof, lift van Benthem's result to quasi-injective bisimulations and versions thereof. The very same proof shows undecidability of invariance under our version of bisimulations, but to make it applicable to arbitrary signatures (in particular, to signatures with nominals) we need a version of disjoint union of models applicable to such cases.

DEFINITION 7. Let $\mathfrak{A}$ and $\mathfrak{B}$ be models of some first-order signature Σ. We define $\mathfrak{A} \uplus \mathfrak{B}$ to be the model whose universe is $A \uplus B$, the interpretation of a relation R is $R^{\mathfrak{A}} \uplus R^{\mathfrak{B}}$, the interpretation of an l-ary function $f(x_1, \dots, x_l)$ is defined by

$$f^{\mathfrak{A}\uplus\mathfrak{B}}(u_1, \dots, u_l) = \begin{cases} f^{\mathfrak{A}}(u_1, \dots, u_l), & \text{if } (u_1, \dots, u_l) \in A^l \\ f^{\mathfrak{B}}(u_1, \dots, u_l), & \text{if } (u_1, \dots, u_l) \in B^l \\ u_i \text{ for the least } i \text{ with } u_i \in A, & \text{otherwise} \end{cases}$$

and finally the interpretation of a constant c is $c^{\mathfrak{A}}$.

LEMMA 3. *Consider a hybrid language over a signature Σ with features from $\mathcal{F}$, for an arbitrary $\mathcal{F}$. It is undecidable whether a formula in the first-order translation Σ° of Σ is invariant under $\mathcal{F}$-$(k, 0)$-bisimulations, for any k.*

PROOF. Let $\mathfrak{A} = (A, R^{\mathfrak{A}}, \{a_i^{\mathfrak{A}} : i \in I\}, V^{\mathfrak{A}})$ and $\mathfrak{B} = (B, R^{\mathfrak{B}}, \{b_i^{\mathfrak{B}} : i \in I\}, V^{\mathfrak{B}})$ be Σ-models such that $A = \{a\}$, $R^{\mathfrak{A}} = \emptyset$, $a_i^{\mathfrak{A}} = a$ for all $i \in I$, and $B = \{0, 1\}$, $R^{\mathfrak{B}} = \{(0, 1)\}$, $b_i^{\mathfrak{B}} = 1$ for all $i \in I$, and $V^{\mathfrak{A}}$, $V^{\mathfrak{B}}$ are vacuous valuations (with no propositional variables). Clearly, $Z_0^k = \{(a, 1)\}$ is a $\mathcal{F}$-$(k, 0)$-bisimulation from $\mathfrak{A}$ to $\mathfrak{B}$. Letting $\psi(x)$ be the formula $\exists y \cdot y\ R\ x$ we see that $\mathfrak{B} \vDash \psi[1]$ but $\mathfrak{A} \nvDash \psi[a]$.

Note that Σ° has unary predicates corresponding to propositional variables. Let P be one of them, and consider the map f from Σ°-sentences to Σ°-formulas, given by

$$f(\sigma) = \psi(x) \vee (\exists x \cdot P(x) \to \sigma^P)$$

where σ^P is the relativisation of σ to P. Clearly, $f(\sigma) \in \Sigma^\circ$. We will show that σ is valid if and only if $f(\sigma)$ is invariant under Z_0^k, thereby showing that invariance under Z_0^k is undecidable.

For the forward direction, if σ is valid, then so is $f(\sigma)$; hence $f(\sigma)$ is true in every model under any valuation, in particular it is invariant under Z_0^k.

For the backward direction, assume σ is not valid. Let $\mathfrak{C}$ be a model with $\mathfrak{C} \vDash \neg\sigma$. Let $\mathfrak{N}$ be $\mathfrak{A} \uplus \mathfrak{C}$, let $\mathfrak{M}$ be $\mathfrak{B} \uplus \mathfrak{C}$, and let $P^{\mathfrak{N}} = P^{\mathfrak{M}} = C$. Then, $f(\sigma)^{\mathfrak{M}}(1) = \psi^{\mathfrak{M}}(1) \vee (\exists x \cdot P(x) \to \sigma^P)^{\mathfrak{M}}$ and thus $\mathfrak{M} \vDash \psi[1]$. On the other hand, $f(\sigma)^{\mathfrak{N}}(a) = \psi^{\mathfrak{N}}(a) \vee (\exists x \cdot P(x) \to \sigma^P)^{\mathfrak{N}}$, and thus $\mathfrak{N} \nvDash \exists x \cdot P(x) \to \sigma^P$, since otherwise we would have $\mathfrak{C} \vDash \sigma$. But $\mathfrak{N}$ and $\mathfrak{M}$ are Z_0^k-related, showing that $f(\sigma)$ is not preserved under Z_0^k. ⊣

COROLLARY 6. *For any $\mathcal{F}$, and for any k and ℓ, invariance under $\mathcal{F}$-(k, ℓ)-bisimulations, as well as invariance under $\mathcal{F}$-ω-bisimulations is undecidable.*

PROOF. Immediate by the definitions of $\mathcal{F}$-(k, ℓ)-bisimulation and $\mathcal{F}$-ω-bisimulation, and an application of Lem. 3. ⊣

§7. Final remarks. Bisimulations in general, and the ones considered by us in particular, are equivalence relations on classes of models. Any reasonable notion of bisimulation is weaker than isomorphism, of course. Weaker conditions imposed on a bisimulation are easier to satisfy, so more models are bisimilar, hence fewer formulas are invariant under weak bisimulations. Conversely, the more we demand of a bisimulation, the *more* formulas will be invariant. The threshold is reached when we demand enough to get a concept at least as strong as elementary equivalence, since then *all* formulas will be invariant. In our case, for $\mathcal{F} \supseteq \{\exists, @\}$ the notion of $\mathcal{F}$-ω-bisimulation is strictly

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stronger than elementary equivalence; hence, for such an $\mathcal{F}$, invariance under $\mathcal{F}$-ω-bisimulations is trivial. Summarising, we obtain the following update of the table in §1:

Hybrid features	Invariance under
$\emptyset$	Bisimulations = $\emptyset$-ω-bisimulations
$\{\downarrow\}$ w/o nominals	$\{\downarrow\}$-ω-bisimulations
$\{\downarrow\}$ with nominals	$\{\downarrow, \textsc{nom}\}$-$\omega$-bisimulations
$\{@\}$	$\{@\}$-ω-bisimulations
$\{@, \downarrow\}$	$\{@, \downarrow\}$-ω-bisimulations
$\{\exists\}$	$\{\exists\}$-ω-bisimulations
$\{@, \exists\}$	equivalent to FOL
full feature set	equivalent to FOL

We end the article with several questions. The first one has to do with the feature set $\{\exists\}$ (or, equivalently, $\{\downarrow, \exists\}$). As we mentioned a number of times, it is well known that hybrid logic in the full feature set is equivalent to first-order logic, and $@$ is the operator that mimics identity. It hence seems of interest to compare $\{\exists\}$-bisimulations with existing equivalence relations on models for first-order logic without identity, such as the ones considered in Casanovas *et al.* [11].

Question 1. *How do $\{\exists\}$-k-bisimulations and $\{\exists\}$-ω-bisimulations compare to various equivalence relations on classes of models for logic without equality?*

Another interesting direction is to consider bisimulation invariance in the finite, more precisely, whether the bisimulation characterisation theorems presented here (or other similar theorems from the literature for hybrid logic) still hold over finite Kripke structures. Rosen [25] (see also Otto [24]) showed that the usual characterisation for modal logic is preserved in moving to a finite context. In this line of research, Ian Hodkinson [19] has recently obtained some results for modal logic with nominals and $@$ using a suitable notion of bisimulation, so there is some hope that the techniques used there could be lifted and generalized to the current setting.

Question 2. *Which of the preservation-under-bisimulations theorems mentioned here still hold on finite structures?*

The following series of questions were all suggested by an anonymous reviewer. We also believe they are of special interest and thus we have decided to include them here. The first one calls for a subtler analysis of (un)decidability than that of Lemma 3.

Question 3. *Given two sets of hybrid features $\mathcal{F}$ and $\mathcal{F}'$ with $\mathcal{F} \subseteq \mathcal{F}'$ is it decidable whether a given $\mathcal{F}'$-formula is expressible in $\mathcal{F}$?*

One may also conceive of hybrid languages with world-variable binders as a language in which one can define new modal operators, such as the binary *until* (which can be defined in hybrid languages containing $\downarrow$ and $@$, as a formula in two variables). From this perspective one can consider another family of extensions of the basic modal language, namely those given by (say, finitely many) modal operators whose definitions are given by a hybrid formula. It would then be natural to ask for a modular van Benthem-type theorem governing all such extensions.

QUESTION 4. *Does a modular van Benthem theorem hold for all such extensions? If not, for which extensions does it hold?*

In practical applications of modal and hybrid logics, additional frame properties are often assumed. The next question is similar in spirit to the one about finite models, but it concerns modally definable frame conditions. If such a condition is definable in first-order logic, then our main result applies, by considering equivalence modulo the condition, and thus provides a bisimulation characterisation.

QUESTION 5. *Do the results of the present article still hold in presence of natural frame conditions that are not definable in first-order logic?*

In [30] and [31] it is shown that van Benthem-type theorems for modal logics can be viewed as a special case of more general Lindström-type characterisations (and conversely).

QUESTION 6. *Are there modular Lindström-type characterisations of the hybrid logics considered in this paper?*

One final problem that we find particularly interesting is the following:

QUESTION 7. *Is there a modular Goldblatt-Thomason theorem for hybrid logic and its fragments? That is, is there a modular model-theoretic characterisation of the classes of frames that are definable by hybrid formulas among the first-order definable classes of frames?*

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