

NONEXPANSIVE MAPPINGS ON THE UNIT SPHERES OF SOME BANACH SPACES

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Abstract

We characterize surjective nonexpansive mappings between unit spheres of $\mathcal{L}^\infty(\Gamma)$ -type spaces. We show that such mappings turn out to be isometries and can be extended to linear isometries on the whole space $\mathcal{L}^\infty(\Gamma)$.

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1. Introduction

Let (X, d_X) and (Y, d_Y) be two metric spaces. A mapping $V_0 : X \rightarrow Y$ is called nonexpansive if it is a 1-Lipschitz map. That is,

$$d_Y(V_0(x), V_0(y)) \leq d_X(x, y) \quad \forall x, y \in X. \quad (1)$$

The mapping V_0 is called an isometry if the equality holds in (1) for all $x, y \in X$.

Freudenthal and Hurewicz [10] stated that every nonexpansive map from a totally bounded metric space onto itself must be an isometry. Rhodes [13] and Brown and Comfort [3] generalized this result to uniform spaces. We wonder whether Freudenthal and Hurewicz's result holds in complete bounded metric spaces which are not compact, in particular, the unit spheres of infinite-dimensional Banach spaces.

On the other hand, in 1972, Mankiewicz [12] proved that an isometry mapping from an open connected subset of a normed space E onto an open subset of another normed space F can be extended to an affine isometry from E onto F . In 1987, Tingley firstly considered isometries between unit spheres of normed spaces. He showed in [14] that isometries between unit spheres of finite-dimensional Banach spaces map antipodal points to antipodal points and he raised the following isometric extension problem: is every onto isometry between unit spheres of two real normed spaces, necessarily the

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restriction of a linear or affine map on the whole space? In recent years, Ding and his students have been working on this topic and have obtained many important results (see [1, 5–9, 11, 15, 16]). Ding [4] was the first to consider the nonexpansive map between unit spheres of Hilbert spaces. He proved that such a map is an isometry on the unit sphere and can also be extended to a linear isometry on the whole space.

In this paper, we generalize Freudenthal and Hurewicz's result to the unit spheres of $\mathcal{L}^\infty(\Gamma)$ -type spaces and give an easy example to show that there exist nonexpansive maps from the unit balls of Banach spaces onto themselves but not isometries. Moreover, applying this result, we give an affirmative answer to Tingley's isometric extension problem in $\mathcal{L}^\infty(\Gamma)$ -type spaces.

Throughout this paper, we consider the spaces over the real field. The following notation for $\mathcal{L}^\infty(\Gamma)$ -type spaces can be found in [7, 11]. The space of all bounded real-valued functions on an index set Γ equipped with the supremum norm is denoted by $\ell^\infty(\Gamma)$ (see [2]) and any of its closed subspaces containing all e_γ 's ($\gamma \in \Gamma$) are called $\mathcal{L}^\infty(\Gamma)$ -type spaces. For example, the spaces $\ell^\infty(\Gamma)$, $c(\Gamma)$ and $c_0(\Gamma)$, particularly, ℓ^∞ , c and c_0 and so on, are all $\mathcal{L}^\infty(\Gamma)$ -type spaces. As usual, $S(\mathcal{L}^\infty(\Gamma)) = \{x \mid x \in \mathcal{L}^\infty(\Gamma), \|x\| = 1\}$.

For every $0 < \varepsilon < 1$ and $x \in S(\mathcal{L}^\infty(\Gamma))$, let $\text{supp } x = \{\gamma \in \Gamma, x(\gamma) \neq 0\}$, $N_x^\pm(\varepsilon) = \{\gamma \in \text{supp } x, \pm x(\gamma) > 1 - \varepsilon\}$ and define the star of x with respect to $S(\mathcal{L}^\infty(\Gamma))$ by

$$St(x) = \{y \in S(\mathcal{L}^\infty(\Gamma)), \|y + x\| = 2\}.$$

2. Some lemmas

We start this section with a simple observation, the proof of which we omit.

LEMMA 1. *Let x, y be in $S(\mathcal{L}^\infty(\Gamma))$. Then $y \in St(x)$ if and only if, for every $0 < \varepsilon < 1$, $N_x^+(\varepsilon) \cap N_y^+(\varepsilon) \neq \emptyset$ or $N_x^-(\varepsilon) \cap N_y^-(\varepsilon) \neq \emptyset$.*

LEMMA 2. *Let x be in $S(\mathcal{L}^\infty(\Gamma))$. If there exists an $x_0 \in St(x)$ satisfying $\|y - x_0\| \leq 1$ for every $y \in St(x)$, then $\text{supp } x_0$ is a singleton.*

PROOF. Suppose that $\text{supp } x_0$ contains more than one point.

Suppose that $|x(\gamma)| = 1$ for all $\gamma \in \text{supp } x_0$. Given $\gamma_1, \gamma_2 \in \text{supp } x_0$ with $\gamma_1 \neq \gamma_2$, put

$$y_1 = x(\gamma_1)e_{\gamma_1} - \text{sign}(x_0(\gamma_2))e_{\gamma_2}.$$

Clearly, we have $y_1 \in St(x)$, but

$$\begin{aligned} \|y_1 - x_0\| &= \|x(\gamma_1)e_{\gamma_1} - \text{sign}(x_0(\gamma_2))e_{\gamma_2} - x_0\| \\ &\geq |\text{sign}(x_0(\gamma_2)) + x_0(\gamma_2)| = 1 + |x_0(\gamma_2)| > 1. \end{aligned}$$

If there is a $\gamma_0 \in \text{supp } x_0$ such that $|x(\gamma_0)| < 1$, then let

$$y_2 = x - x(\gamma_0)e_{\gamma_0} - \text{sign}(x_0(\gamma_0))e_{\gamma_0}.$$

It is also easy to check that $y_2 \in St(x)$ and

$$\|y_2 - x_0\| \geq |\text{sign}(x_0(\gamma_0)) + x_0(\gamma_0)| = 1 + |x_0(\gamma_0)| > 1.$$

Thus it follows from the above two cases that $\text{supp } x_0$ must be a singleton. □

LEMMA 3. *Let $V_0 : S(\mathcal{L}^\infty(\Gamma)) \rightarrow S(\mathcal{L}^\infty(\Delta))$ be a surjective nonexpansive mapping. Then for every $\gamma \in \Gamma$, $\text{supp } V_0(e_\gamma)$ is a singleton and $V_0(-e_\gamma) = -V_0(e_\gamma)$.*

PROOF. We firstly prove that $\text{supp } V_0(e_\gamma)$ is a singleton for every $\gamma \in \Gamma$. By the hypotheses on V_0 , for every $y \in St(-V_0(-e_\gamma))$ and $x \in V_0^{-1}(y)$,

$$2 \geq \|x + e_\gamma\| \geq \|V_0(x) - V_0(-e_\gamma)\| = \|y - V_0(-e_\gamma)\| = 2.$$

It follows that $x \in St(e_\gamma)$ and $V_0^{-1}(St(-V_0(-e_\gamma))) \subseteq St(e_\gamma)$, that is,

$$St(-V_0(-e_\gamma)) \subseteq V_0(St(e_\gamma)). \tag{2}$$

On the other hand, for every $x \in St(e_\gamma)$, it is evident that $\|x - e_\gamma\| \leq 1$. Since V_0 is nonexpansive, we get $\|V_0(x) - V_0(e_\gamma)\| \leq 1$. Together with the relation (2), we get

$$\|y - V_0(e_\gamma)\| \leq 1 \quad \forall y \in St(-V_0(-e_\gamma)). \tag{3}$$

We claim that

$$V_0(e_\gamma) \in St(-V_0(-e_\gamma)). \tag{4}$$

Otherwise, if $\|V_0(e_\gamma) - V_0(-e_\gamma)\| < 2$, then by Lemma 1, we may choose an ε_1 so small that

$$N_{V_0(e_\gamma)}^+(\varepsilon_1) \cap N_{-V_0(-e_\gamma)}^+(\varepsilon_1) = \emptyset \quad \text{and} \quad N_{V_0(e_\gamma)}^-(\varepsilon_1) \cap N_{-V_0(-e_\gamma)}^-(\varepsilon_1) = \emptyset. \tag{5}$$

Equation (3) shows that $\|V_0(e_\gamma) + V_0(-e_\gamma)\| \leq 1$, so we can also find a small enough ε_2 such that

$$N_{V_0(e_\gamma)}^+(\varepsilon_2) \cap N_{V_0(-e_\gamma)}^+(\varepsilon_2) = \emptyset \quad \text{and} \quad N_{V_0(e_\gamma)}^-(\varepsilon_2) \cap N_{V_0(-e_\gamma)}^-(\varepsilon_2) = \emptyset. \tag{6}$$

It is obvious that

$$N_{V_0(-e_\gamma)}^+(\varepsilon_2) = N_{-V_0(-e_\gamma)}^-(\varepsilon_2) \quad \text{and} \quad N_{V_0(-e_\gamma)}^-(\varepsilon_2) = N_{-V_0(-e_\gamma)}^+(\varepsilon_2). \tag{7}$$

Then put

$$\begin{aligned} \varepsilon_0 &= \min(\varepsilon_1, \varepsilon_2); \\ A &= N_{V_0(e_\gamma)}^+(\varepsilon_0) \cup N_{V_0(e_\gamma)}^-(\varepsilon_0); \\ B &= N_{-V_0(-e_\gamma)}^+(\varepsilon_0) \cup N_{-V_0(-e_\gamma)}^-(\varepsilon_0). \end{aligned}$$

By (5), (6) and (7), we conclude that $A \cap B = \emptyset$. Then set

$$y_1 = -V_0(-e_\gamma)\chi_B - V_0(e_\gamma)\chi_A \in S(\mathcal{L}^\infty(\Delta)).$$

By Lemma 1, we obtain

$$\|y_1 - V_0(-e_\gamma)\| = 2 \quad \text{and} \quad \|y_1 - V_0(e_\gamma)\| = 2,$$

which is a contradiction. Therefore the claim is proved and it follows from Lemma 2 that $\text{supp } V_0(e_\gamma)$ is a singleton for every $\gamma \in \Gamma$.

Now we may assume that $\text{supp } V_0(e_\gamma) = \{\delta_1\}$ with $\delta_1 \in \Delta$ and, by (4), we obtain that $\delta_1 \in \text{supp } V_0(-e_\gamma)$. Then we assert that

$$\text{supp } V_0(-e_\gamma) = \{\delta_1\}.$$

Suppose that there exists a $\delta_2 \in \text{supp } V_0(-e_\gamma)$ with $\delta_2 \neq \delta_1$. Since V_0 is surjective, let x_2 be in $S(\mathcal{L}^\infty(\Gamma))$ satisfying

$$V_0(x_2) = -\text{sign}(V_0(-e_\gamma)(\delta_2))e_{\delta_2} - V_0(e_\gamma).$$

Hence

$$\|x_2 + e_\gamma\| \geq \|V_0(x_2) - V_0(-e_\gamma)\| \geq 1 + |V_0(-e_\gamma)(\delta_2)| > 1,$$

which yields $x_2(\gamma) > 0$. It follows that

$$1 \geq \|x_2 - e_\gamma\| \geq \|V_0(x_2) - V_0(e_\gamma)\| = 2.$$

This contradiction proves the assertion. Finally, we apply relation (4) again to obtain that $V_0(-e_\gamma) = -V_0(e_\gamma)$. □

LEMMA 4. *Let V_0 be the same as in Lemma 3. Then there is a family of signs $\{\theta_\delta\}_{\delta \in \Delta}$ and a bijection $\sigma : \Delta \rightarrow \Gamma$ satisfying*

$$e_\delta = \theta_\delta V_0(e_{\sigma(\delta)}) \quad \forall \delta \in \Delta. \tag{8}$$

PROOF. By Lemma 3, we can define a map $\pi : \Gamma \rightarrow \Delta$ satisfying $\{\pi(\gamma)\} = \text{supp } V_0(e_\gamma)$ for each $\gamma \in \Gamma$. Moreover, we shall prove that π is bijective.

If $\pi(\gamma_1) = \pi(\gamma_2)$, then by Lemma 3 and the fact that V_0 is nonexpansive,

$$\begin{aligned} 2 &= \max\{\|V_0(e_{\gamma_1}) - V_0(e_{\gamma_2})\|, \|V_0(e_{\gamma_1}) + V_0(e_{\gamma_2})\|\} \\ &\leq \max\{\|e_{\gamma_1} - e_{\gamma_2}\|, \|e_{\gamma_1} + e_{\gamma_2}\|\} \leq 2. \end{aligned}$$

So $\max\{\|e_{\gamma_1} + e_{\gamma_2}\|, \|e_{\gamma_1} - e_{\gamma_2}\|\} = 2$, which implies that $\gamma_1 = \gamma_2$. To see that π is surjective, suppose, on the contrary, that there exists $\delta_0 \in \Delta/\pi(\Gamma)$.

Choose $\{\varepsilon_\gamma\}_{\gamma \in \Gamma} \in S(\mathcal{L}^\infty(\Gamma))$ with $\varepsilon_\gamma > 0$ for every $\gamma \in \Gamma$ and $y_0 \in S(\mathcal{L}^\infty(\Delta))$ satisfying

$$y_0(\delta) = \begin{cases} V_0(e_\gamma)(\pi(\gamma))\varepsilon_\gamma & \text{if } \delta \in \pi(\Gamma) \text{ with } \delta = \pi(\gamma), \\ 0 & \text{if } \delta \in \Delta/\pi(\Gamma). \end{cases}$$

Since V_0 is surjective, we can find $x_1, x_2 \in S(\mathcal{L}^\infty(\Gamma))$ such that

$$V_0(x_1) = y_0 + e_{\delta_0} \quad \text{and} \quad V_0(x_2) = y_0 - e_{\delta_0}.$$

It follows from Lemma 3 and the property of y_0 that, for every $\gamma \in \Gamma$,

$$\|x_1 + e_\gamma\| \geq \|V_0(x_1) + V_0(e_\gamma)\| = \|y_0 + e_{\delta_0} + V_0(e_\gamma)\| = 1 + \varepsilon_\gamma > 1.$$

This implies that $x_1(\gamma) > 0$. Similarly, we can also get $x_2(\gamma) > 0$ for every $\gamma \in \Gamma$. Hence $\|x_1 - x_2\| \leq 1$, but $\|V_0(x_1) - V_0(x_2)\| = 2$. This is impossible since V_0 is nonexpansive. Therefore π must be onto.

Finally, let $\sigma = \pi^{-1}$ and define $\theta_\delta = \text{sign}(V_0(e_{\sigma(\delta)})(\delta))$ for every $\delta \in \Delta$. We complete the proof of this lemma. \square

3. Main results

THEOREM 5. *Let $V_0 : S(\mathcal{L}^\infty(\Gamma)) \rightarrow S(\mathcal{L}^\infty(\Delta))$ be a surjective nonexpansive mapping. Then V_0 must be an onto isometry and there exists a family of signs $\{\theta_\delta\}_{\delta \in \Delta}$ and a bijection $\sigma : \Delta \rightarrow \Gamma$ such that, for any element $x \in S(\mathcal{L}^\infty(\Gamma))$,*

$$V_0(x)(\delta) = \theta_\delta x(\sigma(\delta)) \quad \forall \delta \in \Delta. \quad (9)$$

PROOF. Let σ and $\{\theta_\delta\}_{\delta \in \Delta}$ be as in Lemma 4. It is easy to see that if (9) holds, then V_0 is an isometry from $S(\mathcal{L}^\infty(\Gamma))$ onto $S(\mathcal{L}^\infty(\Delta))$. Thus we only need to verify (9).

Take $x = \{\xi_\gamma\}_{\gamma \in \Gamma} \in S(\mathcal{L}^\infty(\Gamma))$ and let $V_0(x) = \{\eta_\delta\}_{\delta \in \Delta} \in S(\mathcal{L}^\infty(\Delta))$.

For any $|\eta_\delta| \neq 0$, by Lemma 4, we get

$$\|V_0(x) + \text{sign}(\eta_\delta)\theta_\delta V_0(e_{\sigma(\delta)})\| = \|V_0(x) + \text{sign}(\eta_\delta)e_\delta\| = 1 + |\eta_\delta|.$$

On the other hand,

$$\|x + \text{sign}(\eta_\delta)\theta_\delta e_{\sigma(\delta)}\| = \max\{\sup_{\gamma \neq \sigma(\delta)} |\xi_\gamma|, |\xi_{\sigma(\delta)} + \text{sign}(\eta_\delta)\theta_\delta|\}.$$

Since V_0 is nonexpansive, we conclude that

$$\text{sign}(\xi_{\sigma(\delta)}) = \text{sign}(\eta_\delta)\theta_\delta \quad (10)$$

and

$$|\eta_\delta| \leq |\xi_{\sigma(\delta)}|.$$

Obviously, the inequality holds for $\eta_\delta = 0$, so

$$|\eta_\delta| \leq |\xi_{\sigma(\delta)}| \quad (11)$$

holds for any $\delta \in \Delta$. We shall show, in fact, that $|\eta_\delta| = |\xi_{\sigma(\delta)}|$ for all $\delta \in \Delta$. Assume that there is a $\delta_0 \in \Delta$ such that $|\eta_{\delta_0}| < |\xi_{\sigma(\delta_0)}|$. Then choose a sufficiently small $\varepsilon_0 > 0$ such that

$$|\eta_{\delta_0}| + \varepsilon_0 < |\xi_{\sigma(\delta_0)}|. \quad (12)$$

Define

$$x_0(\gamma) = \begin{cases} (|\eta_{\delta_0}| + \varepsilon_0)\xi_\gamma & \text{if } \gamma \neq \sigma(\delta_0), \\ \text{sign}(\xi_{\sigma(\delta_0)}) & \text{if } \gamma = \sigma(\delta_0), \end{cases}$$

and set $V_0(x_0) = \{\eta_\delta^0\}_{\delta \in \Delta}$. Since $|\xi_{\sigma(\delta)}| \leq 1$ for every $\delta \in \Delta$, we apply the definition of x_0 and (11) to obtain that

$$|V_0(x_0)(\delta)| = |\eta_\delta^0| \leq (|\eta_{\delta_0}| + \varepsilon_0)|\xi_{\sigma(\delta)}| \leq |\eta_{\delta_0}| + \varepsilon_0 < |\xi_{\sigma(\delta_0)}|$$

for every $\delta \in \Delta$ with $\delta \neq \delta_0$. However, $\|V_0(x_0)\| = 1$, therefore $|V_0(x_0)(\delta_0)| = |\eta_{\delta_0}^0| = 1$. It follows that

$$\|V_0(x_0) - V_0(x)\| \geq |V_0(x_0)(\delta_0) - V_0(x)(\delta_0)| \geq |\eta_{\delta_0}^0| - |\eta_{\delta_0}| = 1 - |\eta_{\delta_0}|.$$

On the other hand,

$$\begin{aligned} \|x_0 - x\| &= \sup\{|x_0(\gamma) - x(\gamma)| \mid \gamma \in \Gamma\} \\ &\leq \max\{1 - (|\eta_{\delta_0}| + \varepsilon_0), 1 - |\xi_{\sigma(\delta_0)}|\}. \end{aligned}$$

From (12), it is clear that

$$\|x_0 - x\| = 1 - (|\eta_{\delta_0}| + \varepsilon_0) < 1 - |\eta_{\delta_0}| \leq \|V_0(x_0) - V_0(x)\|.$$

This contradicts the fact that V_0 is nonexpansive. Thus $|\eta_\delta| = |\xi_{\sigma(\delta)}|$ for all $\delta \in \Delta$. Combining this with equality (10), we get the desired characterization of V_0 given by (9). \square

REMARK 6. The surjectivity of V_0 is essential in the Theorem 5. For example, fix $y_0 \in S(\mathcal{L}^\infty(\Delta))$ and define V_0 by $V_0(x) = y_0$ for all $x \in S(\mathcal{L}^\infty(\Gamma))$. It is obvious that V_0 is a nonexpansive mapping, not an isometry. The next simple example shows that we cannot replace the unit sphere by the unit ball and the restriction of V_0 to the unit sphere is also important for us to generalize Freudenthal and Hurewicz's result on the relation between nonexpansive mappings and isometries.

The following example is so easy that we omit the proof.

EXAMPLE 7. Let X be ℓ^p ($1 \leq p \leq \infty$), c or c_0 . Then a map $T : X \rightarrow X$ defined by $T(\xi_1, \xi_2, \xi_3, \dots, \xi_n, \dots) = (\xi_2, \xi_3, \dots, \xi_n, \dots)$, for all $\{\xi_n\}_{n \geq 1}$ in X , is a bounded linear surjective operator with $\|T\| = 1$ and the restriction of T to the unit ball of X , denoted by $T|_{B(X)}$, is a nonexpansive but not isometric map from $B(X)$ onto itself.

Applying Theorem 5, we can get a result for the isometric extension problem as follows.

COROLLARY 8. Let $V_0 : S(\mathcal{L}^\infty(\Gamma)) \rightarrow S(\mathcal{L}^\infty(\Delta))$ be a surjective nonexpansive mapping. Then V_0 can be extended to a linear isometry defined on the whole space $\mathcal{L}^\infty(\Gamma)$.

PROOF. By Theorem 5, there exist a family of signs $\{\theta_\delta\}_{\delta \in \Delta}$ and a bijection $\sigma : \Delta \rightarrow \Gamma$ such that, for any $x \in S(\mathcal{L}^\infty(\Gamma))$,

$$V_0(x)(\delta) = \theta_\delta x(\sigma(\delta)) \quad \forall \delta \in \Delta.$$

Define $V : \mathcal{L}^\infty(\Gamma) \rightarrow \mathcal{L}^\infty(\Delta)$ by

$$V(\bar{x})(\delta) = \theta_\delta \bar{x}(\sigma(\delta)) \quad \text{for each } \bar{x} \in \mathcal{L}^\infty(\Gamma).$$

Clearly, V is a surjective linear isometry on $\mathcal{L}^\infty(\Gamma)$ and the restriction of V to the unit sphere $S(\mathcal{L}^\infty(\Gamma))$ is just V_0 . Hence the proof is complete. \square

By Corollary 8, we have the the following result.

COROLLARY 9. *Let $V_0 : S(\mathcal{L}^\infty(\Gamma)) \rightarrow S(\mathcal{L}^\infty(\Delta))$ be a surjective mapping satisfying $\|V_0(x) - V_0(y)\| \geq \|x - y\|$ for all $x, y \in S(\mathcal{L}^\infty(\Gamma))$. Then V_0 must be an isometry and it can be linearly isometrically extended to the whole space $\mathcal{L}^\infty(\Gamma)$.*

We find that the proof of Theorem 5 relies on the structure properties of $\mathcal{L}^\infty(\Gamma)$ -spaces and we would like to know the following

PROBLEM. Let E be an infinite-dimensional Banach space. Let $T : S(E) \rightarrow S(E)$ be a surjective nonexpansive map. Is T necessarily an isometry?

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References

- [1] G. M. An, 'Isometries on unit sphere of $(\ell^{\beta_n})^*$ ', *J. Math. Anal. Appl.* **301** (2005), 249–254.
- [2] Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Functional Analysis*, Vol. I, Colloquium Publications, Vol. 48 (American Mathematical Society, Providence, RI, 2000).
- [3] T. A. Brown and W. W. Comfort, 'New method for expansion and contraction maps in uniform spaces', *Proc. Amer. Math. Soc.* **11** (1960), 483–486.
- [4] G. G. Ding, 'The 1-Lipschitz mapping between the unit spheres of two Hilbert spaces can be extended to a real linear linear isometry of the whole space', *Sci. China Ser. A* **45**(4) (2002), 479–483.
- [5] ———, 'The isometric extension problem in the unit spheres of $\ell^p(\Gamma)$ ($p > 1$) type spaces', *Sci. China Ser. A* **46**(3) (2003), 333–338.
- [6] ———, 'The representation theorem of onto isometric mapping between two unit spheres of $\ell^1(\Gamma)$ type spaces and the application on isometric extension problem', *Acta Math. Sin. (Engl. Ser.)* **20**(6) (2004), 1089–1094.
- [7] ———, 'The representation of onto isometric mappings between two spheres of ℓ^∞ -type spaces and the application on isometric extension problem', *Sci. China Ser. A* **34**(2) (2004), 157–164 (in Chinese); **47** (5) (2004), 722–729 (in English).
- [8] ———, 'On the extension of isometries between unit spheres of E and $C(\Omega)$ ', *Acta Math. Sin. (Engl. Ser.)* **19**(4) (2003), 793–800.

- [9] ———, ‘The isometric extension of the into mapping from the $\mathcal{L}^\infty(\Gamma)$ -type space to some normed space E ’, *Illinois J. Math.* **52**(2) (2007), 445–453.
- [10] H. Freudenthal and W. Hurewicz, ‘Dehnungen, Verkürzungen, Isometrien’, *Fund. Math.* **26** (1936), 120–122.
- [11] R. Liu, ‘On extension of isometries between the unit spheres of \mathcal{L}^∞ -type space and some Banach space E ’, *J. Math. Anal. Appl.* **333** (2007), 959–970.
- [12] P. Mankiewicz, ‘On extension of isometries in normed linear spaces’, *Bull. Acad. Polon. Sci.* **20** (1972), 367–371.
- [13] F. Rhodes, ‘A generalization of isometries to uniform spaces’, *Proc. Cambridge Philos. Soc.* **52** (1956), 399–405.
- [14] D. Tingley, ‘Isometries of the unit sphere’, *Geom. Dedicata* **22** (1987), 371–378.
- [15] J. Wang, ‘On extension of isometries between unit spheres of AL_p -spaces ($1 < p < \infty$)’, *Proc. Amer. Math. Soc.* **132**(10) (2004), 2899–2909.
- [16] X. Z. Yang, ‘On extension of isometries between unit spheres of $L_p(\mu)$ and $L_p(\nu, H)$ ($1 < p \neq 2$, H is a Hilbert space)’, *J. Math. Anal. Appl.* **323** (2006), 985–992.

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