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Hausdorff–Young Inequalities for Group Extensions

to the memory of Professor W. Roelcke

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Abstract. This paper studies Hausdorff–Young inequalities for certain group extensions, by use of Mackey's theory. We consider the case in which the dual action of the quotient group is free almost everywhere. This result applies in particular to yield a Hausdorff–Young inequality for nonunimodular groups.

Introduction

In this paper we deal with a definition of L^p-Fourier transform on locally compact groups. Recall that for locally compact abelian groups the Hausdorff–Young inequality reads:

Let $1 < p < 2$ *and* $q = p/(p-1)$. *If* $g \in \mathrm{L}^1(G) \cap \mathrm{L}^p(G)$, *then* $\widehat{g} \in \mathrm{L}^q(\widehat{G})$, *with* $\|\widehat{g}\|_q \leq \|g\|_p$.

The inequality allows us to extend the Fourier transform to a continuous operator $\mathcal{F}^p \colon \mathrm{L}^p \to \mathrm{L}^q$ by continuity. It was generalized to type I unimodular groups by Kunze [13]. Over the years, various authors derived Hausdorff–Young inequalities, both for concrete groups [1, 7, 8, 14] and for certain classes of groups [2, 10, 16–18], with the aim of getting a more precise bound in the inequality.

The formulation of the results for nonabelian groups requires a certain amount of notation. Given a locally compact group G, we denote by $\widehat{G}$ its unitary dual, *i.e.*, the set of (equivalence classes of) irreducible unitary representations, endowed with the Mackey Borel structure. The dual space is used to define the operator valued Fourier transform by letting

$$\mathrm{L}^1(G) \ni g \mapsto \mathcal{F}^1(g) := (\sigma(g))_{\sigma \in \widehat{G}},$$

where $\sigma(g)$ is defined by the weak operator integral

$$\sigma(g) = \int_G g(x)\sigma(x)\,dx.$$

The main task for Plancherel theory in the nonabelian setting is to find suitable Banach spaces for these operators, which is what we describe next.

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For a Hilbert space $\mathcal{H}$, we denote by $\mathcal{B}_p(\mathcal{H})$ the set of bounded operators T such that $(T^*T)^{p/2}$ is trace class. Then $\mathcal{B}_p(\mathcal{H})$ is a Banach space with the obvious norm $\|T\|_p = \operatorname{tr}((T^*T)^{p/2})^{1/p}$. The Plancherel theorem for type I unimodular locally compact groups [5, 18.8.2] provides the existence of a *Plancherel measure* ν_G on $\widehat{G}$ with the following properties: given a measurable realization $(\sigma, \mathcal{H}_\sigma)$ of representatives from $\widehat{G}$, denote by $\mathcal{B}_p^\oplus$ $(p > 1)$ the space of operator fields $A = (A_\sigma)_{\sigma\in\widehat{G}}$ such that $A_\sigma \in \mathcal{B}_p(\mathcal{H}_\sigma)$ a.e., and moreover that

$$\|A\|_p := \|(A_\sigma)_\sigma\|_{\mathcal{B}_p^\oplus} := \Big(\int_{\widehat{G}} \|A_\sigma\|_p^p \, d\nu_G(\sigma)\Big)^{1/p}$$

is finite. It is routine to check that $(\mathcal{B}_p^\oplus, \|\cdot\|_{\mathcal{B}_p^\oplus})$ is a Banach space; for $p = 2$ it is even a Hilbert space. It is convenient to use the notation $\mathcal{B}_\infty^\oplus$ for the space of uniformly bounded operator fields. Note that since $\|\pi(g)\| \leq \|g\|_1$, for arbitrary representations of g we have the estimate

$$\|\mathcal{F}(g)\|_\infty \leq \|g\|_1. \tag{0.1}$$

Now the Plancherel theorem for unimodular groups states that for the unique Plancherel measure ν_G, and for all $g \in \mathrm{L}^2(G) \cap \mathrm{L}^1(G)$, we have

$$\|\mathcal{F}(g)\|_{\mathcal{B}_2^\oplus} = \|g\|_2. \tag{0.2}$$

This gives rise to the *Plancherel transform*

$$\mathcal{P} = \mathcal{F}^2 \colon \mathrm{L}^2(G) \to \mathcal{B}_2^\oplus,$$

which is a unitary equivalence. The Hausdorff–Young inequality then states that, for all $1 \leq p \leq 2$ and $g \in \mathrm{L}^p(G) \cap \mathrm{L}^1(G)$, we have

$$\|\mathcal{F}(g)\|_{\mathcal{B}_q^\oplus} \leq \|g\|_p, \tag{0.3}$$

where $q = p/(p-1)$, which uniquely defines a Fourier transform $\mathcal{F}^p \colon \mathrm{L}^p(G) \to \mathcal{B}_q^\oplus$. The proof of the inequality usually involves interpolation techniques to derive the estimate from the inequalities at the "endpoints", *i.e.,* from (0.1) and (0.2). However, for a sharper estimate of the operator norm of $\mathcal{F}_p$, in the following denoted by $A_p(G)$, other techniques are required.

For nonunimodular groups, an additional complication arises. As Khalil showed for the **ax+b**-group [11], there exist $g \in \mathcal{C}_c(G)$ such that for ν_G-almost all σ, $\mathcal{F}^1(g)(\sigma)$ is not a compact operator and hence $\mathcal{F}^1(g) \notin \mathcal{B}_q^\oplus$ for arbitrary q!

However, there is a way to save the Plancherel theorem, first obtained in [11] for the **ax+b**-group and proved later on in full generality [6]. The Plancherel theorem in [6] can be phrased as follows: let g be a nonunimodular locally compact group with type I regular representation. There exists a field of unbounded selfadjoint operators

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$(K_\sigma)_{\sigma\in\widehat{G}}$, called *formal dimension operators*, and a measure ν_G on $\widehat{G}$, such that, for every $g \in \mathrm{L}^2(G)\cap \mathrm{L}^1(G)$ and ν_G-almost every σ, the densely defined operator $\sigma(g)K_\sigma^{1/2}$ extends to a Hilbert–Schmidt operator, denoted $[\sigma(g)K_\sigma^{1/2}]$, and moreover

$$\int_{\widehat{G}} \big\| [\sigma(g)K_\sigma^{1/2}] \big\|_2^2 \, d\nu_G(\sigma) = \|g\|_2^2 \ .$$

The operators are unique up to scalar multiples, and once they are fixed, so is the measure ν_G. Similar results were obtained by [12, 19].

In the literature there does not seem to be an accessible general treatment of the Hausdorff–Young inequality for nonunimodular groups. Terp proved such a result in the preprint [20]; however the paper seems never to have appeared. Several authors established Hausdorff–Young inequalities for the **ax**+**b**-group [7, 16]. To my knowledge, the largest class of groups under consideration were the solvable Lie groups acting on Siegel domains, as investigated by Inoue [10]. The result is fairly intuitive, once we have seen how to treat the L^2-case: letting

$$\mathcal{F}^p(g)(\sigma) := [\sigma(g)K_\sigma^{1/q}]$$

gives a well-defined operator from $\mathrm{L}^1(G)\cap \mathrm{L}^p(G)$ into $\mathcal{B}_q^\oplus$ fulfilling

$$\|\mathcal{F}^p(g)\|_q \le \|g\|_p.$$

The first theorem shows that this result is true for more general nonunimodular groups:

Theorem 1 *Let G be a nonunimodular locally compact group such that λ_G is type* I *and $N = \mathrm{Ker}(\Delta_G)$ is type* I. *Let $(K_\sigma)_{\sigma\in\widehat{G}}$ denote the field of formal degree operators, and ν_G the Plancherel measure of G belonging to that field. Let $1 < p < 2$ and $q = p/(p-1)$.*

Then, for all $g \in \mathrm{L}^1(G)\cap \mathrm{L}^p(G)$ and ν_G-almost all $\sigma \in \widehat{G}$, the operator $\sigma(g)K_\sigma^{1/q}$ has a bounded extension $[\sigma(g)K_\sigma^{1/q}] \in \mathcal{B}_q(\mathcal{H}_\sigma)$, and we have the inequality

$$\Big(\int_{\widehat{G}} \big\| [\sigma(g)K_\sigma^{1/q}] \big\|_{\mathcal{B}_q(\mathcal{H}_\sigma)}^q \, d\nu_G(\sigma) \Big)^{1/q} \le A_p(N)\|g\|_p,$$

i.e., $A_p(G) \le A_p(N)$.

Note that the estimate $A_p(G) \le A_p(N)$ is a sharpening of Terp's result [20, Theorem 4.1], which states $A_p(G) \le 1$.

Theorem 1 is a special case of a more general, somewhat technical result (Theorem 2), which proves a Hausdorff–Young inequality for certain group extensions

$$1 \to N \to G \to H \to 1$$

with the additional properties that N and H are unimodular and the dual action of H on $\widehat{N}$ is free ν_N-almost everywhere.

1 Group Extensions With Free Dual Actions

Now let G be a second countable locally compact group. Let μ_G denote the left-invariant Haar measure and Δ_G the modular function of G. We assume that G is an extension of the unimodular normal subgroup N, and that $H = G/N$ is unimodular as well. Moreover we assume that N is type I and G has a type I regular representation, and we denote the Plancherel measures by ν_N and ν_G, respectively. It is well known that H acts on $\widehat{N}$ via conjugation, denoted by $H \times \widehat{N} \ni (\gamma, \sigma) \mapsto \gamma.\sigma$. Mackey's theory allows the computation of $\widehat{G}$ from this action, at least under certain regularity conditions. The Mackey machine also provides the means to compute the Plancherel measure, under suitable assumptions on N and the dual action [12]. Our main technical assumption is the following:

(A) There exists a Borel ν_N-conull subset $U \subset \widehat{N}$ with the following property: U is H-invariant with U/H standard. Moreover, for all $\sigma_0 \in U$, $\mathrm{Ind}_N^G \sigma_0 \in \widehat{G}$.

In view of Mackey's theory, we obtain the following consequences:

(i) $\mathrm{Ind}_N^G \sigma_0 \in \widehat{G}$ for every $\sigma_0 \in U$ implies that H operates freely on the orbit $H.\sigma_0$. Moreover, Ind_N^G induces an injective map $\overline{\mathrm{Ind}}\colon U/H \hookrightarrow \widehat{G}$.

(ii) Since U/H is standard, there exists a measure decomposition along H-orbits (*e.g.,* [12, Theorem 2.1]). More precisely, there exist measures $\beta_{H.\sigma_0}$ on the orbits, such that the Plancherel measures of N and G satisfy the relation

$$d\nu_N(\rho) = d\beta_{H.\rho}(\rho) d\nu_G(H.\rho). \tag{1.1}$$

That the Plancherel measure of G may be obtained as a quotient measure of ν_N follows from Theorem [12, I, 10.2]. For this we need to check four conditions:

(a) λ_N *is type* I *with* ν_N *concentrated in* $\widehat{N}_t$. This follows from the type I property of N.

(b) ν_N/H *is countably separated.* This holds by (A).

(c) ν_N*-almost every little fixed group is trivial.* This holds by (A).

(d) λ_G *is type* I. This holds by assumption.

(iii) The fact that H acts freely on U entails that U can be seen as a product space $H \times U_0$, by the following arguments: since both U and U/H are standard, (U as a Borel subset of $\widehat{N}$), [15, Theorem 5.2] provides the existence of a Borel transversal, *i.e.,* a Borel subset $U_0 \subset U$ meeting each H-orbit in precisely one point. Hence the freeness of the operation yields a measurable bijection $H \times U_0 \equiv U, (\gamma, \sigma_0) \mapsto \gamma.\sigma_0$, which is a Borel isomorphism [15, Theorem 3.2]. In the following, for $\sigma \in V$ we use the symbol $\sigma_0 \in U_0$ to denote the representative of the associated dual orbit.

(iv) The Borel isomorphism $U \equiv H \times U_0$ allows us to write down the measure disintegration (1.1) much more explicitly. For this purpose we define the mapping $\psi\colon (h, \sigma_0) \mapsto \Delta_G(h)$ on U. Note that $\psi \equiv 1$ iff G is unimodular. Then (1.1) becomes

$$d\nu_N(h, \sigma_0) = \psi(h) d\mu_H(h) d\nu_G(\sigma). \tag{1.2}$$

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Indeed, a straightforward computation shows that $\nu_N(\gamma.A) = \Delta_G(\gamma)\nu_N(A)$. This shows that

$$d\beta_{H.\sigma_0}(\gamma.\sigma_0) = \alpha(H.\sigma_0)\psi(\gamma)d\mu_H(\gamma),$$

for some scalar factor $\alpha(H.\sigma_0)$. Now ν_G can be renormalized to achieve that these factors are one.

(v) For the following calculations, $\sigma = \mathrm{Ind}_N^G \sigma_0$ is realized via cross sections on $\mathcal{H}_\sigma = \mathrm{L}^2(H, d\mu_H; \mathcal{H}_{\sigma_0})$ (see Lemma 4). We define a family of operators K_σ on $\mathcal{H}_\sigma$ given by multiplication with Δ_G:

$$(K_\sigma\eta)(h) = \Delta_G(h)\eta(h),$$

and $\mathrm{dom}(K_\sigma)$ is the set of all $\eta \in \mathrm{L}^2(H, d\mu_H; \mathcal{H}_{\sigma_0})$ for which this product is also square integrable.

Obviously K_σ is the identity operator if G is unimodular. In the other case, K_σ is precisely the formal dimension operator, as can be seen by verifying the semi-invariance relation

$$\sigma(x)K_\sigma\sigma(x)^* = \Delta_G(x)^{-1}K_\sigma,$$

observing that the formal dimension operators obey the same relation [6, Theorem 5], and then applying the uniqueness statement [6, Lemma 1].

2 Hausdorff–Young Inequalities for Group Extensions

The following theorem is the main result of this paper.

Theorem 2 *Let a group extension $1 \to N \to G \to H \to 1$ be given with N, H unimodular, N a type* I *group, and λ_G a type* I *representation. Assume that Assumption* (A) *holds. Let $(K_\sigma)_{\sigma\in\widehat{G}}$ denote the field of multiplication operators given in* (v) *above. Let $1 < p < 2$ and $q = p/(p-1)$.*

Then, for all $g \in \mathrm{L}^1(G) \cap \mathrm{L}^p(G)$ and ν_G-almost all $\sigma \in \widehat{G}$, the operator $\sigma(g)K_\sigma^{1/q}$ has a bounded extension $[\sigma(g)K_\sigma^{1/q}] \in \mathcal{B}_q(\mathcal{H}_\sigma)$, and we have the inequality

$$\Big(\int_{\widehat{G}} \big\| [\sigma(g)K_\sigma^{1/q}] \big\|^q_{\mathcal{B}_q(\mathcal{H}_\sigma)} \, d\nu_G(\sigma)\Big)^{1/q} \le A_p(N)\|g\|_p,$$

i.e., $A_p(G) \le A_p(N)$.

Before we prove this result, let us show how the nonunimodular case follows from it:

Proof of Theorem 1 $N = \mathrm{Ker}(\Delta_G)$ is a normal unimodular subgroup, with $H = G/N$ abelian. Assumption (A) holds by [6, Theorem 6]. Thus Theorem 2 implies Theorem 1. ■

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We now proceed with the proof of Theorem 2. It turns out that it is a quite natural extension of the arguments used in [16] for the **ax+b**-group, essentially by combining it with techniques from [12]. First we show how $\sigma(f)$ acts via an operator valued integral kernel. We then use an estimate of the p-norm of such operators by certain cross norms, as provided by [9].

The first lemma computes the Haar measure of G in terms of μ_N and μ_H, and fixes the normalizations we use in the following. Note that, since N is normal, $\Delta_G|_N = \Delta_N = 1$, hence Δ_G can (and will) be regarded as a function on H.

Lemma 3 *Fix a measurable cross section $\alpha\colon H \to G$. Then the mapping $N \times H \ni (n, h) \mapsto n\alpha(h) \in G$ is an isomorphism of Borel spaces. We use the notation $g = n\alpha(h) \equiv (n, h)$. Then*

$$d\mu_G(n, h) = d\mu_N(n)\Delta_G(h)d\mu_H(h) \tag{2.1}$$

is a left Haar measure.

Proof The map $(n, h) \mapsto n\alpha(h)$ is a measurable bijection between standard Borel spaces, and thus a Borel isomorphism [15, Theorem 3.2]. Fix $g = n\alpha(h), g' = n'\alpha(h') \in G$, then

$$gg' = n\, \alpha(h)n'\alpha(h)^{-1}\, \alpha(h)\alpha(h)'\alpha(hh')^{-1}\, \alpha(hh')$$

with $\alpha(h)n'\alpha(h)^{-1}, \alpha(h)\alpha(h)'\alpha(hh')^{-1} \in N$ (observing $N \lhd G$). Hence right translation on G corresponds to right translation in the variables n, h, though not by n', h'. Now the right invariance of μ_N, μ_H entails that $d\mu_N(n)d\mu_H(h)$ is a right Haar measure on G. But then $\Delta_G d\mu_N d\mu_H$ is a left Haar measure. ■

The following two lemmas provide the integral kernels:

Lemma 4 *Let $\sigma_0 \in U_0$ and $\sigma = \mathrm{Ind}_N^G \sigma_0$. Define the cocycle $\Lambda\colon H \times H \to N$ by*

$$\Lambda(\gamma, \xi) = \alpha(\xi)^{-1}\alpha(\gamma)\alpha(\alpha(\gamma)^{-1}\xi).$$

If we realize σ on $\mathrm{L}^2(H, d\mu_\gamma; \mathcal{H}_\sigma)$ via the cross section α, we obtain for $x = n\alpha(\gamma)$

$$(\sigma(x)f)(\xi) = (\xi.\sigma_0)(n)\sigma_0(\Lambda(\gamma, \xi))f(\gamma^{-1}\xi). \tag{2.2}$$

Proof Since the measure is invariant, the formula for induction via cross sections yields

$$\begin{aligned}(\sigma(n, \gamma)f)(\xi) &= \sigma_0\big(\alpha(\xi)^{-1}n\alpha(\gamma)\alpha(\alpha(\gamma)^{-1}n^{-1}\xi)\big)\ f(\alpha(\gamma)^{-1}n^{-1}\xi)\\ &= \sigma_0\big(\alpha(\xi)^{-1}n\alpha(\xi)\Lambda(\gamma, \xi)\big)\ f(\gamma^{-1}\xi)\\ &= (\xi.\sigma_0)(n)\sigma_0(\Lambda(\gamma, \xi))f(\gamma^{-1}\xi),\end{aligned}$$

where $\alpha(\gamma)^{-1}n^{-1}\xi = \alpha(\gamma)^{-1}\xi$ is due to $N \lhd G$. ■

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The next step consists in integrating this representation:

Lemma 5 *Let $\sigma_0 \in U_0$ and $\sigma = \mathrm{Ind}_N^G \sigma_0$. For $g \in \mathrm{L}^1(G)$ and $\gamma \in H$, let $g_\gamma := g(\cdot, \gamma)$. Then $\sigma(g)\colon \mathrm{L}^2(H, d\mu_H; \mathcal{H}_{\sigma_0}) \to \mathrm{L}^2(H, d\mu_H; \mathcal{H}_{\sigma_0})$ can be written as*

$$\sigma(g) f(\xi) = \int_H k_\sigma(\xi, \gamma) f(\gamma)\, d\gamma,$$

where k_σ is an operator valued integral kernel given by

$$k_\sigma(\xi, \gamma) = (\xi.\sigma_0)(g_{\xi\gamma^{-1}}) \circ \sigma_0(\Lambda(\xi\gamma^{-1}, \xi)) \cdot \Delta_G(\xi\gamma^{-1}).$$

Proof First note that by Fubini's theorem $g_\gamma \in \mathrm{L}^1(N)$, for almost every $\gamma \in H$, which justifies the use of $(\xi.\sigma_0)(g_{\xi\gamma^{-1}})$. The following formal calculations can be made rigorous by plugging them into scalar products, according to the definition of the weak operator integral. Using the previous lemma and unimodularity of H, we see that

$$\begin{aligned}
(\sigma(g) f)(\xi) &= \int_H \int_N g(n, \gamma)(\xi.\sigma_0)(n)\sigma_0(\Lambda(\gamma, \xi)) f(\gamma^{-1}\xi)\, dn \Delta_G(\gamma)\, d\gamma \\
&= \int_H (\xi.\sigma_0)(g_\gamma)\sigma_0(\Lambda(\gamma, \xi)) f(\gamma^{-1}\xi)\Delta_G(\gamma)\, d\gamma \\
&= \int_H (\xi.\sigma_0)(g_{\gamma^{-1}})\sigma_0(\Lambda(\gamma^{-1}, \xi)) f(\gamma\xi)\Delta_G(\gamma^{-1})\, d\gamma \\
&= \int_H (\xi.\sigma_0)(g_{\xi\gamma^{-1}})\sigma_0(\Lambda(\xi\gamma^{-1}, \xi))\Delta_G(\xi\gamma^{-1}) f(\gamma)\, d\gamma,
\end{aligned}$$

which is the desired formula. ∎

Proof of Theorem 2 Let $g \in \mathrm{L}^p(G) \cap \mathrm{L}^1(G)$ be given. By Lemma 5 and the definition of K_σ, we find that $\sigma(g) K_\sigma^{1/q}$ has the operator-valued kernel

$$k_\sigma(\xi, \gamma) = (\xi.\sigma_0)(g_{\xi\gamma^{-1}})\sigma_0(\Lambda(\xi\gamma^{-1}, \xi))\Delta_G(\xi\gamma^{-1})\Delta_G(\gamma)^{1/q} \ .$$

We want to use a result from [9], which gives an estimate of $\|\sigma(g) K_\sigma^{1/q}\|_p$ in terms of the *cross norm*

$$\|\tilde{k}_\sigma\|_{q,p,q} := \Big(\int_H \Big[\int_H \|k(\xi, \gamma)\|_q^p d\xi \Big]^{q/p} d\gamma \Big)^{1/q},$$

for arbitrary operator-valued kernels. Using first [9, Corollary 1] and then the Cauchy–Schwarz inequality, we have

$$\int_{\widehat{G}} \|\sigma(g)K_\sigma^{1/q}\|_q^q \, d\nu_G(\sigma) \le \int_{\widehat{G}} \|k_\sigma\|_{q,p,q}^{q/2} \|k_\sigma^*\|_{q,p,q}^{q/2} \, d\nu_G(\sigma)$$
$$\le \Big(\int_{\widehat{G}} \|k_\sigma\|_{q,p,q}^q \, d\nu_G(\sigma) \Big)^{1/2} \Big(\int_{\widehat{G}} \|k_\sigma^*\|_{q,p,q}^q \, d\nu_G(\sigma) \Big)^{1/2}, \tag{2.3}$$

where $k_\sigma^*(\xi,\gamma) = k_\sigma(\gamma,\xi)^*$. It remains thus to estimate the integral over the cross norms. We have that

$$\begin{aligned}
&\int_{\widehat{G}} \|k_\sigma\|_{q,p,q}^q \, d\nu_G(\sigma) \\
&= \int_{U_0} \int_H \Big[\int_H \|(\xi.\sigma_0)(g_{\xi\gamma^{-1}})\sigma_0(\Lambda(\xi\gamma^{-1},\xi))\Delta_G(\xi\gamma^{-1})\Delta_G(\gamma)^{1/q}\|_q^p d\xi \Big]^{q/p} \\
&\qquad\qquad d\gamma d\nu_G(\sigma) \\
&= \int_{U_0} \int_H \Big[\int_H \|(\xi\gamma.\sigma_0)(g_\xi)\Delta_G(\xi)\Delta_G(\gamma)^{1/q}\|_q^p d\xi \Big]^{q/p} d\gamma d\nu_G(\sigma) \\
&\le \Big(\int_H \Big[\int_{U_0} \int_H \|(\xi\gamma.\sigma_0)(g_\xi)\Delta_G(\xi)\Delta_G(\gamma)^{1/q}\|_q^q d\gamma d\nu_G(\sigma) \Big]^{p/q} d\xi \Big)^{q/p} \\
&= \Big(\int_H \Big[\int_{U_0} \int_H \|(\gamma.\sigma_0)(g_\xi)\Delta_G(\xi)\Delta_G(\xi^{-1}\gamma)^{1/q}\|_q^q \, d\gamma d\nu_G(\sigma) \Big]^{p/q} d\xi \Big)^{q/p} \\
&= \Big(\int_H \Big[\int_{U_0} \int_H \|(\gamma.\sigma_0)(g_\xi)\|_q^q \Delta_G(\gamma) \, d\gamma d\nu_G(\sigma) \Big]^{p/q} \Delta_G(\xi) d\xi \Big)^{q/p}.
\end{aligned}$$

Note that we have tacitly dropped the unitary operators $\sigma_0(\Lambda(\xi\gamma^{-1},\xi))$, since they obviously do not affect the $\|\cdot\|_q$-norm. The inequality is due to Minkowski's generalized inequality. Now, by the measure disintegration (1.2), we find that the inner double integral can be estimated by use of the Hausdorff–Young inequality for N:

$$\begin{aligned}
&\Big(\int_H \Big[\int_{U_0} \int_H \|(\gamma.\sigma_0)(g_\xi)\|_q^q \Delta_G(\gamma) \, d\gamma d\nu_G(\sigma) \Big]^{p/q} \Delta_G(\xi) \, d\xi \Big)^{q/p} \\
&\qquad = \Big(\int_H \|\mathcal{F}^p(g_\xi)\|_q^p \Delta_G(\xi) \, d\xi \Big)^{q/p} \\
&\qquad \le \Big(\int_H A_p^p \|g_\xi\|_p^p \Delta_G(\xi) \, d\xi \Big)^{q/p} = A_p(N)^q \|g\|_p^q,
\end{aligned}$$

which takes care of the first factor in (2.3). For the second factor, we have to compute the cross norms of

$$k_\sigma^*(\xi,\gamma) = \sigma_0(\Lambda(\gamma\xi^{-1},\gamma))^* \circ (\gamma.\sigma_0)(g_{\xi^{-1}\gamma})^* \Delta_G(\gamma\xi^{-1})\Delta_G(\xi)^{1/q}.$$

Here we see that

$$\begin{aligned}
&\int_{\widehat{G}} \|k_\sigma^*\|_{q,p,q}^q \, d\nu_G(\sigma) \\
&= \int_{U_0}\int_H \Big[\int_H \|\sigma_0(\Lambda(\gamma\xi^{-1},\gamma))^*(\gamma.\sigma_0)(g_{\xi^{-1}\gamma})^*\Delta_G(\xi^{-1}\gamma)\Delta_G(\xi)^{1/q}\|_q^p \, d\xi\Big]^{q/p} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad d\gamma d\nu_G(\sigma) \\
&= \int_{U_0}\int_H \Big[\int_H \|(\gamma.\sigma_0)(g_{\xi^{-1}})^*\Delta_G(\xi^{-1})\Delta_G(\xi\gamma)^{1/q}\|_q^p \, d\xi\Big]^{q/p} d\gamma d\nu_G(\sigma) \\
&\le \Big(\int_H \Big[\int_{U_0}\int_H \|(\gamma.\sigma_0)(g_{\xi^{-1}})^*\Delta_G(\xi)^{-1+1/q}\Delta_G(\gamma)^{1/q}\|_q^q \, d\gamma d\nu_G(\sigma)\Big]^{p/q} d\xi\Big)^{q/p} \\
&\le \Big(\int_H \Big[\int_{U_0}\int_H \|(\gamma.\sigma_0)(g_{\xi})^*\Delta_G(\xi)^{1-1/q}\Delta_G(\gamma)^{1/q}\|_q^q \, d\gamma d\nu_G(\sigma)\Big]^{p/q} d\xi\Big)^{q/p} \\
&= \Big(\int_H \Big[\int_{U_0}\int_H \|(\gamma.\sigma_0)(g_{\xi})\|_q^q\Delta_G(\gamma) \, d\gamma d\nu_G(\sigma)\Big]^{p/q} \Delta_G(\xi)\, d\xi\Big)^{q/p} ,
\end{aligned}$$

where the inequality is again the generalized Minkowski inequality, and we have used that taking adjoints and multiplication with unitaries are isometries on $\mathcal{B}_p$. Now we can conclude the proof in the same way as before. ∎

Remark 6 Note that for the case $p = 2$, all inequalities are in fact equalities: that the Hilbert–Schmidt norm of an operator given by an L^2-kernel equals the L^2-norm of the kernel is well known, *i.e.,* instead of (2.3) we have

$$\int_{\widehat{G}} \|\sigma(f)K_\sigma^{1/2}\|_2^2 \, d\nu_G(\sigma) = \int_{\widehat{G}} \|k_\sigma\|_2^2 \, d\nu_G(\sigma).$$

Instead of the generalized Minkowski inequality, we can simply apply Fubini's theorem (since $p = q = 2$), replacing the "$\le$" by "$=$". The last inequality in the argument is now an instance of the Plancherel theorem for N, hence once more an equality. Hence the computation provides a rather concrete proof that $\overline{\nu} = \nu_G$, once the two measures are proven to be equivalent.

Remark 7 The type I assumption on N ensures that we may define the spaces $\mathcal{B}_q^{\oplus}$ as direct integrals of Schatten–von Neumann spaces. In the general case, other traces than the natural operator trace may occur. For this setting, an extension of the norm estimates for Hilbert–Schmidt valued kernels by cross norms, as obtained in [16], to more general traces will be required.

Remark 8 Another situation where Theorem 2 is applicable occurs in the context of simply connected, connected nilpotent Lie groups. Baklouti, Smaoui and Ludwig [2] proved for these groups the estimate

$$A_p(G) \le A_p(\mathbb{R})^{\dim(G)-d^*(G)/2}, \tag{2.4}$$

where $d^*(G)$ is the maximal coadjoint orbit dimension. In particular, let $N \lhd G$ be a connected, codimension 1 normal subgroup of the simply connected, connected Lie group G. Assume that $d^*(N) < d^*(G)$, *i.e.*, $d^*(N) = d^*(G) - 2$. Given $l \in \mathfrak{g}^*$ with maximal orbit dimension, let l_0 denote the restriction to $\mathfrak{n}$, and let

$$\mathfrak{r}_l = \{X \in \mathfrak{g} : l([X,Y]) = 0 \quad \text{for all } Y \in \mathfrak{g}\}$$

denote the *radical* of l in $\mathfrak{g}$. Denote by $\mathfrak{r}_{l_0} \subset \mathfrak{n}$ the radical of l_0 in $\mathfrak{n}$. Then we have that

$$\dim(\mathfrak{r}_l) = \dim(G) - d^*(G)$$

and

$$\dim(\mathfrak{r}_{l_0}) = \dim(N) - \dim(\mathcal{O}_{l_0}) \geq \dim(G) - 1 - (d^*(G) - 2) = \dim(G) - d^*(G) + 1 \ .$$

Thus $\mathfrak{r}_l \subset \mathfrak{n}$ by [4, Proposition 1.3.4], and [4, Theorem 2.5.1(a)] implies that $\pi_l = \mathrm{Ind}_N^G \pi_{l_0}$. Here $\pi_l \in \widehat{G}$ and $\pi_{l_0} \in \widehat{N}$ denote the representations associated to l, l_0 by Kirillov's construction. This holds for all l_0 for which $\dim \mathcal{O}_l = d^*(G)$. But then the set

$$U = \{\pi_{l_0} : \dim(\mathcal{O}_l) = d^*(G)\} \subset \widehat{N}$$

is conull and G-invariant. Here we used the notation π_l for the representation associated to l by the Kirillov construction. Hence, assuming (2.4) for N, we obtain from Theorem 2 that

$$\begin{aligned} A_p(G) \leq A_p(N) = A_p(\mathbb{R})^{\dim(N) - d^*(N)/2} &= A_p(\mathbb{R})^{\dim(G) - 1 - (d^*(G) - 2)/2} \\ &= A_p(\mathbb{R})^{\dim(G) - d^*(G)/2}. \end{aligned}$$

In other words, Theorem 2 provides half of the induction step for the proof of (2.4). The other half would have to deal with the case $d^*(N) = d^*(G)$, which by similar arguments as above implies that the dual action of G/N on $\widehat{N}$ is trivial almost everywhere.

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