FIXED POINTS OF QUASI-NONEXPANSIVE MAPPINGS [1]

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A self-mapping T of a subset C of a normed linear space is said to be *nonexpansive* provided $||Tx-Ty|| \leqq ||x-y||$ holds for all $x, y \in C$. There has been a number of recent results on common fixed points of commutative families of nonexpansive mappings in Banach spaces, for example see DeMarr [6], Browder [3], and Belluce and Kirk [1], [2]. There have also been several recent results concerning common fixed points of two commuting mappings, one of which satisfies some condition like nonexpansiveness while the other is only continuous, for example see DeMarr [5], Jungck [8], Singh [11], [12], and Cano [4]. These results, with the exception of Cano's, have been confined to mappings from the reals to the reals. Some recent results on common fixed points of commuting analytic mappings in the complex plane have also been obtained, for example see Singh [13] and Shields [10].

Our purpose in this paper is to show that similar results can be obtained, in the general setting of a normed linear space, even when the hypothesis of nonexpansiveness is considerably weakened. Essentially, we show that part of the analysis (which is involved in some of the above mentioned results) does not require the full force of nonexpansiveness, but requires only the existence of at least one fixed point together with nonexpansiveness only about each fixed point.

DEFINITION. A self-mapping T of a subset C of a normed linear space is said to be *quasi-nonexpansive* provided T has at least one fixed point in C, and if $p \in C$ is any fixed point of T then $||Tx-p|| \leqq ||x-p||$ holds for all $x \in C$.

This concept which we have labeled quasi-nonexpansiveness was essentially introduced (along with some related ideas) by Diaz and Metcalf [7]. One notes that a nonexpansive mapping $T : C \to C$ with at least one fixed point in C is quasi-nonexpansive, and that a linear quasi-nonexpansive mapping on a subspace is nonexpansive on that subspace; but there exist continuous and discontinuous nonlinear quasi-nonexpansive mappings which are not nonexpansive, e.g. the

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mapping T from the reals to the reals defined by $T(x) = (x/2)\sin(1/x)$, $x \neq 0$, $T(0) = 0$. If T is a self-mapping of a set C, we use $F(T)$ to denote the set of all points in C which are fixed points of T.

THEOREM 1. *If C is a closed convex subset of a strictly convex normed linear space, and $T : C \to C$ is quasi-nonexpansive, then $F(T) = \{p : p \in C \text{ and } Tp = p\}$ is a nonempty closed convex set on which T is continuous.*

PROOF. It follows immediately from the definition of quasi-nonexpansiveness that $F(T) \neq \emptyset$ and that T is continuous at each $p \in F(T)$. Suppose $F(T)$ is not closed. Then there is a limit point x of $F(T)$ which is not in $F(T)$. Since C is closed, $x \in C$; and so $x \notin F(T)$ implies $Tx \neq x$. Let $r = (\frac{1}{3})||Tx-x|| > 0$. There exists $y \in F(T)$ such that $||x-y|| < r$. Since T is quasi-nonexpansive we have

$$||Tx-y|| \leqq ||x-y|| < r,$$

and hence we get

$$3r = ||Tx-x|| \leqq ||Tx-y||+||y-x|| < 2r.$$

This contradiction establishes that $F(T)$ is closed.

We now prove that $F(T)$ is convex. Suppose $a, b \in F(T)$, $a \neq b$, and $0 < t < 1$. Then

$$c = (1-t)a+tb \in C$$

since C is convex. Since T is quasi-nonexpansive we have

$$||Tc-a|| \leqq ||c-a|| \text{ and } ||Tc-b|| \leqq ||c-b||.$$

Noting that $c-a = t(b-a)$ and $c-b = (1-t)(a-b)$, we have

$$||b-a|| \leqq ||b-Tc||+||Tc-a|| \leqq ||c-b||+||c-a|| = ||b-a||.$$

Hence, we get

$$||(b-Tc)+(Tc-a)|| = ||b-Tc||+||Tc-a||.$$

If $b-Tc = 0$, then $||Tc-a|| = ||b-a|| \leqq ||c-a|| = t||b-a||$, whence $1 \leqq t$ which is not true. Similarly, $Tc-a = 0$ implies $1 \leqq 1-t$, whence $t \leqq 0$ which is not true. Thus, since the space is strictly convex, there exists $r > 0$ such that $Tc-a = r(b-Tc)$; whence $Tc = (1-s)a+sb$ where $s = r/(1+r)$. We have $Tc-a = s(b-a)$, and so

$$s||b-a|| = ||Tc-a|| \leqq ||c-a|| = t||b-a||,$$

which gives $s \leqq t$.

Using $Tc-b = (1-s)(a-b)$, a similar argument gives $s \geqq t$.

Thus $s = t$, and so $Tc = (1-t)a+tb = c$, i.e. $c \in F(T)$.

THEOREM 2. *If C is a compact convex subset of a strictly convex normed linear space, and T is a quasi-nonexpansive self-mapping of C, and S is a continuous self-mapping of C, and $TS = ST$, then $F(T) \cap F(S) \neq \emptyset$.*

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PROOF. By Theorem 1, $F(T)$ is a nonempty closed convex subset of the compact set C. Since $TS = ST$ we have $S(F(T)) \subset F(T)$. Hence, by the Tychonoff fixed point theorem, S has a fixed point in $F(T)$.

REMARK 1. As special cases, Theorem 2 contains two theorems of Singh [11] and part of a result of DeMarr [5].

REMARK 2. We here give an example to show that strict convexity is necessary in Theorem 2 even when S is nonexpansive. This is of some interest since DeMarr [6] has shown that if both S and T are nonexpansive then strict convexity is not necessary (to insure the existence of a common fixed point). In the (not strictly convex) Banach space l^∞ (with $\|\{x_n\}\| = \sup_n |x_n|$) let C be the compact convex set

$$\{\{x_n\} : -1 \leqq x_1 \leqq 3,\ -1 \leqq x_2 \leqq 1,\ x_n = 0 \text{ for all } n > 2\}.$$

Define $S : C \to C$ by $S(\{x_n\}) = (2-x_1, x_2, 0, \cdots)$ for all $\{x_n\} \in C$, and define $T : C \to C$ by

$$T(x_1, x_2, 0, \cdots) = (x_1, -x_2, 0, \cdots) \qquad \text{for } x_2 \neq 0,$$

$$T(x_1, 0, 0, \cdots) = \begin{cases} (x_1, |x_1|, 0, \cdots) & \text{for } -1 \leqq x_1 \leqq 1, \\ (x_1, |x_1-2|, 0, \cdots) & \text{for } 1 \leqq x_1 \leqq 3. \end{cases}$$

Then S is nonexpansive, T is quasi-nonexpansive, and $TS = ST$. We have $F(T) = \{(0, 0, 0, \cdots), (2, 0, 0, \cdots)\}$, and $F(S) = \{(1, x_2, 0, \cdots) : -1 \leqq x_2 \leqq 1\}$, so that $F(T) \cap F(S) = \emptyset$.

THEOREM 3. *If C is a closed bounded convex subset of a uniformly convex Banach space, and T is a quasi-nonexpansive self-mapping of C, and S is a self-mapping of C which is either nonexpansive or weakly continuous, and $TS = ST$, then $F(T) \cap F(S) \neq \emptyset$.*

PROOF. Since uniform convexity implies strict convexity, we have by Theorem 1 that $F(T)$ is a nonempty closed convex subset of the bounded set C. Since $TS = ST$, we have $S(F(T)) \subset F(T)$. If S is nonexpansive then by the Browder-Kirk fixed point theorem (Browder [3], Kirk [9]) S has a fixed point in $F(T)$. Suppose now that S is weakly continuous. Since uniformly convex Banach spaces are reflexive, $F(T)$ is weakly compact. Since the weak topology is locally convex Hausdorff, the Tychonoff fixed point theorem gives us that S has a fixed point in $F(T)$.

Our final result is closely related to results of DeMarr [6], Browder [3], and Belluce and Kirk [1].

THEOREM 4. *If C is a weakly compact convex subset of a strictly convex normed linear space, and $\{T_\alpha\}$ is a commutative family of quasi-nonexpansive self-mappings of C, then $\bigcap_\alpha F(T_\alpha) \neq \emptyset$.*

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PROOF. By Theorem 1, each $F(T_\alpha)$ is nonempty, closed, and convex; hence each $F(T_\alpha)$ is weakly closed. Thus, since C is weakly compact, it will be sufficient to show that the collection $\{F(T_\alpha)\}$ has the finite intersection property. With the inductive hypothesis that any n of these sets have nonempty intersection, consider any $n+1$ of the sets $F(T_1), F(T_2), \cdots, F(T_{n+1})$. Let $D = \bigcap_{i=1}^{n} F(T_i) \neq \emptyset$. Clearly D is weakly closed (since closed and convex), and therefore, since $D \subset C$, D is weakly compact. Since $T_{n+1}T_i = T_iT_{n+1}, i = 1, 2, \cdots, n$, we have $T_{n+1}(F(T_i)) \subset F(T_i), i = 1, 2, \cdots, n$, and hence $T_{n+1}(D) \subset D$. Choose any point p in the nonempty set $F(T_{n+1})$. Since D is convex and weakly compact, and since our normed linear space is strictly convex, there is a unique point $q \in D$ nearest to p. Since T_{n+1} is quasi-nonexpansive, we have $||T_{n+1}q-p|| \leqq ||q-p||$; and $q \in D$ implies $T_{n+1}q \in D$. Thus $T_{n+1}q = q$, and so $\bigcap_{i=1}^{n+1} F(T_i) \supset \{q\}$.

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