COMPOSITIO MATHEMATICA

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Compositio Math. **146** (2010), 772–794.

doi:10.1112/S0010437X09004515

FOUNDATION COMPOSITIO MATHEMATICA

The London Mathematical Society

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Abstract

We show that the residue at $s=0$ of the standard intertwining operator attached to a supercuspidal representation $\pi \otimes \chi$ of the Levi subgroup $\mathrm{GL}_2(F) \times E^1$ of the quasisplit group $\mathrm{SO}_6^*(F)$ defined by a quadratic extension E/F of p-adic fields is proportional to the pairing of the characters of these representations considered on the graph of the norm map of Kottwitz–Shelstad. Here π is self-dual, and the norm is simply that of Hilbert's theorem 90. The pairing can be carried over to a pairing between the character on E^1 and the character on $E^\times$ defining the representation of $\mathrm{GL}_2(F)$ when the central character of the representation is quadratic, but non-trivial, through the character identities of Labesse–Langlands. If the quadratic extension defining the representation on $\mathrm{GL}_2(F)$ is different from E the residue is then zero. On the other hand when the central character is trivial the residue is never zero. The results agree completely with the theory of twisted endoscopy and L-functions and determines fully the reducibility of corresponding induced representations for all s.

1. Introduction

One of the major tools in representation theory of reductive groups, over either a local or a global field, is the theory of intertwining operators between parabolically induced representations. These are vector-valued meromorphic functions which are basically a composite of functions of one variable. Harish-Chandra's theory of c-functions, which is merely another name for intertwining operators, connects these important objects to both reducibility questions for the inducing spaces as well as Plancherel measures [Har84, Sil79].

On the other hand, the work developed by the first author (cf. [Sha90]) connects these questions to arithmetic through L-functions [Sha08] and their poles. In fact, one can use the information gathered from harmonic analysis (poles of intertwining operators) to define these L-functions and conversely [Sha92]. Following [Sha92], in a series of papers [GS98, GS01, GS], Goldberg and the first author computed the residues at $s=0$ for these operators in the cases where the group is a quasisplit classical group, the parabolic subgroup is maximal and the inducing data is supercuspidal. This gives the rank one setting necessary to determine the Plancherel measure and R-groups of Knapp–Stein and Harish-Chandra, as well as the L-functions (when the inducing data is generic).

One particularly important feature of this case is its connection with the theory of (twisted) endoscopy of Kottwitz–Shelstad [KS99] and Langlands, which reflects itself in the

Received 18 March 2009, accepted in final form 12 August 2009, published online 20 April 2010.
2000 Mathematics Subject Classification 22E50 (primary).
Keywords: p-adic representation theory, intertwining operators, character identities, L-functions.
The first author was supported by NSF Grant DMS-0700280.

functorial transfer from quasisplit classical groups to GL_n, as established in [CKPS04] in the generic case and by Arthur [Art] in general. The corresponding L-functions are then those of Artin [HT01, Hen93, Sou05].

While significant progress was made in [GS98, GS01, GS], the connection with endoscopy remained conjectural. In fact, the residue was reduced to a sum of two terms R_G and R_{sing}, with R_G as an integral of products of twisted orbital integrals on GL_n with orbital integrals on the classical group, related by the norm map, and R_{sing} as a limit in the boundary [GS98, GS01, GS]. But, how the non-vanishing of either term implies the connection with the inducing data being transferred from each other, as predicted by endoscopy, was not clear.

On the other hand, the efforts of the second author who pursued the results in [GS98, GS01, GS], with the goal of a precise interpretation of these residues, led to a very promising reformulation of these in [Spa08].

The purpose of this paper is to completely verify these conjectures in a low-dimensional but important case. The results are under no assumptions since the character identities needed for the (twisted) endoscopic transfer in this case were already proved by Labesse–Langlands [LL79, Lan80]. One remarkable identity appears when the central character of the supercuspidal representation on the GL(2) part of the inducing Levi subgroup is quadratic, but non-trivial. It expresses these residues precisely as Harish-Chandra's orthogonality pairing, as we explain below. The results are in complete agreement with what is predicted by the theory of L-functions as explained in [Sha08] (appendix to [Spa08]).

Let E/F be a quadratic extension of a p-adic local field of characteristic zero. Let $\tilde{G} = \mathrm{SO}_6^*$ be the quasisplit special orthogonal group of (split) rank two determined by E/F. Let $\tilde{B} = \tilde{T}\tilde{U}$ be the Borel subgroup of upper triangular elements in $\tilde{G}$, where $\tilde{T}$ is the Cartan subgroup of diagonals in $\tilde{B}$ and $\tilde{U}$ its unipotent radical. Let $M = \mathrm{GL}_2 \times \mathrm{SO}_2^*$ be the Levi subgroup of $\tilde{G}$ generated by the root $e_1 - e_2$ of $\tilde{T}$ (or A_0). If P is the parabolic subgroup of $\tilde{G}$ with a Levi decomposition $P = MN$, $N \subset \tilde{U}$, then the simple root α in N is simply the restriction of either $e_2 - e_3$ or $e_2 + e_3$ to A_0. We let $\tilde{\alpha} = \langle \rho_P, \alpha \rangle^{-1} \rho_P$ as in [Sha90], where ρ_P is half the sum of roots in N.

Let π be an irreducible supercuspidal representation of $G(F) = \mathrm{GL}_2(F)$ and χ_H a character of the torus $H(F) = \mathrm{SO}_2^*(F) = E^1$, the subgroup of elements of norm one in $E^\times$. We are interested in

$$I(s\tilde{\alpha}, \pi \otimes \chi_H) = \mathrm{Ind}_{M(F)N(F)}^{\tilde{G}(F)} \pi \otimes \chi_H \otimes q^{\langle s\tilde{\alpha}, H_M(\ )\rangle} \otimes \mathbf{1},$$

where $s \in \mathbb{C}$. In particular, we would like to determine its points of reducibility by means of Plancherel measures [Sha90]. This simply means determining the poles of the intertwining operators $A(s\tilde{\alpha}, \pi \otimes \chi_H, w_0)$ at $s = 0$, where $w_0 = w_\ell \cdot w_{\ell,\theta}^{-1}$ and $\theta = \{e_1 - e_2\}$.

The operator will have poles only if $w_0(\pi) \simeq \pi$, i.e. π is self-dual or equivalently $\omega_\pi^2 = 1$, where $\omega = \omega_\pi$ is the central character of π. Consequently, $I(s\tilde{\alpha}, \pi \otimes \chi_H)$ will have reducibility points only if $\omega^2 = 1$.

When $\omega \neq 1$, then π will be the Weil representation attached to $\mathrm{Ind}_{W_{E'}}^{W_F} \chi'$, where E' is a quadratic extension of F and χ' is a character of $(E')^\times$. Moreover, if G_+ is the subgroup of $\mathrm{GL}_2(F)$ consisting of elements g for which $\det(g)$ is a norm from E', then the restriction $\pi|_{G_+}$ decomposes into the sum $\pi_+ \oplus \pi_-$ of two inequivalent representations. Write $\Theta_{\pi_\pm}$ for the character of $\pi_\pm$, and σ for the non-trivial automorphism of E.

https://doi.org/10.1112/S0010437X09004515 Published online by Cambridge University Press

The main result of this paper is the following.

THEOREM 1. *The operator $A(s\tilde{\alpha}, \pi \otimes \chi_H)$ is holomorphic at $s = 0$, unless $\omega^2 = 1$.*

(a) *Assume that $\omega = 1$. Then $A(s\tilde{\alpha}, \pi \otimes \chi_H)$ has a simple pole at $s = 0$. (We assume that the residual characteristic of F is odd in this case, for simplicity.)*

(b) *Assume that $\omega^2 = 1$, but $\omega \neq 1$. Let E' and χ' be as above. The residue of the operator $A(s\tilde{\alpha}, \pi \otimes \chi_H)$ at $s = 0$ is zero unless $E = E'$ and the central character of π is sgn_E. Then $\chi'(x) = \chi_G(x/\sigma(x))$ for some character χ_G of E^1. In this case the residue is proportional to*

$$\int_{E^1} \chi_H(\gamma) \cdot \Delta_E(\tilde{\gamma})(\Theta_{\pi_+}(\tilde{\gamma}) - \Theta_{\pi_-}(\tilde{\gamma}))\, d\gamma, \tag{1.1}$$

where $\gamma = \sigma(\tilde{\gamma})/\tilde{\gamma}$ is the norm of the element $\tilde{\gamma} \in E^\times$ through the norm map $\tilde{\gamma} \mapsto \gamma$ of Kottwitz–Shelstad, which in this case is the map $F^\times \backslash E^\times \twoheadrightarrow E^1$ of Hilbert's theorem 90. Here Δ_E is the discriminant for $E^\times$ as a Cartan subgroup in $\mathrm{GL}_2(F)$. In particular, using the character identities of Labesse–Langlands for the transfer $\chi_G \to \pi$, the residue is proportional to

$$\int_{E^1} \chi_H(\gamma)(\chi_G(\gamma) + \chi_G^{-1}(\gamma))\, d\gamma,$$

and therefore the residue is non-zero precisely when π is attached to $\operatorname{Ind}_{W_E}^{W_F} \chi_H$ as predicted by endoscopy and L-functions.

An immediate consequence of this theorem is the following reducibility criterion (cf. [Sha90]).

PROPOSITION 1. *The induced representation $I(s\tilde{\alpha}, \pi \otimes \chi_H)$ is irreducible unless $\omega^2 = 1$.*

(a) *Suppose that $\omega = 1$. Then $I(\pi \otimes \chi_H)$ is irreducible. In this case $I(\frac{1}{2}\tilde{\alpha}, \pi \otimes \chi_H)$ is reducible and there are no other points of reducibility for $s \geqslant 0$.*

(b) *Suppose that $\omega^2 = 1$, but $\omega \neq 1$. Then $I(\pi \otimes \chi_H)$ is reducible unless $E = E'$ and π is attached to $\operatorname{Ind}_{W_E}^{W_F} \chi_H$. If π is attached to $\operatorname{Ind}_{W_E}^{W_F} \chi_H$, then $I(\tilde{\alpha}, \pi \otimes \chi_H)$ is reducible and there are no other points of reducibility for $s \geqslant 0$.*

COROLLARY 1. *Our results are in complete agreement with those given in [Sha08] using L-functions.*

We remark that this paper gives a purely local proof of these results, whereas the method of L-functions is necessarily global.

The proof of the theorem is to apply the general formula obtained in [Spa08] to this case. This is fairly non-trivial. The non-vanishing in the case $\omega = 1$ requires a bulky proof, as one needs to use character values for π. In general, one clearly needs a more efficient way of proving the non-vanishing by relating the residue to the non-vanishing of the corresponding singular twisted orbital integral in [Sha92], which gives the poles for the second L-functions $L(s, \pi, \Lambda^2)$ or $L(s, \pi, \mathrm{Sym}^2)$, where π is the representation on $\mathrm{GL}_n(F)$.

We conclude our discussion of the case $\omega = 1$ by pointing out that in this case the residue becomes a pairing between the character of π, a representation of $\mathrm{GL}_2(F)$, and χ_H, a representation of $\mathrm{SO}_2^*(F)$, which is not a natural one. This could justify the bulky calculations one has to deal with in proving the non-vanishing.

On the other hand, when $\omega \neq 1$, the pairing (1.1) comes out very naturally, and assuming the character identities generalizing those in [LL79, Lan80], (1.1) seems to be amenable to

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generalization. It seems that the residue formulas proven in [GS98, GS01, GS], as reformulated in [Spa08], are naturally suitable to detect the poles of the first L-function $L(s, \pi \times \chi_H)$ and their generalizations, rather than the second L-functions $L(s, \pi, \Lambda^2)$ or $L(s, \pi, \mathrm{Sym}^2)$. In this paper, we have found that $R_{\mathrm{sing}} = 0$, and the residue comes entirely from R_G. In cases of higher rank, we do not expect R_{sing} to vanish.

Although the present work only deals with a low-rank case, it still brings in many features of the general case, and is the first case where such subtle character identities appear so explicitly as a residue. With the reformulation presented in [Spa08], and the present example complete, we plan to complete this project in our future work. In general, one has a contribution to the residue for each maximal torus in H. One may treat the terms corresponding to compact tori through the methods of this paper. A more difficult matter is to study the weight factors of [Spa08], which diverge for non-compact tori. We expect to apply a more delicate limiting process for these, and that *weighted* orbital integrals [Art88] will play the role that ordinary orbital integrals do in the present paper. The result should again be a pairing of a twisted character value on G with an ordinary character value on H at non-elliptic elements. One appeals to the theory of twisted endoscopic transfer in this situation; if π_G arises from π_H via endoscopic transfer, this should point to the non-vanishing of the residue. This is work in progress.

We now describe the layout of this paper.

In § 2, in addition to setting up notation, we describe the norm correspondence and compute its Jacobian, for our particular case. We emphasize that the calculation of measures through the Jacobian in particular is very delicate but vital for the matching that must occur with the character identities.

In § 3, we recall the formula from [Spa08] for the residue. In our case it simplifies considerably. The conclusion of this section is that the residue problem reduces to the integral

$$R(\pi, \chi_H) = \frac{1}{2 \log q} \int_{E^1} \chi_H(\gamma) \Theta_\pi^\varepsilon(S(\gamma)^{-1})\, d^*\gamma.$$

Here Θ_π^ε is the twisted character associated with the self-dual representation π. The measure $d^*\gamma$ is proportional to $|\mathrm{Tr}(\gamma) - 2|^{\frac{1}{2}}\, d\gamma$, where $d\gamma$ is a normalized Haar measure on E^1.

The case of non-trivial central character is dealt with in § 4. In this case the twisted character value is a difference of two character values, and lines up directly with the work of Labesse and Langlands as mentioned above.

Section 5 sets up the case of trivial central character. Here the twisted character value is merely the usual character value at $\tilde{\gamma}$. However, the Jacobian weights the pairing to preclude the vanishing for any choice of χ_H and χ'. Section 6 treats the case for which $E \neq E'$, and § 7 treats the case for which $E = E'$. These sections provide a direct annulus computation using the character formulas of [Shi77, Sil70].

2. Preliminaries and notation

2.1 The group SO_2^*

Let F be a p-adic field with ring of integers $\mathcal{O}_F$, maximal ideal $\mathfrak{p}_F$ and residue field k of order q. Let $G = \mathrm{GL}_2(F)$. Write w for the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. For $g \in M_2(F)$, write $g^\vdash$ for $w^t g w^{-1}$. Let ε be the involution of G given by $\varepsilon(g) = (g^\vdash)^{-1}$. We say that elements $x, y \in G$ are ε-conjugate if there is a $g \in G$ so that $g x g^\vdash = y$.

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Fix a quadratic extension E of F. Write σ for the non-trivial automorphism of E, and Nm and Tr for the norm and trace maps from E to F. Let E^1 denote the elements of E of norm one. We now specify a subgroup H of G isomorphic to E^1. Assume the orthogonal form $J = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -\tau \end{smallmatrix}\right)$ for some non-square $\tau \in \mathcal{O}_F - \mathfrak{p}_F^2$. Then our group $H = \mathrm{SO}_2^* = \mathrm{SO}(J)$ is given by matrices of the form

$$\begin{pmatrix} a & b \\ b\tau & a \end{pmatrix} \tag{2.1}$$

corresponding to $a + b\sqrt{\tau} \in E^1$. Note that if $h \in H$ then $h^{\vdash} = h$. Throughout this paper we will often identify E with the set of matrices of the form (2.1) with $a, b \in F$. Write $g_\theta = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$; then the full orthogonal group is $H^+ = O^*(J) = H \cup g_\theta H$. Note that $g_\theta h g_\theta^{-1} = \sigma(h)$ for $h \in H$. Since H is commutative in this case, we also write $T = H$. Put $T' = T - \{1\}$.

2.2 The norm correspondence for SO_6^*

Put

$$\tilde{J} = \begin{pmatrix} & & w \\ & J & \\ w & & \end{pmatrix} \in \mathrm{GL}_6(F)$$

and $\tilde{G} = \mathrm{SO}(\tilde{J}) = \{g \in \mathrm{SL}_6(F) \mid g\tilde{J}{}^t g = \tilde{J}\}$.

Let M be the Levi subgroup of $\tilde{G}$ consisting of matrices of the form

$$\begin{pmatrix} g & & \\ & h & \\ & & \varepsilon(g) \end{pmatrix},$$

with $g \in G$ and $h \in H$. Write P for the parabolic subgroup generated by M and the Borel subgroup $\tilde{B}$ of upper triangular matrices in $\tilde{G}$. Then $P = MN$, where N is the subgroup of matrices of the form

$$n(X, Y) = \begin{pmatrix} I & X & Y \\ & I & X' \\ & & I \end{pmatrix}$$

in $\tilde{G}$. Here $X, X', Y \in M_2(F)$. The condition that $n(X, Y) \in \tilde{G}$ gives the equations

$$X' = -J{}^t X w \quad \text{and} \quad Y + Y^{\vdash} = XX'. \tag{2.2}$$

The integration over N defining our intertwining operator is decomposed into orbits via the action of the Levi subgroup M in [GS98]. The set of these orbits is equivalent to the set of ε-conjugacy classes of elements Y for which there is a solution to (2.2). We may parameterize these classes very simply by means of the the norm correspondence from [GS98].

Lemma 1. *Let (X, Y) be a pair of invertible 2×2 matrices satisfying (2.2), with Y invertible, and put* $\mathrm{Norm}(X, Y) = I - X'Y^{-1}X$.

(i) *We have* $\mathrm{Norm}(X, Y) \in T'$.

(ii) *If $g \in G$, then the pair $(g^{-1}X, g^{-1}Y\varepsilon(g))$ satisfies (2.2), and*

$$\mathrm{Norm}(X, Y) = \mathrm{Norm}(g^{-1}X, g^{-1}Y\varepsilon(g)).$$

(iii) *If X is invertible, and (X_1, Y) is also a solution to (2.2), then* $\mathrm{Norm}(X_1, Y) = \mathrm{Norm}(X, Y)$ *or* $\mathrm{Norm}(X_1, Y) = \mathrm{Norm}(Xg_\theta, Y) = \mathrm{Norm}(X, Y)^{-1}$.

Proof. It is straightforward to see that $\mathrm{Norm}(X,Y)\in H^+$. Since each $\mathrm{Norm}(X,Y)-I$ is invertible, we obtain (i). Part (ii) is a computation.

The hypothesis for part (iii) implies that $XJ^tX = X_1J^tX_1$, which implies that $X_1 = Xh$ with $h\in H^+$. It follows that $\mathrm{Norm}(X_1,Y) = h^{-1}\,\mathrm{Norm}(X,Y)h$, and the conclusion follows. $\square$

Write $\mathcal{N}_r$ for the set of ε-conjugacy classes $[Y]$ of invertible matrices Y for which (2.2) have a solution for an invertible matrix X. Write T'/σ for the quotient of T' under inversion (also conjugation, since $T=E^1$). Then Lemma 1 shows that N_ε gives a well-defined map from $\mathcal{N}_r$ to T'/σ, given by $N_\varepsilon([Y]) = \{\mathrm{Norm}(X,Y),\mathrm{Norm}(X,Y)^{-1}\}$. We will show that this is a bijection. To understand the fibres of this map over T'/σ, we relate it to the map $\nu: G\to G$ given by $\nu(g) = g^{\vdash}g^{-1}$.

LEMMA 2. *Suppose that* $[Y_1],[Y_2]\in\mathcal{N}_r$, *and that* $N_\varepsilon([Y_1]) = N_\varepsilon([Y_2])$. *Then* $[Y_2]$ *contains an element* Y_3 *so that* $\nu(Y_1)=\nu(Y_3)$.

Proof. For $i=1,2$, pick invertible X_i satisfying condition (2.2) with Y_i. Let $\gamma_i = \mathrm{Norm}(X_i,Y_i)$. The hypothesis implies that $\gamma_2 = h\gamma_1h^{-1}$, with $h=1$ or $h=g_\theta$.

By Lemma 3.3 of [GS98], we have

$$X_i\gamma_i = -\nu(Y_i)X_i.$$

Put $Y_3 = (X_1h^{-1}X_2^{-1})Y_2(X_1h^{-1}X_2^{-1})^{\vdash}$; then a calculation shows that $\nu(Y_1)=\nu(Y_3)$. $\square$

We therefore compute the fibres of the map ν.

LEMMA 3. *Fix a quadratic extension* E *over* F, *and let* $g_1\in E^\times$ *with* $\sigma(g_1)\neq\pm g_1$. *Let* $\delta_1 = g_1g_\theta$ *and* $\gamma=\nu(\delta_1)$. *Then the fibre of* ν *over* γ *is equal to* Zg_1, *where* Z *is the center of* G.

Proof. Note that $\gamma\in E^1-\{\pm1\}$. Suppose that $\delta_2\in G$ with $\nu(\delta_2)=\gamma$. This can be rewritten as $\delta_2^{\vdash}=\gamma\delta_2$. Since $\gamma^{\vdash}=\gamma$, we obtain $\delta_2=\delta_2^{\vdash}\gamma$. Substituting, it follows that $\delta_2=\gamma\delta_2\gamma$. Now let $\delta_2 = g_2g_\theta$, for some $g_2\in G$. This gives $g_2g_\theta = \gamma g_2g_\theta\gamma = \gamma g_2\sigma(\gamma)g_\theta$. This implies that $g_2=\gamma g_2\gamma^{-1}$, and therefore $g_2\in E^\times$. Now this gives

$$\frac{\sigma(g_1)}{g_1} = \frac{\sigma(g_2)}{g_2} = \gamma,$$

and it follows that $(g_1/g_2)\in F^\times$, as desired. $\square$

Note that if $z\in Z$, then g is ε-conjugate to z^2g. Actually, we know a little bit more.

LEMMA 4. *Let* $\delta\in E^\times\cdot g_\theta$, *with* $\nu(\delta)=\gamma\in E^1-\{\pm1\}$, *and* $\alpha\in F^\times$. *Then* $\alpha\delta$ *is* ε*-conjugate to* δ *if and only if* α *is a norm of* $E^\times$.

Proof. Suppose that $\alpha=\mathrm{Nm}(\beta)$ with $\beta\in E^\times$. Then combining the facts that $\beta^{\vdash}=\beta$, and that $\beta g_\theta = g_\theta\sigma(\beta)$, we deduce that $\beta\delta\beta^{\vdash}=\alpha\delta$. For the other direction, suppose that

$$g\delta g^{\vdash} = \alpha\delta, \tag{2.3}$$

and apply ν to both sides. We obtain $g\gamma g^{-1}=\gamma$, so that g commutes with γ, and it follows that $g\in E^\times$. Viewing g as an element of $E^\times$, we write $\sigma(g)$ for its conjugate. Now, as above, (2.3) implies that $g\sigma(g)\delta=\alpha\delta$, and therefore α is the norm of g. $\square$

In other words, there are two ε-conjugacy classes in the fibre of ν over such γ, corresponding to the two elements of $F^\times/\mathrm{Nm}(E^\times)$. However, we will soon see that there is only one ε-conjugacy class for the norm correspondence.

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Definition 1. For $\gamma \in T'$, we define $S(\gamma) = wJ^{-1}(\gamma - 1)$.

The significance of $S(\gamma)$ comes from the following proposition.

Proposition 2. *The norm correspondence $N_\varepsilon : \mathcal{N}_r \to T'/\sigma$ is a bijection. More precisely, if $\gamma \in T'$, then the fibre of the norm correspondence over γ is the singleton $[S(\gamma)^{-1}]$.*

Proof. Let $Y = S(\gamma)^{-1} = (\gamma - 1)^{-1} J w^{-1}$. Then (I, Y) satisfies (2.2), and $\mathrm{Norm}(I, Y) = \gamma$, since $Y + Y^{\vdash} = -Jw$ and $I - I'Y^{-1} = \gamma$, as the reader may verify.

By Lemma 2, if $[Y_1] \in \mathcal{N}_r$ with $N_\varepsilon(Y_1) \ni \gamma$, then we may assume that $\nu(Y_1) = \nu(Y)$. Then, by Lemma 3, there is an $\alpha \in F^\times$ so that $Y_1 = \alpha Y$. Therefore, we must look for solutions X to $\alpha Y + \alpha Y^{\vdash} = XX'$. This leads to the equation

$$\alpha J = XJX^T;$$

in other words, such an X exists if and only if the quadratic forms $x_1^2 - \tau x_2^2$ and $\alpha x_1^2 - \alpha\tau x_2^2$ are F-equivalent. Following [Ser73], we compute the Hasse–Witt invariants. The invariant for the first is $(1, -\tau) = 1$ and the invariant for the second is $(\alpha, -\alpha\tau) = (\alpha, -\alpha)(\alpha, \tau) = (\alpha, \tau)$. We are using here the Hilbert symbol and its elementary properties. Therefore, such an X exists if and only if α is a norm of $E^\times$. But then, by Lemma 4, αY is ε-conjugate to Y. $\square$

The element $S(\gamma)^{-1}$ is not in $E^\times$, but can be written in the form $\tilde{\gamma} g_\theta$, with $\tilde{\gamma} \in E^\times$. The relationship between γ and $\tilde{\gamma}$ will play an important role in this paper, so we gather together a few properties. Let $E = F[\sqrt{\tau}\,]$ be any quadratic extension of F, with τ a non-square in $\mathcal{O}_F - \mathfrak{p}_F^2$. Suppose that $\gamma = a + b\sqrt{\tau} \in E^1$, with $a, b \in F$ and $\gamma \neq \pm 1$. Throughout this paper, we set

$$\tilde{\gamma} = \frac{\sqrt{\tau}}{1 - \gamma}.$$

Lemma 5. *We have the following facts about γ and $\tilde{\gamma}$.*

(i) $\widetilde{\sigma(\gamma)} = -\sigma(\tilde{\gamma})$, $\tilde{\gamma} - \sigma(\tilde{\gamma}) = \sqrt{\tau}$ *and* $\sigma(\tilde{\gamma})/\tilde{\gamma} = \gamma$.

(ii) *We have* $\mathrm{Nm}(\tilde{\gamma}) = \tau/(\mathrm{Tr}(\gamma) - 2)$ *and* $\mathrm{Tr}(\tilde{\gamma}) = -b\tau/(a - 1)$.
For the next three items, suppose further that $|\gamma - 1| < 1$, and let $x = a - 1$.

(iii) *If* $\mathrm{ord}(\tau) = 1$, *then* $\mathrm{ord}(x)$ *is odd, and* $\mathrm{ord}(x) = 2k + 1 \Leftrightarrow \mathrm{ord}_E(\gamma - 1) = 2k + 1$.

(iv) *If τ is a unit, then* $\mathrm{ord}(x)$ *is even, and* $\mathrm{ord}(x) = 2k \Leftrightarrow \mathrm{ord}_E(\gamma - 1) = k$.

(v) *There is a square root $\lambda \in F^\times$ of* $\mathrm{Nm}(\tilde{\gamma})$, *so that* $\lambda^{-1} = (-b + x^2/2b\tau) \bmod x^2$.

Proof. The first two, (i) and (ii), are immediate. The condition $|\gamma - 1| < 1$ implies that $x \in \mathfrak{p}_F$. Since $\gamma \in E^1$, we have

$$\tau b^2 = 2x + x^2. \tag{2.4}$$

Therefore, we see that $|x| = |\tau b^2|$, from which it follows that $|\gamma - 1| = |\tau b^2|^{\frac{1}{2}} = |x|^{\frac{1}{2}}$. This gives (iii) and (iv). As for (v), it is straightforward to check that $\mathrm{Nm}(\tilde{\gamma})^{-1} = 2x/\tau$ has a square root: one may divide both sides of (2.4) by $\varpi^{2\,\mathrm{ord}(b)}\tau$ and then use Hensel's lemma. The rest is an easy calculation. $\square$

2.3 Jacobians and measures

As we shall see later, to get the residues as an orthogonality relation as predicted in [Sha08] when the central character is non-trivial, the Jacobians and measures must match precisely. The purpose of this section is to verify that this is in fact the case. We start by recalling the measures that show up in [GS98]. With notation as in § 2.2 and the discussion in [GS98], the invariant measure for integration in (4.2) of [GS98] is

$$d^*(X,Y) = |\det Y|_F^{-\langle \rho_P, \tilde{\alpha}\rangle}\, d(X,Y).$$

Here $d(X,Y)$ is the euclidean measure on $N(F)$. For us, $Y = \tilde{\gamma} g_\theta$ and $d(X,Y) = dg\, d\tilde{\gamma}$.

We first recall that in the situation in hand, M is generated by $e_1 - e_2$ and one can take either $\alpha' = e_2 - e_3$ or $e_2 + e_3$ as the non-restricted root restricting to α. Note that $\rho_P = \frac{3}{2}(e_1 + e_2)$. Then $\langle \rho_P, \alpha\rangle = \frac{3}{2}$, so $\tilde{\alpha} = \langle \rho_P, \alpha\rangle^{-1}\rho_P = e_1 + e_2$. Consequently, $\langle \rho_P, \tilde{\alpha}\rangle = 3$. (We remark that for $\mathrm{GL}_n \times \mathrm{SO}_n^*$ as Levi in SO_{3n}^*, $\langle \rho_P, \tilde{\alpha}\rangle = n - \frac{1}{2}$, when $n > 2$ is even.) This gives

$$\begin{aligned} d^*(X,Y) &= |\det Y|_F^{-3}\, d(X,Y) \\ &= |\det(\tilde{\gamma})|_F^{-3}\, dg\, d\tilde{\gamma}. \end{aligned}$$

Now $|\det(\tilde{\gamma})|_F = |\mathrm{Nm}(\tilde{\gamma})|_E^{\frac{1}{2}}$, so, by Lemma 5,

$$|\det(\tilde{\gamma})|_F^{-3} = \left|\frac{\mathrm{Tr}(\gamma) - 2}{\tau}\right|_E^{\frac{3}{2}}.$$

From the identity $\tilde{\gamma}(1-\gamma) = \sqrt{\tau}$, we obtain

$$\begin{aligned} \frac{d\tilde{\gamma}}{d\gamma} &= \left|\frac{\tilde{\gamma}}{1-\gamma}\right|_E \\ &= \left|\frac{\sqrt{\tau}}{\mathrm{Tr}(\gamma) - 2}\right|_E. \end{aligned}$$

For the rest of this paper, we will drop the subscript E from the norms.

Definition 2. Let $d^*\gamma = |\tau|^{-1}|\,\mathrm{Tr}(\gamma) - 2|^{\frac{1}{2}}\, d\gamma$.

We are therefore able to write

$$d^*(X,Y) = d^*\gamma\, dg.$$

We need to compare this measure to other Jacobians which arise in the subject. Recall the definition of D_ε.

Definition 3. Let $\delta \in G$. We write $G_{\delta,\varepsilon} = \{g \in G \mid g\delta g^{\vdash} = \delta\}$. Let $d\varepsilon : \mathrm{Lie}(G) \to \mathrm{Lie}(G)$ be the differential of ε, given by $d\varepsilon(X) = -X^{\vdash}$. Then

$$D_\varepsilon(\delta) = \det(\mathrm{Ad}(\delta) \circ d\varepsilon - 1; \mathrm{Lie}(G)/\mathrm{Lie}(G_{\delta,\varepsilon})).$$

Proposition 3. *We have* $D_\varepsilon(\tilde{\gamma} g_\theta) = 2(\mathrm{Tr}(\gamma) - 2)$.

Proof. Let $G' = \mathrm{SL}_2(F)$. It is easy to see that $\mathrm{Lie}(Z)$ is an eigenspace for $\mathrm{Ad}(\delta) \circ d\varepsilon - 1$ with eigenvalue -2; therefore, we restrict our attention to $\mathrm{Lie}(G')$. Write $\varepsilon_0(g) = (1/\det(g))g$; recall that $\varepsilon(g) = \mathrm{Ad}(g_\theta) \circ \varepsilon_0$. Therefore, on $\mathrm{Lie}(G')$, $d\varepsilon$ is equal to $\mathrm{Ad}(g_\theta)$. We are reduced to computing $\det(\mathrm{Ad}(\tilde{\gamma}) - 1; \mathrm{Lie}(G)/\mathrm{Lie}(T))$, where T is the usual centralizer of $\tilde{\gamma}$. The matrix of the adjoint action of $\tilde{\gamma}$ has eigenvalues γ and $\sigma(\gamma)$. Therefore,

$$\begin{aligned} D_\varepsilon(\delta) &= -2(\gamma - 1)(\sigma(\gamma) - 1) \\ &= -2(\gamma\sigma(\gamma) - \gamma - \sigma(\gamma) + 1) \\ &= 2(\mathrm{Tr}(\gamma) - 2). \end{aligned}$$

□

It follows that

$$d^*(X,Y) = |\tau|^{-1}|2|^{-\frac{1}{2}}|D_\varepsilon(S(\gamma))|^{\frac{1}{2}}\, d\gamma\, dg.$$

Here is the definition of the discriminant Δ_E mentioned in the introduction.

DEFINITION 4. Let g be a matrix in $\mathrm{GL}_2(F)$ with distinct eigenvalues a, b in a quadratic extension E. Then set

$$\Delta_E(g) = \left|\frac{(a-b)^2}{ab}\right|^{\frac{1}{2}}.$$

The following is straightforward.

PROPOSITION 4. *Let $\gamma \in E^1$ and $\tilde{\gamma} = ((\sigma(\gamma)-1)\sqrt{\tau})/(\mathrm{Tr}(\gamma)-2)$. Then*

$$\Delta_E(\tilde{\gamma}) = |\mathrm{Tr}(\gamma)-2|^{\frac{1}{2}} = |\tfrac{1}{2}D_\varepsilon(\tilde{\gamma})|^{\frac{1}{2}}.$$

So, we may write $d^*(X,Y)$ in a third way, as

$$d^*(X,Y) = |\tau|^{-1}\Delta_E(\tilde{\gamma})\, d\gamma\, dg.$$

3. The residue

3.1 Recollection

In this paper we are treating a low-dimensional example of a more general theory. Its history includes [GS98, GS01, GS], which treat the quasisplit classical groups, in particular the case of even orthogonal groups. The present set-up may be generalized by letting $\tilde{G}$ be an orthogonal group SO_{6n}, and the Levi subgroup $M \tilde{=} \mathrm{GL}_{2n} \times \mathrm{SO}_{2n}$ of three equal-sized blocks. Again one takes a self-dual supercuspidal representation π_G of $G = \mathrm{GL}_{2n}(F)$ and a supercuspidal representation π_H of $H = \mathrm{SO}_{2n}(F)$, and studies parabolic induction to $\tilde{G}$ as in the introduction.

If the intertwining operator has a pole at zero, this pole will occur along a flat section of functions $f \in I(s\tilde{\alpha}, \pi_G \otimes \pi_H)$ assembled from some choice of matrix coefficients ψ of π_G and f_H of π_H, and a pair of compact subsets L, L' of $M_n(F)$. (See [GS98] for details.) Denote the central character of π_G by ω. Choose a compactly supported function f_G so that

$$\psi(g) = \int_Z \omega(z)^{-1} f_G(zg)\, dz.$$

The residue obtained from these choices is denoted by $R(f_G, f_H)$. The main result of [Spa08] is the following theorem: the residue $R(f_G, f_H)$ is equal to

$$\operatorname*{Res}_{s=0} \sum_T |W(T)|^{-1} \sum_{k=0}^{\infty} q^{-2nks} \int_T \sum_S \int_{G/T}\int_{T\backslash H^+} f_G(gS(\gamma)^{-1}g^{\vdash}) f_H(h^{-1}\gamma h) W_k(g,h)\, dh\, dg\, d^*\gamma.$$

Here S runs over sections of the norm correspondence over γ, and $W_k(g,h)$ is a certain 'weight function' defined in [Spa08]. For us, the sum over S is a singleton by Proposition 2. Another sum runs over conjugacy classes of maximal tori T in H, and $|W(T)|$ denotes the order of the Weyl group of T in H. By H^+ we denote the full orthogonal group O_{2n}. A slight variation on the methods gives

$$\mathrm{Res}_{s=0} \textstyle\sum_T |W(T)|^{-1} \sum_{k=0}^{\infty} q^{-2nks} \int_T \int_{G/T} \int_{T\backslash H^+} \displaystyle\sum_{\alpha\in A} \omega(\alpha)^{-1} f_G(\alpha g S(\gamma)^{-1} g^{\vdash})$$
$$\times\ f_H(h^{-1}\gamma h) w_k(g,h)\, dh\, dg\, d^*\gamma,$$

where $w_k(g,h) = \mathrm{vol}_T(T \cap \varpi^{-k} g^{-1} L h^{-1})$ and A is a set of representatives for $F^\times/F^{\times 2}$.

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In our case, where $H = T = \mathrm{SO}_2^*$ and $G = \mathrm{GL}_2(F)$, a few simplifications occur. We have $w_k(g,h) = w_k(g,1)$, and so may write $w_k(g) = w_k(g,h)$. Since T is compact and L contains a neighborhood of 0, we have $\lim_{k\to\infty} w_k(g) = \mathrm{vol}(T) = 1$ for all $g \in G$. The quotient $T\backslash H^+ = H\backslash H^+$ has order two; we write the quotient as $\{1, g_\theta\}$. Write $\gamma' = g_\theta^{-1}\gamma g_\theta$. Moreover, we may take f_H to be a character χ_H on H. Write π for π_G and f for f_G. The residue then simplifies to

$$\operatorname*{Res}_{s=0} \sum_{k=0}^{\infty} q^{-4ks} \int_T (\chi_H(\gamma) + \chi_H(\gamma')) \sum_{\alpha\in A} \omega(\alpha)^{-1} \int_{G/T} f(\alpha g S(\gamma)^{-1} g^{\vdash}) w_k(g)\, dh\, dg\, d^*\gamma.$$

PROPOSITION 5. *In this case, we have*

$$\begin{aligned}\lim_{k\to\infty} \int_T (\chi_H(\gamma) + \chi_H(\gamma')) \sum_{\alpha\in A} \omega(\alpha)^{-1} \int_{G/T} f(\alpha g S(\gamma)^{-1} g^{\vdash}) w_k(g)\, dg\, d^*\gamma \\ = \int_T (\chi_H(\gamma) + \chi_H(\gamma')) \sum_{\alpha\in A} \omega(\alpha)^{-1} \int_{G/T} f(\alpha g S(\gamma)^{-1} g^{\vdash})\, dg\, d^*\gamma.\end{aligned}$$

Proof. The switching of limits follows from the usual reasoning: normalized twisted orbital integrals are bounded and have compact support on T. The result then follows by Lebesgue's dominated convergence theorem. □

This combines nicely with the following elementary analysis.

PROPOSITION 6. *Let $a_k \in \mathbb{C}$ be a sequence, and suppose that $\lim_{k\to\infty} a_k = a$. Then*

$$\lim_{s\to 0} s \cdot \sum_{k=0}^{\infty} a_k q^{-2nks} = \frac{a}{2n \log q}.$$

Proof. The change of variables $x = q^{-2ns}$ reduces the problem to computing

$$C \cdot \lim_{x\to 1^-} (\log x) \sum_{k=0}^{\infty} a_k x^k,$$

where $C = -1/(2n \log q)$. By comparison with geometric series, we see that the sum has radius of convergence at least one, and thus converges absolutely for $|x| < 1$. So, in this interval we may perform rearrangements. Since $\lim_{x\to 1}(\log x/(x-1)) = 1$, we may replace $\log x$ with $x - 1$. Therefore, we have

$$(x-1) \sum_{k=0}^{\infty} a_k x^k = -a_0 + \sum_{k=0}^{\infty} (a_k - a_{k+1}) x^k.$$

By Abel's limit theorem, this approaches $-a$ as $x \to 1^-$. □

COROLLARY 2. *The residue $R(f_G, f_H)$ is independent of the choice of lattice L and equal to*

$$\frac{1}{4\log q} \int_T (\chi_H(\gamma) + \chi_H(\gamma')) \sum_{\alpha\in A} \omega(\alpha)^{-1} \int_{G/T} f(\alpha g S(\gamma)^{-1} g^{\vdash})\, dg\, d^*\gamma.$$

3.2 The matrix coefficient

PROPOSITION 7. *We have*

$$\sum_{\alpha\in A} \omega(\alpha)^{-1} \int_{G/T} f(\alpha g S(\gamma)^{-1} g^{\vdash})\, dg = |2|^{-1} \int_{G/TZ} \psi(g S(\gamma)^{-1} g^{\vdash})\, dg.$$

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Proof. Putting $\delta = S(\gamma)^{-1}$, this follows from the equalities

$$\begin{aligned}\sum_{\alpha\in A}\omega(\alpha)^{-1}\int_{G/T} f(\alpha g\delta g^{\vdash})\,dg &= \sum_{\alpha\in A}\omega(\alpha)^{-1}\int_{G/TZ}\int_{Z/\{\pm1\}} f(\alpha z g\delta g^{\vdash} z^{\vdash})\,dz\,dg\\ &= \frac{1}{2}\int_{G/TZ}\int_{Z}\sum_{\alpha\in A}\omega(\alpha)^{-1} f(\alpha z^2 g\delta g^{\vdash})\,dz\,dg\\ &= |2|^{-1}\int_{G/TZ}\int_{Z}\omega(z)^{-1} f(zg\delta g^{\vdash})\,dz\,dg\\ &= |2|^{-1}\int_{G/TZ}\psi(g\delta g^{\vdash})\,dg\end{aligned}$$

since $\omega^2 = 1$ and $T\cap Z = \{\pm 1\}$. □

Corollary 3. *The residue $R(f_G, f_H)$ is equal to*

$$\frac{1}{2|2|\log q}\int_T \chi_H(\gamma)\int_{G/Z}\psi(gS(\gamma)^{-1}g^{\vdash})\,dg\,d^*\gamma.$$

Proof. A simple computation shows that $S(\gamma')^{-1} = g_\theta S(\gamma)^{-1} g_\theta^{\vdash}$, and therefore the orbital integrals for ψ over the orbits of $S(\gamma)^{-1}$ and $S(\gamma')^{-1}$ agree. Moreover, since $g_\theta w g_\theta^{-1} = -w$, we have $g_\theta\varepsilon(g)g_\theta^{-1} = \varepsilon(g_\theta g g_\theta^{-1})$, which implies that $\mathrm{Ad}(g_\theta)\circ\varepsilon = \varepsilon\circ\mathrm{Ad}(g_\theta)$, and so

$$\det(\mathrm{Ad}(g_\theta\delta g_\theta^{-1})\circ\varepsilon - 1) = \det(\mathrm{Ad}(\delta)\circ\varepsilon - 1).$$

In particular, $D_\varepsilon(S(\gamma')) = D_\varepsilon(S(\gamma))$. This allows us to replace the factor $(\chi_H(\gamma)+\chi_H(\gamma'))/2$ with $\chi_H(\gamma)$. We may drop the quotient by T since it is compact with normalized measure. □

At this point, we may discard some cases of non-trivial ω.

Proposition 8. *If ω restricted to the subgroup $\mathrm{Nm}(E^\times)$ is non-trivial, then $R(f_G, f_H) = 0$. Thus, when ω is non-trivial, we conclude that $R(f_G, f_H) = 0$ unless $E = E'$.*

Proof. Let $\delta\in E^\times g_\theta$. Suppose that $\beta\in E^\times$, and that $\omega(\beta\sigma(\beta)) = \omega(\alpha)\neq 1$. Then

$$\begin{aligned}\int\psi(g\delta g^{\vdash})\,dg &= \int\psi(g\beta\delta\beta^{\vdash}g^{\vdash})\,dg\\ &= \int\psi(g\beta\sigma(\beta)\delta g^{\vdash})\,dg\\ &= \int\psi(\alpha g\delta g^{\vdash})\,dg\\ &= \omega(\alpha)\int\psi(g\delta g^{\vdash})\,dg,\end{aligned}$$

the integrals being over G/Z. It follows that

$$\int_{G/Z}\psi(g\delta g^{\vdash})\,dg = 0.$$ □

Definition 5. Let G be a p-adic reductive group, and $\varepsilon: G\to G$ an involution. Suppose that (π, V) is an irreducible admissible representation of G with a non-zero intertwining operator $I_\varepsilon: \pi\to\pi\circ\varepsilon$ satisfying $I_\varepsilon^2(v) = v$.

Let $f \in C_c^\infty(G)$ and write $\pi^\varepsilon(f) : V \to V$ for the operator defined by

$$\pi^\varepsilon(f)v = \int_G f(x)\pi(x)I_\varepsilon v\, dx.$$

By [Clo87], there is a locally integrable function Θ_π^ε defined on the regular elements of G so that for all such f,

$$\mathrm{Tr}(\pi^\varepsilon(f)) = \int_G \Theta_\pi^\varepsilon(x) f(x)\, dx.$$

Note that if I is any intertwining operator from π to $\pi \circ \varepsilon$, then I^2 intertwines π. Therefore, $I^2 v = cv$ for some $c \in \mathbb{C}^\times$; by dividing by a square root of c we obtain an involution I_ε as above.

PROPOSITION 9. *Suppose in the above situation that π is supercuspidal. Pick $v, \tilde{v} \in V$, and a G-invariant inner product $(\ ,\ )$ which is also I_ε-invariant.*

Let ψ be the matrix coefficient defined by $\psi(g) = (\tilde{v}, gv)$. Let x be a regular element of G. Then

$$\int_{G/Z} \psi(gx\varepsilon(g)^{-1})\, dg = \Theta_\pi^\varepsilon(x)(\tilde{v}, I_\varepsilon v)\, d(\pi)^{-1},$$

where $d(\pi)$ is the formal degree of π.

Note that if $(\ ,\ )_1$ is a G-invariant inner product, then the inner product $(\ ,\ )_2$ defined by $(v, w)_2 = (v, w)_1 + (I_\varepsilon v, I_\varepsilon w)_1$ is both G-invariant and I_ε-invariant.

We follow the proof of Theorem 9 in [Har70] closely.

Proof. Let $f \in C_c^\infty(G)$. We have

$$\int_{G/Z} \psi(gx\varepsilon(g)^{-1})\, dg = \int_{G/Z} (g^{-1}\tilde{v}, x\varepsilon(g)^{-1}v)\, dg.$$

Note that $\pi(\varepsilon(g)^{-1})v = I_\varepsilon \pi(g)^{-1} I_\varepsilon v$ for all v. Let $\{\phi_i\}_{i\in I}$ be an orthonormal basis of V. Then

$$\begin{aligned}(g^{-1}\tilde{v}, x\varepsilon(g)^{-1}v) &= \sum_{i,j\in I} (g^{-1}\tilde{v}, \phi_i)(\phi_i, xI_\varepsilon\phi_j)(\phi_j, g^{-1}I_\varepsilon v) \\ &= \sum_{i,j} Q_{ij}(\tilde{v}, g\phi_i)\overline{(I_\varepsilon v, g\phi_j)},\end{aligned}$$

where $Q_{ij} = (\phi_i, \pi(x)I_\varepsilon\phi_j)$. So, our orbital integral becomes

$$\int_{G/Z} \sum_{i,j} Q_{ij}(\tilde{v}, g\phi_i)\overline{(I_\varepsilon v, g\phi_j)}\, dg.$$

Integrating this over G against f gives

$$\int_G \int_{G/Z} \sum_{i,j} Q_{ij}(\tilde{v}, g\phi_i)\overline{(I_\varepsilon v, g\phi_j)} f(x)\, dg\, dx = \int_{G/Z} \sum_{i,j} R_{ij}(\tilde{v}, g\phi_i)\overline{(I_\varepsilon v, g\phi_j)}\, dg, \tag{3.1}$$

where

$$R_{ij} = \int_G (\phi_i, \pi(x)I_\varepsilon\phi_j) f(x)\, dx.$$

By the orthogonality of matrix coefficients, (3.1) becomes

$$\left(\sum_i R_{ii}\right) d(\pi)^{-1}(\tilde{v}, I_\varepsilon v).$$

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We note that

$$\begin{aligned}\sum_i R_{ii} &= \sum_i \int_G (\phi_i, \pi(x) I_\varepsilon \phi_i) f(x)\, dx \\ &= \sum_i (\phi_i, \pi^\varepsilon(f)\phi_i) \\ &= \operatorname{Tr} \pi^\varepsilon(f).\end{aligned}$$

The conclusion follows, since we have shown that

$$\int_G \left(\int_{G/Z} \psi(gx\varepsilon(g)^{-1})\, dg \right) f(x)\, dx = \operatorname{Tr} \pi^\varepsilon(f)\, d(\pi)^{-1}(\tilde{v}, I_\varepsilon v). \qquad \square$$

Corollary 4. *If we choose our ψ with $v, \tilde{v}$ so that $(\tilde{v}, I_\varepsilon v) = d(\pi)$, then*

$$\int_{G/Z} \psi(gS(\gamma)^{-1}\varepsilon(g)^{-1})\, dg = \Theta_\pi^\varepsilon(S(\gamma)^{-1}).$$

Let us keep this assumption on ψ throughout the paper, for simplicity. Another choice of ψ would involve multiplying the residue by a (possibly zero) scalar.

Returning to Corollary 3, note that $D_\varepsilon(\alpha S(\gamma))$ depends only on $\operatorname{Ad}(\alpha S(\gamma)) = \operatorname{Ad}(S(\gamma))$, so that $D_\varepsilon(\alpha S(\gamma)) = D_\varepsilon(S(\gamma))$. Moreover, $\psi(g\alpha S(\gamma)^{-1} g^{\vdash}) = \omega(\alpha)\psi(gS(\gamma)^{-1}g^{\vdash})$, where ω is the central character of π. We deduce the following theorem. Recall that $d^*\gamma = |\tau|^{-1}|\operatorname{Tr}(\gamma) - 2|^{\frac{1}{2}}\, d\gamma$, where $d\gamma$ is the normalized Haar measure on T.

Theorem 2. *If ω is trivial on the subgroup $\operatorname{Nm}(E^\times)$ of $F^\times$, then*

$$R(f_G, f_H) = \frac{1}{|2|\log(q)} \int_T \chi_H(\gamma)\Theta_\pi^\varepsilon(S(\gamma)^{-1})\, d^*\gamma. \tag{3.2}$$

Since the right-hand side only depends on π and χ_H, we make the following definition.

Definition 6. Write $R(\pi, \chi_H)$ for the right-hand side of (3.2).

We will compute $R(\pi, \chi_H)$ in the sequel.

4. Case of non-trivial central character

By Proposition 8, we may assume that $E = E'$. To compute $R(\pi, \chi_H)$, we need to find an intertwining operator I_ε as in Definition 5. Write $\varepsilon_0(g) = (1/\det(g))g$; recall that $\varepsilon = \operatorname{Ad}(g_\theta) \circ \varepsilon_0$. This means that $\pi \circ \varepsilon_0(g) = \omega(\det(g))\pi(g)$. Moreover, since π is self-dual with a non-trivial central character, ω is the sign character sgn_E associated with the extension E.

The following is from Theorem 4.8.6 of [Bum98]. Write G_+ for the subgroup of matrices in G whose determinant is a norm from $E^\times$; it is a subgroup of G of index two.

Proposition 10. *The representation (π, V) is induced from a representation (π_+, V_+) of G_+.*

The following lemma is familiar from Clifford theory.

Lemma 6. *Let G be a group, and G_+ a subgroup of index two. Pick an element $s \in G - G_+$. Let (π_+, V_+) be a representation of G_+. Write (π_-, V_-) for the G_+-module whose representation space is again V_+ but where the action is given by $\pi_-(g) = \pi_+(sgs^{-1})$. Now let $(\pi, V) = \operatorname{Ind}_{G_+}^G \pi_+$. Then V is isomorphic to $V_+ \oplus V_-$ as a G_+-module. Let χ be the non-trivial one-dimensional*

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character of G which is trivial on G_+. Consider the operator $I_\chi : V \to V$ which is the identity on V_+ and -1 on V_-. Then I_χ is an intertwining operator from (π, V) to $(\chi \cdot \pi, V)$. Here $\chi \cdot \pi$ is the representation on V given by $(\chi \cdot \pi)(g) = \chi(g)\pi(g)$.

We may also write $I_\chi = P_+ - P_-$, where $P_\pm$ is the projection from V to $V_\pm$ orthogonal to $V_\mp$.

PROPOSITION 11. *We have*

$$\Theta_\pi^\varepsilon(S(\gamma)^{-1}) = \Theta_{\pi_+}(S(\gamma)^{-1}g_\theta) - \Theta_{\pi_-}(S(\gamma)^{-1}g_\theta),$$

where $\pi_-(g)$ is the representation of G_+ given by $\pi_-(g) = \pi_+(sgs^{-1})$, where $s \in G - G_+$.

Proof. Pick an inner product on V which is G- and I_ε-invariant. By the above, we may take $I_\varepsilon = \pi(g_\theta) \circ (P_+ - P_-)$. Let $I_{\varepsilon_0} = P_+ - P_-$; we will first compute the twisted character $\Theta_\pi^{\varepsilon_0}$ with respect to this intertwining operator. Note that $(\ ,\ )$ is also I_{ε_0}-invariant.

Suppose that $f \in C_c(G)$ has support in G_+. For $v \in V$, we have

$$\pi^{\varepsilon_0}(f)v = \int_G f(x)\pi(x)P_+v\,dx - \int_G f(x)\pi(x)P_-v\,dx.$$

Let $v^+ \in V_+$ and $v^- \in V_-$. Since $(\ ,\ )$ is I_{ε_0}-invariant we have $(v^+, v^-) = (v^+, -v^-)$, and therefore the vectors are orthogonal. Therefore, there is an orthonormal basis of V, written as a union $\{e_i^+\} \cup \{e_i^-\}$, with $e_i^+ \in V_+$ and $e_i^- \in V_-$. Then for all e_i^+ we have

$$\begin{aligned}(\pi^{\varepsilon_0}(f)e_i^+, e_i^+) &= \left(\int_G f(x)\pi(x)e_i^+\,dx, e_i^+\right)\\ &= \left(\int_{G_+} f_+(x)\pi(x)e_i^+\,dx, e_i^+\right)\\ &= (\pi_+(f_+)e_i^+, e_i^+),\end{aligned}$$

where we write f_+ for the restriction of f to G_+. Similarly, for all e_i^- we have

$$\begin{aligned}(\pi^{\varepsilon_0}(f)e_i^-, e_i^-) &= -\left(\int_{G_+} f(x)\pi(x)e_i^-\,dx, e_i^-\right)\\ &= -(\pi_-(f_+)e_i^-, e_i^-).\end{aligned}$$

It follows that $\operatorname{tr}\pi^{\varepsilon_0}(f) = \operatorname{tr}\pi_+(f_+) - \operatorname{tr}\pi_-(f_+)$, and so for $g \in G_+$ we have

$$\Theta_\pi^{\varepsilon_0}(g) = \Theta_{\pi_+}(g) - \Theta_{\pi_-}(g).$$

By translating by g_θ, we find that for $g \in G_+g_\theta$ we have

$$\Theta_\pi^\varepsilon(g) = \Theta_{\pi_+}(gg_\theta) - \Theta_{\pi_-}(gg_\theta).$$

The result follows, since $S(\gamma)^{-1}g_\theta \in E^\times \subset G_+$. □

Let us summarize this as the following proposition.

PROPOSITION 12. *Suppose that $E = E'$, and the central character of π is sgn_E. Then we have*

$$R(\pi, \chi_H) = \frac{1}{|2|\log q}\int_T \chi_H(\gamma)(\Theta_{\pi_+}(\tilde\gamma) - \Theta_{\pi_-}(\tilde\gamma))\,d^*\gamma.$$

Next, we turn to [Lan80, Lemma 7.19], where Langlands computed

$$\Theta_{\pi_+}(\tilde\gamma) - \Theta_{\pi_-}(\tilde\gamma) = \pm\lambda(E/F,\psi)\operatorname{sgn}_E\left(\frac{\tilde\gamma - \sigma(\tilde\gamma)}{\sqrt{\tau}}\right)\frac{\chi'(\tilde\gamma) + \chi'(\sigma(\tilde\gamma))}{\Delta_E(\tilde\gamma)}.$$

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(This is also in [LL79].) Here $\lambda(E/F,\psi)$ is the Gauss sum attached to sgn_E, as in [Lan70]. In fact, we have $\tilde{\gamma}-\sigma(\tilde{\gamma})=\sqrt{\tau}$, so the sgn_E term is 1. Since π has central character sgn_E, we know that χ' must be trivial on $F^\times$. Therefore, it factors through the homomorphism $\alpha^\vee : E^\times \to E^1$ given by $\alpha^\vee(x)=x/\sigma(x)$ by Hilbert's theorem 90. This defines a character χ_G of E^1 for which $\chi'=\chi_G\circ\alpha^\vee$. Also, note that since $\chi'(\mathrm{Nm}(\tilde{\gamma}))=1$ we have $\chi'(\sigma(\tilde{\gamma}))=\overline{\chi'(\tilde{\gamma})}$.

Proposition 13. *We have $\chi'(\tilde{\gamma})=\overline{\chi_G(\gamma)}$ in the above situation.*

Proof. In fact, one has $\alpha^\vee(\tilde{\gamma})=\gamma^{-1}$ by Lemma 5. □

Using Definition 2 and applying Proposition 4, we see that the discriminant terms cancel, and we obtain the following theorem.

Theorem 3. *Suppose that $E=E'$, and the central character of π is sgn_E. Then $R(\pi,\chi_H)$ is a non-zero constant multiple of*

$$\int_T \chi_H(\gamma)(\chi_G(\gamma)+\overline{\chi_G(\gamma)})\,d\gamma.$$

Corollary 5. *This verifies (b) in [Sha08, Proposition 2].*

5. Case of trivial central character

Let us henceforth assume that the central character ω of π is trivial. This is equivalent to the condition that the restriction of χ' to $F^\times$ is the sign character $\mathrm{sgn}_{E'}$ of $F^\times$ associated with E'. For the rest of this paper we restrict ourselves to the case of odd residual characteristic for simplicity. Again, our first step is to find an intertwining operator I_ε as in Definition 5. This is simpler than in the case of non-trivial ω; here $\varepsilon=\mathrm{Ad}(g_\theta)\circ\varepsilon_0$. This means that $\pi(\varepsilon(g))=\pi(\mathrm{Ad}(g_\theta)(g))$, and therefore $I_\varepsilon=\pi(g_\theta)$ intertwines π and $\pi\circ\varepsilon$. We obtain the following proposition. Recall that $\tilde{\gamma}=S(\gamma)^{-1}g_\theta$.

Proposition 14. *The residue $R(\pi,\chi_H)$ is equal to*

$$\frac{1}{|2|\log q}\int_T \chi_H(\gamma)\Theta_\pi(\tilde{\gamma})\,d^*\gamma.$$

For the values of Θ_π, we turn to the explicit character value computations which are in both Shimizu [Shi77, Proposition 2] and Silberger [Sil70, § 2.6]. But we set up the integration first.

5.1 Measures of annuli

We follow [Shi77] and [Adl97] with the following definitions.

Definition 7. For an integer $n\geqslant 0$, write

$$C_n=\begin{cases} E^1\cap(1+\mathfrak{p}_E^{2n+1}) & \text{if } E \text{ is ramified,}\\ E^1\cap(1+\mathfrak{p}_E^{n}) & \text{if } E \text{ is unramified and } n \text{ is positive,}\\ E^1 & \text{if } E \text{ is unramified and } n=0.\end{cases}$$

We write A_n for the 'annulus' C_n-C_{n-1}. Similarly, we write C_n', A_n' for the corresponding subsets of E'. For a non-trivial character χ of the norm-one group E^1 of a quadratic extension E of F, write $\ell_E(\chi)$ for the minimum n so that χ is trivial on C_n. We may drop the subscript E if it is understood. For us, $\ell_{E'}(\chi')\geqslant 1$ since χ' induces a supercuspidal representation with trivial central character.

As in § 2.3, we put $d^*\gamma = |\tau|^{-1}|\mathrm{Tr}(\gamma) - 2|^{\frac{1}{2}}\, d\gamma$. Much of the forthcoming integral computations reduce to the following lemmas.

DEFINITION 8. Suppose that E is unramified over F, and χ is a character on $E^\times$ with $\ell = \ell_E(\chi)$. Then let

$$I(\ell, n) = \int_{C_n} \chi(\gamma)\, d^*\gamma.$$

The following follows from the usual annulus computation.

LEMMA 7.

$$I(\ell, n) = \begin{cases} -\dfrac{q^{-2\ell+3}}{(q+1)^2} & \text{if } n < \ell, \\ -\dfrac{q^{-2n+2}}{(q+1)^2} & \text{if } n \geqslant \ell. \end{cases} \tag{5.1}$$

This expression is non-zero. If we choose instead a ramified extension E, the result is proportional to this, and also non-zero.

DEFINITION 9. Let E be ramified, and χ a character on $E^\times$ with $\ell = \ell(\chi)$. Then let

$$I_r(\ell, n) = \int_{C_n} \chi(\gamma)\, d^*\gamma.$$

LEMMA 8.

$$I_r(\ell, n) = \frac{q+1}{2\sqrt{q}} I(\ell, n).$$

6. Trivial central character, different tori

THEOREM 4. *Suppose that E is not isomorphic to E'. Then $R(\pi, \chi_H) \neq 0$.*

We will require some delicate information in the case when E and E' are both ramified but not isomorphic.

DEFINITION 10. Let E be a quadratic extension of F. Given $\gamma \equiv 1 \bmod \mathfrak{p}$, let $g(\gamma) = \tilde{\gamma}/\lambda$, where λ is as in Lemma 5(v).

This is a rescaling of $\tilde{\gamma}$ so that $g(\gamma) \in E^1$.

LEMMA 9. *Suppose that $E = F[\sqrt{\tau}]$ is a ramified quadratic extension of F, with $\mathrm{ord}(\tau) = 1$. Then:*

(i) $\mathrm{Tr}(g(\gamma)) \equiv (2 + x/2) \bmod xb\tau$, *where* $\gamma = (1 + x) + b\sqrt{\tau} \in E^1$;

(ii) *the map from C_n to $\mathfrak{p}_F^n$ taking γ to $(\gamma - \sigma(\gamma))/\sqrt{\tau}$ induces an isomorphism of groups*

$$B_n : C_n/C_{n+1} \xrightarrow{\sim} \mathfrak{p}_F^n/\mathfrak{p}_F^{n+1}.$$

Through this identification, we may view C_n/C_{n+1} as a k-module. If $\mu \in k$, and γ is in the quotient, we write this action as ${}^\mu\gamma$;

(iii) $$\mathrm{Tr}({}^\mu\gamma) - 2 \equiv \mu^2(\mathrm{Tr}(\gamma) - 2) \mod \mathfrak{p}_F^{2n+2};$$

(iv) $$\mathrm{Tr}(g({}^\mu\gamma)) - 2 \equiv \mu^2(\mathrm{Tr}(g(\gamma)) - 2) \mod \mathfrak{p}_F^{2n+2}.$$

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Proof. This is left to the reader. Note that if $a+b\sqrt{\tau}\in C_n$, then $x=a-1\in\mathfrak{p}_F^{2n+1}$ and $b\in\mathfrak{p}_F^n$. □

Definition 11. Let $E=F[\sqrt{\tau}]$ be a ramified extension of F, with $\mathrm{ord}(\tau)=1$. Write $\theta_n:\mathfrak{p}_F^n/\mathfrak{p}_F^{n+1}\xrightarrow{\sim}k$ for the isomorphism taking b to $b/(-\tau)^n$. Write $\phi_n:C_n/C_{n+1}\xrightarrow{\sim}k$ for $\theta_n\circ B_n$.

Note that if $b\notin\mathfrak{p}_F^{n+1}$, then $\mathrm{sgn}_E(b)=\mathrm{sgn}_E(\theta_n(b))$ is equal to the Legendre symbol $\left(\frac{\theta_n(b)}{k}\right)$ and $\phi_n(\sigma(\gamma))=-\phi_n(\gamma)$. Also note that $\phi_n(\gamma)\neq 0$ if $\gamma\in A_n$. We now prove the theorem.

Proof.

Case I. E' unramified and E ramified.

Let $\gamma\in H=E^1$. As in [Shi77], we take $E=F[\sqrt{-\varpi}]$. Let $\ell=\ell_E(\chi_H)$ and $\ell'=\ell_{E'}(\chi')$. We have

$$\Theta_\pi(\tilde{\gamma})=\begin{cases}-2q^{\ell'-1} & \text{if } \gamma\in C_{\ell'-1},\\ 0 & \text{otherwise.}\end{cases}$$

Therefore, following Proposition 14, we obtain

$$\begin{aligned}R(\pi,\chi_H)&=\frac{1}{|2|\log q}\int_T\chi_H(\gamma)\Theta_\pi(\tilde{\gamma})\,d^*\gamma\\ &=-\frac{2q^{\ell'-1}}{|2|\log q}\cdot I_r(\ell,\ell'-1)\neq 0.\end{aligned}\tag{6.1}$$

Case II. E' ramified and E unramified.

This time we have

$$\Theta_\pi(\tilde{\gamma})=\begin{cases}-(q+1)q^{\ell'-1} & \text{if } \gamma\in C_{\ell'},\\ 0 & \text{otherwise.}\end{cases}$$

As in the previous proof, we obtain

$$R(\pi,\chi_H)=-\frac{1}{|2|\log q}(q+1)q^{\ell'-1}\cdot I(\ell,\ell')\neq 0.\tag{6.2}$$

Case III. E' and E both ramified and $\ell>\ell'$.

The convention is that $E'=F[\sqrt{-\varpi}\,]$ and $E=F[\sqrt{-\epsilon_0\varpi}\,]$, where $\epsilon_0\in\mathcal{O}_F^\times$ is a non-square. We write Tr' for the trace map from E' to F. Let $\gamma\in H=E^1$ and $\ell'=\ell_{E'}(\chi')$. We have

$$\Theta_\pi(\tilde{\gamma})=\begin{cases}-(q+1)q^{\ell'-1} & \text{if } \gamma\in C_{\ell'},\\ q^{\ell'-1}\displaystyle\sum_{\beta\in C'_{\ell'-1}/C'_{\ell'}}\chi'(\beta)\,\mathrm{sgn}_{E'}(\mathrm{Tr}(g(\gamma))-\mathrm{Tr}'(\beta)) & \text{if } \gamma\in A_{\ell'-1},\\ 0 & \text{otherwise.}\end{cases}\tag{6.3}$$

The new feature here is the sum over $\beta\in C'_{\ell'-1}/C'_{\ell'}$. Since χ_H is non-trivial on $C_{\ell'}$, there is an element $\gamma_0\in C_{\ell'}$ with $\chi_H(\gamma_0)\neq 1$. For all $\gamma\in C_{\ell'-1}$, we have $\mathrm{Tr}'(g(\gamma\gamma_0))\equiv\mathrm{Tr}'(g(\gamma))\bmod\mathfrak{p}^{2\ell}$. We also have $\mathrm{Tr}'(\beta)-2\in\mathfrak{p}^{2\ell'-1}$ and $\mathrm{Tr}(g(\gamma))\in\mathfrak{p}^{2\ell'-1}-\mathfrak{p}^{2\ell'}$ for $\gamma\in A_{\ell'-1}$, and it follows that

$$\mathrm{Tr}(g(\gamma))-\mathrm{Tr}'(\beta)\in\mathfrak{p}^{2\ell'-1}-\mathfrak{p}^{2\ell'}.$$

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Therefore, we have

$$\begin{aligned}\int_{A_{\ell'-1}} \chi_H(\gamma) \sum_{\beta\in C'_{\ell'-1}/C'_{\ell'}} \chi'(\beta)\,\mathrm{sgn}_{E'}(\mathrm{Tr}(g(\gamma)) - \mathrm{Tr}'(\beta))\,d\gamma \\ = \int_{A_{\ell'-1}} \chi_H(\gamma\gamma_0) \sum_{\beta\in C'_{\ell'-1}/C'_{\ell'}} \chi'(\beta)\,\mathrm{sgn}_{E'}(\mathrm{Tr}(g(\gamma)) - \mathrm{Tr}'(\beta))\,d\gamma,\end{aligned}$$

and it follows that this term is 0. As above, we are done because

$$R(\pi, \chi_H) = -\frac{1}{|2|\log q}(q+1)q^{\ell'-1} \cdot I_r(\ell, \ell') \neq 0. \tag{6.4}$$

Case IV. E' and E both ramified and $\ell \leqslant \ell'$.

Equation (6.3) is still valid, but here the integral over $A_{\ell'-1}$ will be non-zero. In this case χ_H is trivial on $C_{\ell'}$, and its restriction to $C_{\ell'-1}$ may be viewed as a character on $C_{\ell'-1}/C_{\ell'}$. Note that χ' restricts to a character on $C'_{\ell'-1}/C'_{\ell'}$. Given $\mu \in k$, write χ'^{μ} for the character

$$\chi'^{\mu}(\beta) = \chi'({}^{\mu}\beta),$$

defined on this quotient. Of course, if $\mu = 0$, then χ'^0 is the trivial character.

Fix an element $\gamma_0 = a_0 + b_0\sqrt{-\varpi\varepsilon_0} \in A_{\ell'-1}$, and let

$$f(\beta) = \mathrm{sgn}_{E'}(\mathrm{Tr}(g(\gamma_0)) - \mathrm{Tr}'(\beta)),$$

viewed as a function on $C'_{\ell'-1}/C'_{\ell'}$. (In fact, $\mathrm{Tr}(g(\gamma_0)) \neq \mathrm{Tr}'(\beta)$ in all cases.) Write $\hat{f}(\chi'^{\mu})$ for the Fourier coefficient of f with respect to the character χ'^{μ}. That is,

$$\hat{f}(\chi'^{\mu}) = \sum_{\beta\in C'_{\ell'-1}/C'_{\ell'}} \chi'^{\mu}(\beta)\,\mathrm{sgn}_{E'}(\mathrm{Tr}(g(\gamma_0)) - \mathrm{Tr}'(\beta)).$$

For $\mu \neq 0$, we have

$$\hat{f}(\chi'^{\mu}) = \sum_{\beta} \chi'(\beta)\,\mathrm{sgn}_{E'}(\mathrm{Tr}(g(\gamma_0)) - \mathrm{Tr}'({}^{\mu^{-1}}\beta)).$$

But then by Lemma 9, we have

$$\begin{aligned}\mathrm{sgn}_{E'}(\mathrm{Tr}(g(\gamma_0)) - \mathrm{Tr}'({}^{\mu^{-1}}\beta)) &= \mathrm{sgn}_{E'}(\mathrm{Tr}(g(\gamma_0)) - 2 + \mu^{-2}(2 - \mathrm{Tr}'(\beta))) \\ &= \mathrm{sgn}_{E'}(\mathrm{Tr}(g({}^{\mu}\gamma_0)) - \mathrm{Tr}'(\beta)).\end{aligned}$$

Thus,

$$\sum_{\beta\in C'_{\ell'-1}/C'_{\ell'}} \chi'(\beta)\,\mathrm{sgn}_{E'}(\mathrm{Tr}(g(\gamma)) - \mathrm{Tr}'(\beta)) = \hat{f}(\chi'^{\mu}), \tag{6.5}$$

where $\gamma = {}^{\mu}\gamma_0$. On the other hand,

$$\begin{aligned}\hat{f}(\chi'^{0}) &= \sum_{\beta\in C'_{\ell'-1}/C'_{\ell'}} \mathrm{sgn}_{E'}(\mathrm{Tr}(g(\gamma_0)) - \mathrm{Tr}'(\beta)) \\ &= \sum_{\beta} \mathrm{sgn}_{E'}\left(\frac{x_0}{2} - 2y\right) \\ &= \sum_{\beta} \mathrm{sgn}_{E'}(2x_0 - 8y).\end{aligned}$$

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Here we write $\beta = c + d\sqrt{-\varpi}$, and put $x_0 = a_0 - 1$ and $y = d - 1$. This is equal to

$$\begin{aligned}\sum_{\beta} \operatorname{sgn}_{E'}(-\varpi\epsilon_0 b_0^2 + 4\varpi d^2) &= \sum_{\beta} \operatorname{sgn}_{E'}(4d^2 - \epsilon_0 b_0^2) \\ &= \sum_{\beta} \operatorname{sgn}_{E'}\left(\frac{4d^2 - \varepsilon_0 b_0^2}{\varpi^{2n}}\right).\end{aligned}$$

Now $(4d^2 - \varepsilon_0 b_0^2)/\varpi^{2n}$ is a norm of $F[\sqrt{\varepsilon_0}]$, and so it is a norm of E' if and only if it is a perfect square in F. As it is a unit, we may replace $\operatorname{sgn}_{E'}$ with the Legendre symbol and obtain

$$\sum_{\mu \in k} \left(\frac{\mu^2 - \epsilon_0 \phi_{\ell-1}(\gamma_0)^2}{k}\right).$$

This is -1 (see [IR90, Exercise 5.8]). Therefore, we have

$$\hat{f}(\chi'^0) = -1.$$

Consider the integral

$$\int_{A_{\ell'-1}} \chi_H(\gamma) \sum_{\beta \in C'_{\ell'-1}/C'_{\ell'}} \chi'(\beta) \operatorname{sgn}_{E'}(\operatorname{Tr}(g(\gamma)) - \operatorname{Tr}'(\beta))\, d\gamma.$$

Since χ_H is trivial on $C_{\ell'}$, this is equal to

$$\frac{1}{2} q^{-\ell'} \sum_{\gamma \in A_{\ell'-1}/C_{\ell'}} \chi_H(\gamma) \hat{f}(\chi'^{\mu}). \tag{6.6}$$

Here $\mu = \mu(\gamma)$ is defined by $\gamma = {}^{\mu}\gamma_0$. Fix an element $\beta_0 \in A'_{\ell'-1}$. There is a $\mu_0 \in k$ so that

$$\chi_H(\gamma_0) = \chi'^{\mu_0}(\beta_0).$$

So, we may write (6.6) as

$$\begin{aligned}\frac{1}{2} q^{-\ell'} \sum_{\mu \in k^\times} \chi'^{\mu_0 \mu}(\beta_0) \hat{f}(\chi'^{\mu}) &= \frac{1}{2} q^{-\ell'} \left[-\hat{f}(\chi'^0) + \sum_{\mu \in k} \chi'^{\mu_0 \mu}(\beta_0) \hat{f}(\chi'^{\mu})\right] \\ &= \frac{1}{2} q^{-\ell'} (q \cdot f({}^{\mu_0}\beta_0) + 1),\end{aligned}$$

by Fourier inversion. We have

$$f({}^{\mu_0}\beta_0) = \operatorname{sgn}_{E'}(\operatorname{Tr}(g(\gamma_0)) - \operatorname{Tr}'({}^{\mu_0}\beta_0)) = \pm 1,$$

with the sign depending on the relationship between χ' and χ_H.

Putting this together, we obtain

$$\int_T \chi_H(\gamma) \Theta_\pi(\tilde{\gamma})\, d^*\gamma = -(q+1) q^{\ell'-1} I_r(\ell, \ell') + \frac{\sqrt{q}}{2} q^{-\ell'+1} q^{-\ell'} (1 \pm q).$$

This is equal to

$$\frac{\sqrt{q}}{2}[q^{-\ell'} + q^{-2\ell'+1}(1 \pm q)],$$

and so we have the residue

$$R(\pi, \chi_H) = \frac{\sqrt{q}}{2|2| \log q}(q^{-\ell'} + q^{-2\ell'+1}(1 \pm q)) \neq 0. \tag{6.7}$$

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7. Trivial central character, same tori

Next we handle the cases where $E = E'$.

PROPOSITION 15. *Let $E = F[\sqrt{\tau}]$ be unramified over F. Let $\chi : E^\times \to \mathbb{C}^\times$ be a character whose restriction to $F^\times$ is sgn_E. Then there is a character $\chi_G : E^1 \to \mathbb{C}^\times$ so that for all $\gamma \in E^1 - \{\pm 1\}$,*

$$\chi_G(\gamma^{-1}) \operatorname{sgn}_E(b) = \chi'(\tilde{\gamma}).$$

As usual, $\gamma = a + b\sqrt{\tau}$.

Proof. By Lemma 5, we have $\sigma(\tilde{\gamma})/\tilde{\gamma} = \gamma$. By Lemma 5(iv), either $\mathrm{Nm}(\tilde{\gamma})$ is a square in $F^\times$ or the product of a square and τ. In the first case, we may write

$$\begin{aligned}\chi(\tilde{\gamma}) &= \chi\left(\frac{\tilde{\gamma}}{\sqrt{\tilde{\gamma}\sigma(\tilde{\gamma})}}\right) \operatorname{sgn}_E(\sqrt{\mathrm{Nm}(\tilde{\gamma})})\\ &= \chi\left(\sqrt{\frac{\tilde{\gamma}}{\sigma(\tilde{\gamma})}}\right) \operatorname{sgn}_E(b)\\ &= \chi(\sqrt{\gamma^{-1}}) \operatorname{sgn}_E(b).\end{aligned}$$

Similarly, in the second case, $\chi(\tilde{\gamma}) = \chi(\sqrt{\gamma^{-1}\tau}) \operatorname{sgn}_E(b)$. Therefore, the proposition holds with

$$\chi_G(\gamma) = \begin{cases} \chi'(\sqrt{\gamma}) & \text{if } \gamma \text{ is a square,}\\ \chi'(\sqrt{\gamma\tau}) & \text{if } \gamma\tau \text{ is a square.}\end{cases}$$

This defines a character on E^1. $\square$

Remark. In fact, if $\ell' = \ell_E(\chi') \geqslant 1$ as in our case, then $C_{\ell'}^2 = C_{\ell'}$, and it follows that $\ell_E(\chi_G) = \ell_E(\chi')$. Also, note that $\chi_G(\gamma)^2 = \chi'(\gamma)$ for all $\gamma \in E^1$.

THEOREM 5. *Suppose that $E = E'$ is the unramified quadratic extension of F. Then $R(\pi, \chi_H) \neq 0$.*

Proof. Let χ_G be as in Proposition 15, applied to χ'. Let $\ell' = \ell_E(\chi')$ and $\ell = \ell_E(\chi_H)$. We have, by [Shi77],

$$\Theta_\pi(\tilde{\gamma}) = \begin{cases} -2q^{\ell'-1} & \text{if } \gamma \in C_{\ell'},\\ (-1)^{\ell'} |\mathrm{Tr}(\gamma) - 2|^{-\frac{1}{2}} (\chi_G(\gamma) + \overline{\chi_G(\gamma)})(-1)^{\mathrm{ord}(\gamma-1)} & \text{otherwise.}\end{cases}$$

Our integral $\int_T \chi_H(\gamma)\Theta_\pi(\tilde{\gamma})\, d^*\gamma$ is equal to

$$(-1)^{\ell'} \sum_{k=0}^{\ell'-1} (-1)^k \int_{A_k} \chi_H(\gamma)(\chi_G(\gamma) + \overline{\chi_G(\gamma)})\, d\gamma - 2q^{\ell'-1} I(\ell, \ell'). \tag{7.1}$$

Write T_k for $\int_{A_k} \chi_H(\gamma)(\chi_G(\gamma) + \overline{\chi_G(\gamma)})\, d\gamma$. Note that each T_k is a rational number by orthogonality, since $\mathrm{vol}(A_k)$ is rational. Also, $I(\ell, \ell')$ is rational by Lemma 7, and therefore (7.1) is a rational number. We simply wish to show that it is non-zero. Pick a prime number r dividing $q + 1$, and say that $\mathrm{ord}_r(q + 1) = e$. Then $\mathrm{ord}_r(I(\ell, \ell')) = -2e$ but $\mathrm{ord}_r(T_k) \geqslant -e$ for all k. Suppose that $r \neq 2$. Then (7.1) has $\mathrm{ord}_r = -2e$ and is therefore non-zero. (The case in which $q + 1$ is a power of two is similar but left to the reader.) $\square$

We will do something similar for the ramified case, but it is a little more complicated. Let $E^\times$ be a ramified extension of F. By Hensel's lemma, any element $x \in 1 + \mathfrak{p}_E$ has a unique square root in $1 + \mathfrak{p}_E$. It follows that there is a well-defined square root function $r : C_0 \to C_0$, so that $r(\gamma^2) = \gamma$ for $\gamma \in C_0$.

Proposition 16. *Let E be ramified over F. Suppose that $\chi : E^\times \to \mathbb{C}^\times$ is a linear character whose restriction to $F^\times$ is sgn_E. Then:*

(i) *if $\gamma \in E^1 - C_0$, then $\chi(\tilde{\gamma}) = \operatorname{sgn}_E(2)\chi(\sqrt{\tau})(\chi \circ r)(-\gamma^{-1})$;*

(ii) *if $\gamma \in C_0$, then $\chi(\tilde{\gamma}) = \operatorname{sgn}_E(-b)(\chi \circ r)(\gamma^{-1})$.*

Proof. As usual, write $\gamma = a + b\sqrt{\tau}$. Then $\operatorname{Nm}(\tilde{\gamma}) = \tau/(2a-2) \equiv -\tau/4 \bmod \mathfrak{p}\tau$. Let $\lambda \in F^\times$ be the square root of $-\operatorname{Nm}(\tilde{\gamma})/\tau$ which is congruent to $\frac{1}{2} \bmod \mathfrak{p}$. Note that

$$\left(\frac{\tilde{\gamma}}{\lambda\sqrt{\tau}}\right)^2 = -\frac{\tilde{\gamma}^2}{\tilde{\gamma}\sigma(\tilde{\gamma})} = -\gamma^{-1}.$$

A computation shows that $\tilde{\gamma}/\lambda\sqrt{\tau} \equiv 1 \bmod \mathfrak{p}$, and it follows that

$$\frac{\tilde{\gamma}}{\lambda\sqrt{\tau}} = r(-\gamma^{-1}).$$

Therefore,

$$\chi(\tilde{\gamma}) = \operatorname{sgn}_E(\lambda)\chi(\sqrt{\tau})\chi(r(-\gamma^{-1})),$$

as desired. The second part is easier (use Lemma 5(v)). □

Gauss sums make an appearance in the ramified case.

Definition 12. Let $\chi : E^1 \to \mathbb{C}^\times$ be a character, with $\ell = \ell_E(\chi) \geqslant 1$. Let $\Lambda_\chi : k \to \mathbb{C}^\times$ be the additive character given by $\chi \circ \phi_{\ell-1}^{-1}$, where $\phi_{\ell-1}$ is given in Definition 11. We put

$$\tau(\chi) = \sum_{a \in k} \left(\frac{a}{k}\right)\Lambda_\chi(a) \quad \text{and} \quad \varepsilon(\chi) = \frac{\tau(\chi)}{\sqrt{q}}.$$

It is well known (see [IR90]) that $\varepsilon(\chi)^2 = (-1/k)$.

Theorem 6. *Suppose that $E = E'$ is a ramified quadratic extension of F, and the central character of π is trivial. Then $R(\pi, \chi_H) \neq 0$.*

Proof. Let $\ell' = \ell_E(\chi')$ and $\ell = \ell_E(\chi_H)$. The convention is that $E = F[\sqrt{-\varpi}\,]$. We have, by [Shi77],

$$\Theta_\pi(\tilde{\gamma}) = \begin{cases} -(q+1)q^{\ell'-1} & \text{if } \gamma \in C_{\ell'}, \\ q^{\ell'-1}\displaystyle\sum_\beta \chi'(\beta)\operatorname{sgn}_E(\operatorname{Tr}(g(\gamma)-\beta)) & \text{if } \gamma \in A_{\ell'-1}, \\ \varepsilon(\chi')(-1)^{\ell'}|\operatorname{Tr}(\gamma)-2|^{-\frac{1}{2}}(\chi'(\tilde{\gamma}) + \operatorname{sgn}_E(-1)\chi'(\sigma(\tilde{\gamma}))) & \text{otherwise.} \end{cases}$$

The sum is over $\beta \in C_{\ell'-1}/C_{\ell'}$ with $\beta \neq g(\gamma), g(\sigma(\gamma)) \bmod C_{\ell'}$. Here $\varepsilon(\chi')$ is the root of unity attached to χ' as in [Shi77]. Also, let $\tau(\chi') = \varepsilon(\chi')\sqrt{q}$. Put $\chi_1 = \chi_H \cdot (\chi \circ r)^{-1}$, $\chi_2 = \chi_H \cdot (\chi^\sigma \circ r)^{-1}$, and $\ell_i = \ell(\chi_i)$. Using Proposition 16 and Fourier inversion on $C_{\ell'-1}/C_{\ell'}$ again, the integral

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$\int_T \chi_H(\gamma)\Theta_\pi(\tilde\gamma)\, d^*\gamma$ is equal to the sum of the three terms

$$\begin{cases} \left(\dfrac{\sqrt{q}}{2}\right)q^{-2\ell+\ell'+1} & \text{if } \ell' < \ell, \\ \left(\dfrac{\sqrt{q}}{2}\right)q^{-\ell'} & \text{if } \ell' \geqslant \ell, \end{cases} \tag{7.2}$$

$$\begin{cases} 0 & \text{if } \ell' < \ell, \\ \dfrac{\sqrt{q}}{2}q^{-\ell'}(1\pm q) & \text{if } \ell' \geqslant \ell \text{ and} \end{cases} \tag{7.3}$$

$$\tfrac{1}{2}\varepsilon(\chi')(-1)^{\ell'}((q-1)[\mathrm{sgn}_E(-1)\tau(\chi_1)q^{-\ell_1}]_1 + [\tau(\chi_2)q^{-\ell_2}]_2 + [\pm q]_3). \tag{7.4}$$

Here the $[\ ]_1$ term only appears if $\ell_1 \leqslant \ell' - 1$, the $[\ ]_2$ term only appears if $\ell_2 \leqslant \ell' - 1$ and the $[\ ]_3$ term only appears if ℓ_1 or $\ell_2 = 0$. Of these, it is clear that ord_q of (7.4) is $\geqslant -\ell' + 1 + \frac{1}{2}$. The term (7.2) has $\mathrm{ord}_q \leqslant -\ell' + \frac{1}{2}$. If $\ell \leqslant \ell'$, then ord_q of (7.3) is also $-\ell' + \frac{1}{2}$, but it is clear that the sum of (7.2) and (7.3) will not have a lower ord_q than (7.2). Therefore, the sum cannot be zero, concluding the proof. □

COROLLARY 6. *This agrees with (a) in Proposition 2 in [Sha08].*

Theorem 1 is now proved.

Acknowledgements

The first author thanks Laurent Clozel and Université de Paris-Sud for their hospitality during his one month visit while some of these ideas were discussed. The second author worked on this project while a post-doc at Purdue University and the University of Oklahoma. He is grateful for the support of these universities. Both authors would like to thank David Goldberg for helpful conversations.

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