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Composition operators on a functional Hilbert space

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Let T be a mapping from a set X into itself and let $H(X)$ be a functional Hilbert space on the set X. Then the composition operator C_T on $H(X)$ induced by T is a bounded linear transformation from $H(X)$ into itself defined by $C_T f = f \circ T$. In this paper composition operators are characterized in the case when $H(X) = H^2(\pi^+)$ in terms of the behaviour of the inducing functions in the vicinity of the point at infinity. An estimate for the lower bound of $\|C_T\|$ is given. Also the invertibility of C_T is characterized in terms of the invertibility of T.

1. Introduction and preliminaries

Let $H(X)$ denote a functional Hilbert space on a set X, and let T be a mapping from X into itself. Then the composition mapping C_T, defined as

$$C_T f = f \circ T ,$$

maps $H(X)$ into the vector space of all complex-valued functions on X. This mapping C_T is a linear transformation. If for every f in $H(X)$, $C_T f$ is also in $H(X)$, then by the closed graph theorem C_T is a bounded linear transformation on $H(X)$. The Banach algebra of all bounded linear transformations from $H(X)$ into itself is denoted by $B(H(X))$. If

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$C_T \in B(H(X))$, we call it a composition operator on $H(X)$ induced by T . Some of the pertinent questions about these operators are: for which T is the mapping $C_T \in B(H(X))$; when is an element $A \in B(H(X))$ a composition operator; if $C_T \in B(H(X))$, when is it invertible, compact, Fredholm, or normal? The answers of most of these questions are given by Nordgren [5] and Caughran and Schwartz [1]. If $H(X)$ is taken to be a nice well known functional Hilbert space, then some of the results obtained turn out to be very interesting. For example, if $H(X) = H^2(D)$, the classical Hardy space of the unit disc D , then every analytic mapping from D into itself induces a composition operator [6].

In this note we are interested in the case when $H(X)$ is equal to $H^2(\pi^+)$, the Hilbert space of all those functions f analytic on the upper half-plane π^+ for which

$$\sup_{y>0} \left\{\int_{-\infty}^{\infty} |f(x+iy)|^2 dx\right\} < \infty .$$

A characterization of all analytic mappings T from π^+ into itself for which C_T is an operator on $H^2(\pi^+)$ is reported in this paper in terms of the behaviour of T in the neighbourhood of the point at infinity. An estimate of the norm of C_T is given and the invertibility of C_T is characterized.

The symbols P and $\tilde{P}$ will stand for the Poisson integrals in the disc and in the upper half-plane respectively. The linear fractional transformation $L(z) = i(1+z)/(1-z)$ maps D onto π^+ and the unit circle onto the real line with L^{-1} defined as $L^{-1}(w) = (w-i)/(w+i)$. The linear transformation Q on $L^2(m)$, where m is the normalized Lebesgue measure on the unit circle, defined by

$$(Qf)(x) = (1/\sqrt{\pi})\left(f \circ L^{-1}\right)(x)/(x+i) ,$$

is a well established isometric isomorphism of $L^2(m)$ onto $L^2(\mu)$, where μ is the Lebesgue measure on the real line [4]. The set of all analytic functions $T : \pi^+ \to \pi^+$ such that the only singularity that T can have is a pole at ∞ will be denoted by $A(\pi^+)$.

2. Boundedness of composition operators

If t is an analytic mapping from the unit disc D into itself, then it is shown by Schwartz [6] that C_t is a bounded operator on $H^2(D)$. But this is not true in the case of analytic mappings on π^+, as is shown later in an example in this section. In the following theorem a necessary and sufficient condition for an analytic mapping to induce a composition operator on $H^2(\pi^+)$ is given.

THEOREM 2.1. *Let* $T \in A(\pi^+)$. *Then* C_T *is a bounded operator on* $H^2(\pi^+)$ *if and only if the point at infinity is a pole of* T.

Proof. We first suppose that the point at infinity is a pole of T. Since T is analytic in a neighbourhood of ∞, the function $t = L^{-1} \circ T \circ L$ is analytic in a neighbourhood of 1, and $t(1) = 1$. So by Corollary 2 of [8], C_T belongs to $B\left(H^2(\pi^+)\right)$.

For necessity we suppose that C_T is a bounded operator on $H^2(\pi^+)$. Then $f \circ T \in H^2(\pi^+)$ for every $f \in H^2(\pi^+)$. Hence by Corollary 2 of [2, p. 191], $f\left(T(w)\right)$ tends to zero as w tends to infinity within each half-plane $\operatorname{im} w \geq \delta > 0$, where $\operatorname{im} w$ stands for the imaginary part of w. Since the function $1/(i+w)$ belongs to $H^2(\pi^+)$, it follows that $1/\left(i+T(w)\right)$ tends to zero as w tends to infinity, which in turn implies that $T(w)$ tends to infinity as w tends to infinity. This shows that the point at infinity is a pole of T.

COROLLARY 2.2. *Let* $T \in A(\pi^+)$. *If* C_T *is a bounded operator on* $H^2(\pi^+)$, *then* M_β *belongs to* $B\left(H^2(D)\right)$, *where* M_β *is the multiplication operator on* $H^2(D)$ *induced by* $\beta(z) = \left(1-t(z)\right)/(1-z)$, *and* $t = L^{-1} \circ T \circ L$.

Proof. This follows trivially from Theorem 2.1.

EXAMPLES. (1) Let $T(w) = aw + w_0$, where a is a non-zero positive real number and $w_0 \in \pi^+$. Then T induces a composition operator on

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$H^2(\pi^+)$.

(2) Let

$$T(w) = i\big((w+i)^{n+1}+w(w-i)^n\big)/\big((w+i)^{n+1}-w(w-i)^n\big) ,$$

where n is a positive integer. Since the mapping

$$t(z) = \big(L^{-1} \circ T \circ L\big)(z) = \tfrac{1}{2}\big(z^n+z^{n+1}\big)$$

maps D into itself, T maps π^+ into π^+ . Also the point at infinity is a pole of T . Hence by Theorem 2.1, $C_T \in B\big(H^2(\pi^+)\big)$.

(3) Let $T(w) = (aw+b)/(cw+d)$, where $a, b, c, d \in R$, $ad - bc > 0$ and $c \neq 0$. Then T maps π^+ onto itself, and by Theorem 2.1 it does not define a composition operator.

In [9] it is proved that if t is an inner function from D into itself, then $C_T \in B\big(H^2(\pi^+)\big)$ implies that $t_*(1) = 1$, where $T = L \circ t \circ L^{-1}$ and t_* denotes the non-tangential limit of t . We prove this result in the following theorem for any analytic t , not necessarily an inner function.

THEOREM 2.3. *Let t be an analytic function from D into itself and let $T = L \circ t \circ L^{-1}$. Then $C_T \in B\big(H^2(\pi^+)\big)$ implies that $t_*(1) = 1$.*

Proof. Since T is analytic and C_T is a bounded operator on $H^2(\pi^+)$, $|T(w)| \to \infty$ as $|w| \to \infty$ within each half-plane $\operatorname{im} w \geq \delta > 0$, as is established in the proof of the necessary part of Theorem 2.1. Hence $t_*(1) = 1$.

The converse of this theorem is not true as is obvious from the following example.

EXAMPLE. Let $t(z) = 1 - (1-z)^{\frac{1}{2}}$. Then t induces a Hilbert-Schmidt composition operator on $H^2(D)$ [7]. Clearly $t_*(1) = 1$ and

$$T(w) = \big(L \circ t \circ L^{-1}\big) = \big(2(iw-1)\big)^{\frac{1}{2}} - i .$$

But C_T is not bounded. This is because the function $f(w) = 1/(w+i)$ is a member of $H^2(\pi^+)$, but the function

$$(f \circ T)(w) = 1/\sqrt{2(iw-1)}$$

is not in $H^2(\pi^+)$.

Let N denote the set of all non-negative integers. For $n \in N$, define S_n on π^+ as

$$S_n(w) = \left((w-i)^n\right)/\left(\sqrt{\pi}(w+i)^{n+1}\right) .$$

Then it is well known that the family $\{S_n : n \in N\}$ is an orthonormal basis for $H^2(\pi^+)$ [4].

If $\alpha \in \pi^+$, then the reproducing kernel k_α for $H^2(\pi^+)$ is defined by the equation

$$\langle f, k_\alpha \rangle = f(\alpha)$$

for all $f \in H^2(\pi^+)$. Using the above orthonormal basis it can be shown that

$$k_\alpha(w) = \sum_{n=0}^{\infty} \left(\left((w-i)^n\right)/\left(\sqrt{\pi}(w+i)^{n+1}\right)\right)\overline{\left(\left((\alpha-i)^n\right)/\left(\sqrt{\pi}(\alpha+i)^{n+1}\right)\right)}$$

[3, Problem 30].

A simple computation gives

$$k_\alpha(w) = i/2\pi(w-\bar{\alpha})$$

for every $w \in \pi^+$. Furthermore, the norm of k_α is given by

$$\begin{aligned} \|k_\alpha\|^2 &= \langle k_\alpha, k_\alpha \rangle \\ &= k_\alpha(\alpha) \\ &= 1/(4\pi \text{ im } \alpha) . \end{aligned}$$

If C_T is a composition operator, then the set of kernel functions is invariant under C_T^*, and in fact $C_T^* k_\alpha = k_{T(\alpha)}$ [5]. This result is used

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in the following theorem to find a lower estimate for the norm of C_T .

THEOREM 2.4. *If* C_T *is a composition operator on* $H^2(\pi^+)$ *, then*

$$\sup_{w\in\pi^+} \{(\operatorname{im} w)/(\operatorname{im} T(w))\} \le \|C_T\|^2 .$$

Proof. For every $w \in \pi^+$, we have

$$\begin{aligned}((\operatorname{im} w)/(\operatorname{im} T(w))) &= \|k_{T(w)}\|^2/\|k_w\|^2 \\ &= \|C_T^* k_w\|^2/\|k_w\|^2 \\ &\le \|C_T^*\|^2 \\ &= \|C_T\|^2 .\end{aligned}$$

Since $w \in \pi^+$ is arbitrary, it follows that

$$\sup_{w\in\pi^+} \{(\operatorname{im} w)/(\operatorname{im} T(w))\} \le \|C_T\|^2 .$$

3. Invertibility of composition operators

The invertibility of composition operators on $H^2(D)$ was studied by Schwartz [6]. He has shown that a composition operator is invertible if and only if it is induced by a conformal automorphism of the unit disc. We shall prove an analogous theorem on the invertibility of composition operators on $H^2(\pi^+)$ by using an argument similar to that of Schwartz.

THEOREM 3.1. *Suppose* $T \in A(\pi^+)$ *and* $C_T \in B(H^2(\pi^+))$. *Then* C_T *is invertible if and only if* T *is invertible.*

Proof. Suppose T is invertible. Since by Theorem 2.1 the point at infinity is a pole of T and T is invertible, it follows that the point at infinity is also a pole of T^{-1}, which shows that $C_{T^{-1}} \in B(H^2(\pi^+))$. Clearly

$$C_T C_{T^{-1}} = C_{T^{-1}} C_T = I .$$

Therefore,

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$$(C_T)^{-1} = C_{T^{-1}} .$$

Conversely, suppose C_T is invertible. From Theorem 1 of [8] we have

$$M_\beta C_t = PQ^{-1}\tilde{P}^{-1}C_T\tilde{P}QP^{-1} ,$$

where $t = L^{-1} \circ T \circ L$, C_t is the composition operator on $H^2(D)$ induced by t , and M_β is the multiplication operator on $H^2(D)$ induced by $\beta(z) = \big(1-t(z)\big)/(1-z)$. Hence we can conclude that $M_\beta C_t$ is invertible. Since M_β is subnormal and surjective, it is an invertible operator on $H^2(D)$. This is enough to conclude that C_t is invertible on $H^2(D)$. From a theorem of [6] we get that t is invertible, and consequently T is invertible. This completes the proof of the theorem.

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