A NOTE ON HENSELIAN VALUATION RINGS

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Let K be a field and K_a its algebraic closure. A valuation ring A of K is called henselian, if there is only one valuation ring C of K_a which lies over A (i.e. such that $C \cap K = A$) or, equivalently, if Hensel's Lemma is valid for K, A (see [5], F). In the following, we shall consider only rank one valuation rings.

Let $L|K$ be an algebraic field extension, B a valuation ring of L, and $A = B \cap K$ the valuation ring of K lying under B. If A is henselian, then obviously so is B. It is natural to ask for conditions such that the converse is true, i.e. that B henselian implies A henselian. This is true, for instance, whenever $L|K$ is purely inseparable (see [2], (10.7)). We intend to show that also each of the following conditions is sufficient:

1) $L|K$ is normal and $L_s \neq L$

2) $[L:K]_s < \infty$ and $L_s \neq L$

where L_s is the separable closure of L and $[\ :\ ]_s$ the separability degree. Condition 2) is weaker than the condition

2') $[L:K] < \infty$ and $[L_s:L] = \infty$

recently presented by Ribenboim([4], C), which in turn is weaker than the condition

2'') $[L:K] < \infty$, and for every $n \geq 1$ there exists exactly one separable extension $K_n|K$ of degree n,

used by Kaplansky and Schilling [3], Theorem 4.

To prove the sufficiency of condition 1) (theorem 1) we shall need only the conjugation theorem for valuation rings and the following well known fact (see [3], Theorem 2, or [2], (27.7)): Any field having

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more than one henselian valuation ring is separably closed. To prove the sufficiency of condition 2) (theorem 2) we shall need only theorem 1 and Artin-Schreier's theorem in a slightly generalized form. Hence, our proof will be much easier than the proof of the analogous theorem in [4].

Before proving these theorems, we want to mention two examples which show that none of the conditions

i) $L_s \neq L$; ii) $[L_s:L] = \infty$; iii) $L|K$ is normal ; iv) $[L:K]_s < \infty$

alone is sufficient.

Example 1. Let A be the p-adic valuation ring of $K = Q$, and let L be the decomposition field of some valuation ring C of K_a which lies over A. Then $L_s = K_a$, $[L_s:L] = \infty$, $B = C \cap L$ is henselian, but $A = B \cap K$ is non-henselian.

Example 2. Let L (resp. K) be the field of all algebraic (resp. real algebraic) numbers. Then $L|K$ is normal, $[L:K]_s = [L:K] = 2 < \infty$, every valuation ring of L is henselian, but no valuation ring of K is henselian (see [4], A)).

Now we prove:

THEOREM 1. *Let* $L|K$ *be normal and* $L_s \neq L$. *If* B *is a henselian valuation ring of* L, *then* $A = B \cap K$ *is a henselian valuation ring of* K.

Proof. Let C be the unique valuation ring of L_a lying over B. Suppose that A is not henselian. Then there exists some valuation ring $C' \neq C$ of L_a which lies over A, and we have $B' \neq B$, where $B' = C' \cap L$. There exists some K-automorphism σ of L_a such that $C = \sigma C'$, and we have $\sigma L = L$ since $L|K$ is normal. Since B is the only henselian valuation ring of L, there exists some valuation ring $C'' \neq C'$ of L_a which lies over B', and we have $\sigma C'' \neq C$, $\sigma C'' \cap L = \sigma(C'' \cap L) = \sigma B' = \sigma(C' \cap L) = C \cap L = B$, which is a contradiction.

COROLLARY. *Let* A *be a non-henselian valuation ring of* K, $L|K$ *a separable extension, and* B *a henselian valuation ring of* L *lying over* A. *Then* $L_s = K_s$ *is the least field that contains* L *and is normal over* K.

Proof. $L_s = K_s$ is obviously normal over K. On the other hand, let $N|K$ be normal and $L \subseteq N \subseteq L_a$. Then the unique valuation ring

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C of N that lies over B is henselian, and from theorem 1, it follows that $N = N_s$, hence $L_s \subseteq N$.

In particular, applying this corollary to a henselisation (L, B) of a non-henselian valued field (K, A)[1], we see that no field between L and L_s and distinct from L_s is normal over K.

We shall use Artin-Schreier's theorem in the following form:

LEMMA. Let S be a separably closed field and let K be a subfield of S such that $1 < [S:K]_s < \infty$. Then K is really closed, S is algebraically closed, and $S = K(\sqrt{-1})$.

Proof. The algebraic closure S_a of S is purely inseparable over the field $L = \{a \in S \mid a$ separable over $K\}$, and the fixed field K' of the Galois group of $S_a|K$ is purely inseparable over K. We have $S_a = L \cdot K'$, hence $[S_a:K'] \leq [L:K]_s = [S:K]_s < \infty$. On the other hand $[S_a:K]_s \geq [S:K]_s > 1$, hence $S_a \neq K'$. By Artin-Schreier's theorem (see [1], theorem 4), K' is really closed, S_a is algebraically closed, and $S_a = K'(\sqrt{-1})$. Since K' has characteristic zero, we have $S_a = S = L$ and $K' = K$.

Now we prove:

THEOREM 2.[2] Let $L|K$ be algebraic, $[L:K]_s < \infty$, and $L_s \neq L$. If B is a henselian valuation ring of L, then $A = B \cap K$ is a henselian valuation ring of K.

Footnotes

1) A henselisation of a valued field (K, A) is a valued field (L, B), consisting of the decomposition field L over K of some valuation ring C of K_s that lies over A and the valuation ring $B = C \cap L$. The henselisation of (K, A) is unique up to an K-isomorphism. In particular, $(L, B) = (K, A)$ if and only if A is henselian. (See [5], F).

2) I was told by Mr. Ribenboim that another proof of theorem 2 was communicated to him by Mr. Neukirch, some months ago. For the case of a perfect field K see J. Neukirch, Bonner Math. Schriften Nr. 25, (4.12).

Proof. Without loss of generality we may assume $[L:K]_s > 1$. Let N be the least field that contains L and is normal over K; then $[L:K]_s \leq [N:K]_s < \infty$. Suppose that $N_s = N$; then from the lemma it follows that $[N:K] = 2$, hence $[N:L] = [N:K].[L:K]^{-1} \leq 1$, hence $L = N = N_s \supseteq L_s$, in contradiction to $L_s \neq L$; therefore $N_s \neq N$. Since the valuation ring C of N that lies over B is henselian, we conclude from theorem 1 that A is henselian.

COROLLARY. *Let* A *be a non-henselian valuation ring of* K, $L|K$ *an algebraic extension, and* B *a henselian valuation ring lying over* A. *If* $[L:K]_s < \infty$, *then* K *is really closed*, L *is algebraically closed, and* $L = K(\sqrt{-1})$.

Proof. $[L:K]_s > 1$, since $L|K$ is not purely inseparable. If $[L:K]_s < \infty$, then $L_s = L$ by theorem 2. Now the corollary results from the lemma.

Applying this corollary to a henselisation (L, B) of a non-henselian valued field (K, A), we see that $L|K$ is never finite unless K is really closed and $L = L_s = L_a = K(\sqrt{-1})$. One should note that the field L of a henselisation (L, B) of a non-henselian valued field (K, A) may be separably closed also in other cases. Indeed, this happens whenever K has a henselian valuation ring A_h (since then L has more than one henselian valuation ring). One knows that in this case any valuation ring $A \neq A_h$ of K is *saturated*, i.e. its value group Γ_A is divisible and its residue field Λ_A is algebraically closed (see [2], (27.6)). Moreover, we prove:

THEOREM 3. *Let* A *be a valuation ring of* K *such that* Λ_A *has characteristic zero, and let* (L, B) *be a henselisation of* (K, A). *Then*:

L *is algebraically closed* $\Longleftrightarrow$ A *is saturated*.

Proof. L is the decomposition field over K of some valuation ring C of K_a that lies over A. Let M be the inertia field of C over K. Then $K_a|M$ and $M|L$ are Galois extensions, with Galois groups G_1 and G_2, say. G_1 is isomorphic to the character group of the value group extension Γ_C/Γ_A, and G_2 is isomorphic to the Galois group of the Galois extension $\Lambda_C | \Lambda_A$ (see [2], (20.1) and (20.20)). A is saturated if and only if $\Gamma_C = \Gamma_A$ and $\Lambda_C = \Lambda_A$ (see [2], (22.7)). These equations hold if and only if G_1 and G_2 are trivial, if and only if $L = K_a$.

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