Glasgow Math. J. **56** (2014) 681–689. © Glasgow Mathematical Journal Trust 2014.
doi:10.1017/S001708951400010X.

ON LENGTH DISTORTIONS WITH RESPECT TO QUADRATIC DIFFERENTIAL METRICS

ZONGLIANG SUN
Department of Mathematics, Shenzhen University, Shenzhen 518060, P. R. China
E-mail: moonshoter@163.com

(Received 29 December 2012; accepted 7 September 2013)

Abstract. In this paper, we consider the question about length distortions under quasiconformal mappings with respect to quadratic differential metrics. More precisely, let X and Y be closed Riemann surfaces with genus at least 2, and $f : X \to Y$ being a K-quasiconformal mapping. Given two quadratic differential metrics $|q_1|$ and $|q_2|$ with unit areas on X and Y respectively, whether there exists a constant $\mathcal{C}$ depending only on K such that $\frac{1}{\mathcal{C}} l_{q_1}(\gamma) \leq l_{q_2}(f(\gamma)) \leq \mathcal{C} l_{q_1}(\gamma)$ holds for any simple closed curve $\gamma \subset X$. Here $l_{q_i}(\alpha)$ denotes the infimum of the lengths of curves in the homotopy class of α with respect to the metric $|q_i|$, $i = 1, 2$. We give positive answers to this question, including the aspects that the desired constant $\mathcal{C}$ explicitly depends on q_1, q_2 and K, and that the constant $\mathcal{C}$ is universal for all the quantities involved.

2000 *Mathematics Subject Classification.* Primary 30F45; Secondary 51M25.

1. Introduction. Let us fix some notations. Let S be a closed surface of genus $g \geq 2$. Denote by $T(S)$ the Teichmüller space of S, and denote by $M(S)$ the moduli space of S. For $X \in T(S)$, let $Q(X)$ be the space of holomorphic quadratic differentials on X. Each non-zero $q \in Q(X)$ induces a metric $|q(z)||dz|^2$ on X, which is called the flat metric or the quadratic differential metric. The theory of holomorphic quadratic differentials and its connections with the Teichmüller theory have been studied extensively (see, e.g. [**4**, **6**, **9**]). In recent years, the theory of quadratic differentials and quadratic differential metrics has been studied actively from some new perspectives (see, e.g. [**2**, **8**, **10**]).

In this paper, we consider the question of length distortions under quasiconformal mappings with respect to quadratic differential metrics. Before stating our main results, we recall the following classical result of Wolpert [**11**, Lemma 3.1] on distortions under quasiconformal mappings of hyperbolic length, length with respect to the hyperbolic metric.

THEOREM A. ([**11**]) *Let $f : X \to Y$ be a K-quasiconformal mapping from X to Y. Then for any closed curve $\gamma \subset X$, we have*

$$\frac{l_X(\gamma)}{K} \leq l_Y(f(\gamma)) \leq K l_X(\gamma),$$

where $l_X(\gamma)$ and $l_Y(f(\gamma))$ are the infima of hyperbolic lengths of curves homotopic to γ and $f(\gamma)$ on X and Y respectively.

In this paper, we consider the following question about length distortions with respect to the quadratic differential metrics: Let $f : X \to Y$ be a K-quasiconformal mapping between Riemann surfaces X and Y with genus $g \geq 2$. Endow $Q(X)$ and $Q(Y)$ with the L_1-norm, and let $Q_1(X)$ and $Q_1(Y)$ be the unit spheres in $Q(X)$ and $Q(Y)$ respectively. Then the question is as follows: Given two holomorphic quadratic differentials $q_1 \in Q_1(X)$ and $q_2 \in Q_1(Y)$ and hence two quadratic differential metrics $|q_1|$ and $|q_2|$ on X and Y respectively, whether there exists a constant $\mathcal{C}$ depending only on K such that

$$\frac{l_{q_1}(\gamma)}{\mathcal{C}} \leq l_{q_2}(f(\gamma)) \leq \mathcal{C} l_{q_1}(\gamma)$$

holds for any simple closed curve $\gamma \subset X$, where $l_{q_1}(\gamma)$ and $l_{q_2}(f(\gamma))$ denote the infima of the lengths of curves in the homotopy class of γ and $f(\gamma)$ with respect to metrics $|q_1|$ and $|q_2|$ respectively?

We will give three positive answers to this question. Firstly, in Theorem 1, we give a specific answer, where the desired constant $\mathcal{C}$ depends explicitly on q_1, q_2 and K. Recall that for $q \in Q(X)$, $||q||_\infty = \sup_X \frac{|q(z)|}{\lambda(z)}$ is Bers' sup-norm of q, where the supremum is taken over X and $\lambda(z)|dz|^2$ is the hyperbolic area element on X.

THEOREM 1. *Let $f : X \to Y$ be a K-quasiconformal mapping. Then given two quadratic differentials $q_1 \in Q_1(X)$ and $q_2 \in Q_1(Y)$, there exist constants $C_1 = 2\mathcal{M}(X)\sqrt{\pi(g-1)||q_2||_\infty||q_1||_\infty}$ and $C_2 = 2\mathcal{M}(Y)\sqrt{\pi(g-1)||q_2||_\infty||q_1||_\infty}$ depending only on q_1 and q_2 such that*

$$\frac{l_{q_1}(\gamma)}{C_2 K} \leq l_{q_2}(f(\gamma)) \leq C_1 K l_{q_1}(\gamma)$$

holds for any simple closed curve $\gamma \subset X$, where $\mathcal{M}(X)$ and $\mathcal{M}(Y)$ are constants depending only on X and Y respectively.

Next, in Theorem 2, we give a general answer where the desired constant $\mathcal{C}$ depends only on K and is universal for all the $q_1 \in Q_1(X)$ and $q_2 \in Q_1(Y)$.

THEOREM 2. *Let $f : X \to Y$ be a K-quasiconformal mapping. Then there exists a constant $C = C(X, Y)$ depending only on X and Y such that for any two quadratic differentials $q_1 \in Q_1(X)$ and $q_2 \in Q_1(Y)$,*

$$\frac{l_{q_1}(\gamma)}{CK} \leq l_{q_2}(f(\gamma)) \leq CK l_{q_1}(\gamma)$$

holds for any simple closed curve $\gamma \subset X$.

Finally, in Theorem 3, we improve Theorem 2 so that the desired constant $\mathcal{C}$ is universal for all the ingredients involved.

THEOREM 3. *For any $\epsilon > 0$, there exists a constant C_ϵ depending only on ϵ such that for any $X, Y \in M_\epsilon(S)$, any K-quasiconformal mapping $f : X \to Y$, and any two quadratic differentials $q_1 \in Q_1(X)$ and $q_2 \in Q_1(Y)$,*

$$\frac{l_{q_1}(\gamma)}{C_\epsilon K} \leq l_{q_2}(f(\gamma)) \leq C_\epsilon K l_{q_1}(\gamma)$$

holds for any simple closed curve $\gamma \subset X$.

2. Preliminaries. In this section, we give some necessary backgrounds on related topics. For references, see [**3**, **4**, **7**, **9**].

Teichmüller space, moduli space. Let S be a closed surface of genus $g \geq 2$. The Teichmüller space $T(S)$ of S is the space of equivalence classes of marked Riemann surfaces $(X, f : S \to X)$, where two markings $f_1 : S \to X_1$ and $f_2 : S \to X_2$ are equivalent if there exists a conformal mapping $c : X_1 \to X_2$, which is homotopic to $f_2 \circ f_1^{-1}$. The moduli space $M(S)$ of S is obtained by forgetting the markings in $T(S)$. Throughout the paper, we often drop the marking notations for points in $T(S)$, remembering that a marked surface is the same as a surface where we 'know the names of the curves'. For any $\epsilon > 0$, let $M_\epsilon(S)$ be the ϵ-thick part of $M(S)$, which consists of all the Riemann surfaces whose injectivity radius is at least ϵ. By Mumford's compactness theorem, $M_\epsilon(S)$ is compact.

For $X \in T(S)$, the space $Q(X)$ of holomorphic quadratic differentials on X is a complex vector space of complex dimension $3g - 3$ (real dimension $6g - 6$), and the unit sphere $Q_1(S)$ is a compact space with real dimension $6g - 7$. Let $\mathcal{Q}(S) \to M(S)$ be the moduli space of holomorphic quadratic differentials, where the projection $\pi : \mathcal{Q}(S) \to M(S)$ sends (X, q) to X. Denote by $\mathcal{Q}_1(S) \to M(S)$ the moduli space of holomorphic quadratic differentials with unit L_1-norm. Equivalently, we may view $\mathcal{Q}_1(S) = \bigcup_{X \in M(S)} Q_1(X)$. For $\epsilon > 0$, let $\mathcal{Q}_\epsilon(S) = \{q \in \mathcal{Q}_1(S) : \pi(q) \in M_\epsilon(S)\}$. Then it is well known that [**5**] $\mathcal{Q}_\epsilon(S)$ is compact.

Quadratic differential metric. Given a non-zero $q = q(z)dz^2 \in Q(X)$, there are $4g - 4$ zeros of q counted with multiplicities. We call a point $p \in X$ a critical point of q if it is a zero of q, otherwise it is called a regular point. Near each regular point, there is the so-called natural parameter w of X such that $q(z)dz^2 = dw^2$.

A non-zero $q \in Q(X)$ induces a metric on X, which is locally given, in terms of the local parameter z of X, by $|q(z)|^{\frac{1}{2}}|dz|$. We call this metric the flat metric, or the quadratic differential metric. The quadratic differential metric is complete on X. For a closed curve $\gamma \subset X$, denote

$$|\gamma|_q = \int_\gamma |q(z)|^{\frac{1}{2}}|dz|,$$
$$|\gamma|_v = \int_\gamma |\Im\{q(z)^{\frac{1}{2}}dz\}|$$

and

$$|\gamma|_h = \int_\gamma |\Re\{q(z)^{\frac{1}{2}}dz\}|,$$

where $\Im z$ and $\Re z$ represent the imaginary and real parts of z respectively. Let

$$l_q(\gamma) = \inf_{\alpha \sim \gamma}\{|\alpha|_q\},$$
$$v_q(\gamma) = \inf_{\alpha \sim \gamma}\{|\alpha|_v\}$$

and

$$h_q(\gamma) = \inf_{\alpha \sim \gamma}\{|\alpha|_h\},$$

where the infimum is taken over all curves α in the homotopy class of γ. Then $l_q(\gamma)$ is called the q-length or quadratic differential length of γ, $v_q(\gamma)$ is called the vertical length (or q-height) of γ and $h_q(\gamma)$ is called the horizontal length (or q-width) of γ.

A saddle connection of $q = q(z)dz^2$ is by definition a geodesic segment whose endpoints are critical points without passing through any critical point in its interior.

The existence and uniqueness of the geodesic in each freely homotopy class of a closed curve is as follows. Let $q \in Q(X)$ and $\gamma \subset X$ be a closed curve, then there exists a unique q-geodesic in the freely homotopy class of γ, except for the case that γ is one of the continuous families of closed geodesics which sweep out a flat cylinder. In the latter case, each of the two boundary curves of the cylinder consists of a finite number of saddle connections. To sum up, in both cases, the geodesic representative in the free homotopy class of a closed curve γ consists of a finite number of saddle connections.

Measured foliation. A measured foliation $\mathcal{F}$ on a topological surface S of genus $g \geq 2$ is a singular foliation on S where the singularities are isolated and k-pronged ($k \geq 3$), equipped with a measure μ on transverse arcs, which is invariant under translation along leaves (see [**3**, Sections 5.3, 5.4, 6.5 and 6.6] for more details). The space $\mathcal{MF}$ of equivalence classes of measured foliations is defined where two measured foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are equivalent if $i(\gamma, \mathcal{F}_1) = i(\gamma, \mathcal{F}_2)$ for every simple closed curve γ, where $i(\cdot, \cdot)$ is the intersection number. Two classes of measured foliations $[\mathcal{F}_1]$ and $[\mathcal{F}_2]$ are projectively equivalent if there is a constant $r > 0$ so that $i(\gamma, \mathcal{F}_1) = ri(\gamma, \mathcal{F}_2)$ for every simple closed curve γ. The space of projective equivalence classes of measured foliations is denoted by $\mathcal{PMF}$.

We have the following descriptions of $\mathcal{MF}$ and $\mathcal{PMF}$.

THEOREM B. ([**10**]) *$\mathcal{MF}$ is homeomorphic to a $6g-6$-dimensional ball. $\mathcal{PMF}$ is homeomorphic to a $6g-7$-dimensional sphere.*

There is a special class of measured foliations $\mathcal{F}$ with the property that the complement of critical leaves (those passing through singularities) is homeomorphic to a cylinder. Foliation's leaves on the cylinder are all freely homotopic to a single simple closed curve γ. Such a foliation is completely determined as a point in $\mathcal{MF}$ by height h of the cylinder ($h = i(A, \mathcal{F})$, the infimum of $\int_A \mathcal{F}$ for arcs A with endpoints on the boundary of the cylinder) and the homotopy class of γ. Denote this foliation by $\mathcal{F}_{\gamma,h}$. Let $\mathcal{S}(S)$ be the set of all homotopy classes of simple closed curves on S. Then we have the following.

THEOREM C. ([**10**]) *There is an embedding $\mathcal{S}(S) \times \mathbb{R}^+ \to \mathcal{MF}$ which sends (γ, h) to $\mathcal{F}_{\gamma,h}$. The image of this embedding is dense in $\mathcal{MF}$. The image of $\mathcal{S}(S)$ in $\mathcal{PMF}$ is dense.*

Extremal length, Jenkins–Strebel differential. Let X be a Riemann surface and γ be a simple closed curve. Then the extremal length $\mathrm{ext}_X(\gamma)$ of γ is defined by [**1**]

$$\sup_{\rho} \left\{ \frac{l_\rho^2(\gamma)}{A_\rho} \right\},$$

where the supremum is taken over all conformal metrics $\rho = \rho(z)|dz|^2$ on X with area $0 < A_\rho = \int_X \rho < \infty$, and $l_\rho(\gamma)$ is the infimum of ρ-length of simple closed curves homotopic to γ.

Note that [**7**] both hyperbolic length and extremal length can be extended continuously from simple closed curves to measured foliations.

Jenkins and Strebel [9] showed that there is an extremal metric which realises the supremum in the definition of extremal length, and that such a metric is a quadratic differential metric $|q|$ for some $q \in Q(X)$. Such a quadratic differential is called the Jenkins–Strebel differential of γ on X, and is denoted by $\phi(\gamma, X)$.

3. Some lemmas. For a given Riemann surface X, we give a uniform comparison, for all $q \in Q_1(X)$, between the corresponding hyperbolic lengths and the quadratic differential lengths of simple closed curves.

LEMMA 1. *There exists a constant $\kappa = \kappa(X)$ depending only on X such that for any $q \in Q_1(X)$ and any $\gamma \in \mathcal{S}(X)$, we have*

$$\frac{l_X(\gamma)}{\kappa} \le l_q(\gamma) \le \kappa l_X(\gamma).$$

Proof. The proof follows from a compactness argument. Consider the function

$$F(q, \mathcal{F}) = \frac{l_q(\mathcal{F})}{l_X(\mathcal{F})}.$$

By Theorem B, it is well defined on the compact space $Q_1(X) \times \mathcal{PMF}$. $F(q, \mathcal{F})$ is positive and continuous, and hence is uniformly bounded. Consequently, from the embedding of $\mathcal{S}(X)$ into $\mathcal{MF}$ as in Theorem C, $l_q(\gamma)/l_X(\gamma)$ is uniformly bounded for all $q \in Q_1(X)$ and $\gamma \in \mathcal{S}(X)$. □

The following result improves Lemma 1. It gives uniform comparisons on $M_\epsilon(S)$ and $\mathcal{Q}_\epsilon(S)$.

LEMMA 2. *For any $\epsilon > 0$, there exists a constant $\mathcal{C}_\epsilon$ depending only on ϵ such that for any $X \in M_\epsilon(S)$ and any $q \in Q_1(X)$,*

$$\frac{l_X(\gamma)}{\mathcal{C}_\epsilon} \le l_q(\gamma) \le \mathcal{C}_\epsilon l_X(\gamma)$$

holds for all simple closed curve γ.

Proof. Consider the positive and continuous function

$$F(X, q, \mathcal{F}) = \frac{l_q(\mathcal{F})}{l_X(\mathcal{F})}$$

defined on $M_\epsilon(S) \times \mathcal{Q}_\epsilon(S) \times \mathcal{PMF}$. From Theorem B and the compactness of $M_\epsilon(S)$ and $\mathcal{Q}_\epsilon(S)$, $F(X, q, \mathcal{F})$ is bounded uniformly. □

Since saddle connections are Euclidean straight line segments measured in the quadratic differential metric, we have the following description of the relation of q-length, q-height and q-width. This can be taken as a 'Pythagoras theorem' for quadratic differential metric.

LEMMA 3. *Let $q \in Q(X)$ and $\gamma \subset X$ be a simple closed curve. Suppose $\tilde{\gamma} = \bigcup_{i=1}^n \gamma_i$ is the geodesic representative (in the quadratic differential metric $|q|$) of γ, where γ_i are saddle connections. Then for each γ_i, we have*

$$l_q(\gamma_i) = \sqrt{(|\gamma_i|_v)^2 + (|\gamma_i|_h)^2}, \quad i = 1, 2, \ldots, n.$$

Recall that a quasiconformal mapping $f : X \to Y$ is the Teichmüller mapping if its Beltrami coefficient $\mu_f = f_z/f_{\bar{z}}$ is of the form $\mu_f = k\bar{q}/|q|$ for some constants $k \in [0, 1)$ and $q \in Q(X)$. These mappings are important since they are unique dilatation minimizing quasiconformal mappings in their homotopy classes. To end this section, we record the following result (see e.g. [**6**]) for later use. It says that the Teichmüller mappings are essentially affine with respect to natural choices of coordinates.

LEMMA 4. *Let $f : X \to Y$ be a Teichmüller mapping with maximal dilatation K. Then f determines a unique* (*up to a multiplicative positive constant*) *$q_1 \in Q(X)$, which is called the initial differential of f, and correspondingly a unique $q_2 \in Q(Y)$, which is called the terminal differential of f, with the following properties:*

(i) *if $w = \xi + i\eta$ is a natural parameter of q_1 at p, then there exists a natural parameter $\zeta = \sigma + i\tau$ of q_2 at $f(p)$ such that f is locally represented as*

$$\xi \mapsto \sigma = K^{\frac{1}{2}}\xi, \quad \eta \mapsto \tau = K^{-\frac{1}{2}}\eta. \tag{1}$$

(ii) *f maps the zeros of q_1 to those of q_2, and the order of q_1 at a zero $p \in X$ equals the order of q_2 at $f(p)$.*

4. Proof of the main results.

THEOREM 1. *Let $f : X \to Y$ be a K-quasiconformal mapping. Then given two quadratic differentials $q_1 \in Q_1(X)$ and $q_2 \in Q_1(Y)$, there exist constants $C_1 = 2\mathcal{M}(X)\sqrt{\pi(g-1)||q_2||_\infty||q_1||_\infty}$ and $C_2 = 2\mathcal{M}(Y)\sqrt{\pi(g-1)||q_2||_\infty||q_1||_\infty}$ depending only on q_1 and q_2 such that*

$$\frac{l_{q_1}(\gamma)}{C_2K} \le l_{q_2}(f(\gamma)) \le C_1 K l_{q_1}(\gamma)$$

holds for any simple closed curve $\gamma \subset X$, where $\mathcal{M}(X)$ and $\mathcal{M}(Y)$ are constants depending only on X and Y respectively.

Proof. We need two inequalities on comparisons of hyperbolic length and quadratic differential length. Generally, let R be a Riemann surface, $q \in Q_1(R)$, $\gamma \subset R$ be a simple closed curve and $\lambda(z)|dz|^2$ be the hyperbolic metric on R.

Firstly, we have the following estimate:

$$\begin{aligned} |\gamma|_q &= \int_\gamma \sqrt{|q(z)|}|dz| \\ &= \int_\gamma \frac{\sqrt{|q(z)|}}{\sqrt{\lambda(z)}}\sqrt{\lambda(z)}|dz| \\ &\le \sqrt{||q||_\infty}\int_\gamma \sqrt{\lambda(z)}|dz| \\ &= \sqrt{||q||_\infty}|\gamma|_\lambda, \end{aligned}$$

where $||q||_\infty = \sup_R \frac{|q(z)|}{\lambda(z)}$ is Bers' sup-norm, which is finite since R is closed. Consequently,

$$l_q(\gamma) \le \sqrt{||q||_\infty}\, l_R(\gamma). \tag{2}$$

https://doi.org/10.1017/S001708951400010X Published online by Cambridge University Press

Secondly, from the definition of extremal length and the extremality of the metric induced by the Jenkins–Strebel differential $\phi(\gamma, R) \in Q_1(R)$ of γ, we get

$$\frac{l_R^2(\gamma)}{2\pi(2g-2)} \leq \text{ext}_R(\gamma) = \frac{l^2_{\phi(\gamma,R)}(\gamma)}{||\phi(\gamma, R)||_1} = l^2_{\phi(\gamma,R)}(\gamma), \tag{3}$$

where, by Gauss–Bonnet's formula, $2\pi(2g-2)$ is a hyperbolic area of R. Therefore, in order to compare $l_R(\gamma)$ with $l_q(\gamma)$, we need to compare $l_{\phi(\gamma,R)}(\gamma)$ with $l_q(\gamma)$. For this, observe that since $Q(R)$ is of finite dimension, the norms $||\cdot||_1$ and $||\cdot||_\infty$ are equivalent, i.e. there exists a constant $L = L(R)$ such that

$$\frac{1}{L}||q||_1 \leq ||q||_\infty \leq L||q||_1 \tag{4}$$

holds for any $q \in Q(R)$. Then (2) and (4) imply that

$$l_{\phi(\gamma,R)}(\gamma) \leq \sqrt{||\phi(\gamma, R)||_\infty}\, l_R(\gamma) \leq L\sqrt{||q||_\infty}\, l_R(\gamma) \tag{5}$$

holds for any $q \in Q_1(R)$ and any γ. Thus, from Lemma 1 and (5) we get the following comparison of $l_{\phi(\gamma,R)}(\gamma)$ with $l_q(\gamma)$,

$$\frac{l_{\phi(\gamma,R)}(\gamma)}{l_q(\gamma)} = \frac{l_R(\gamma)}{l_q(\gamma)}\frac{l_{\phi(\gamma,R)}(\gamma)}{l_R(\gamma)} \leq \kappa(R)L(R)\sqrt{||q||_\infty}. \tag{6}$$

Consequently, we deduce from (3) and (6) that

$$l_R(\gamma) \leq 2\kappa(R)L(R)\sqrt{\pi(g-1)||q||_\infty}\, l_q(\gamma) \tag{7}$$

holds for any $q \in Q_1(R)$ and any γ. This finishes our general comparisons between $l_R(\gamma)$ and $l_q(\gamma)$.

Now we are ready to give the proof of the theorem. We conclude from Theorem A, (2) and (7) that

$$\begin{aligned} l_{q_2}(f(\gamma)) &\leq \sqrt{||q_2||_\infty}\, l_Y(f(\gamma)) \\ &\leq \sqrt{||q_2||_\infty}\, K l_X(\gamma) \\ &\leq 2\kappa(X)L(X)\sqrt{\pi(g-1)||q_2||_\infty||q_1||_\infty}\, K l_{q_1}(\gamma). \end{aligned}$$

Similarly, we have

$$\begin{aligned} l_{q_1}(\gamma) &\leq \sqrt{||q_1||_\infty}\, l_X(\gamma) \\ &\leq \sqrt{||q_1||_\infty}\, K l_Y(f(\gamma)) \\ &\leq 2\kappa(Y)L(Y)\sqrt{\pi(g-1)||q_2||_\infty||q_1||_\infty}\, K l_{q_2}(f(\gamma)). \end{aligned}$$

Consequently, the desired result follows from the above two inequalities. □

As a special case, when considering the distortion under the Teichmüller mapping with respect to the metrics induced by initial and terminal differentials, we note the following observation.

https://doi.org/10.1017/S001708951400010X Published online by Cambridge University Press

PROPOSITION 1. *Let $f : X \to Y$ be a Teichmüller mapping whose maximal dilatation is K. Let $q_1 \in Q_1(X)$ and $q_2 \in Q_1(Y)$ be the initial and terminal differentials of f, respectively. Then for any simple closed curve $\gamma \subset X$, we have*

$$\frac{l_{q_1}(\gamma)}{\sqrt{K}} \leq l_{q_2}(f(\gamma)) \leq \sqrt{K} l_{q_1}(\gamma).$$

Proof. Let $\tilde{\gamma} = \bigcup_{i=1}^{n} \gamma_i$ be the q_1-geodesic representative of γ, where γ_i are saddle connections of q_1, $i = 1, 2, \ldots, n$. Then,

$$l_{q_1}(\gamma) = \sum_{i=1}^{n} l_{q_1}(\gamma_i). \tag{8}$$

By Lemma 4, $f(\tilde{\gamma})$ is the q_2-geodesic representative of $f(\gamma)$, with $f(\gamma_i)$ being the saddle connections of q_2, $i = 1, 2, \ldots, n$. Consequently,

$$l_{q_2}(f(\gamma)) = \sum_{i=1}^{n} l_{q_2}(f(\gamma_i)). \tag{9}$$

From Lemma 3, we have

$$l_{q_1}(\gamma_i) = \sqrt{(|\gamma_i|_v)^2 + (|\gamma_i|_h)^2}, \quad i = 1, 2, \ldots, n, \tag{10}$$

$$l_{q_2}(f(\gamma_i)) = \sqrt{(|f(\gamma_i)|_v)^2 + (|f(\gamma_i)|_h)^2}, \quad i = 1, 2, \ldots, n. \tag{11}$$

In view of (1), (11) becomes

$$l_{q_2}(f(\gamma_i)) = \sqrt{K^{-1}(|\gamma_i|_v)^2 + K(|\gamma_i|_h)^2}, \quad i = 1, 2, \ldots, n. \tag{12}$$

Consequently, from (10) and (12) we conclude that

$$K^{-\frac{1}{2}} l_{q_1}(\gamma_i) \leq l_{q_2}(f(\gamma_i)) \leq K^{\frac{1}{2}} l_{q_1}(\gamma_i), \quad i = 1, 2, \ldots, n. \tag{13}$$

Therefore, we get the desired result from (8), (9) and (13). □

THEOREM 2. *Let $f : X \to Y$ be a K-quasiconformal mapping. Then there exists a constant $C = C(X, Y)$ depending only on X and Y, such that for any two quadratic differentials $q_1 \in Q_1(X)$ and $q_2 \in Q_1(Y)$,*

$$\frac{l_{q_1}(\gamma)}{CK} \leq l_{q_2}(f(\gamma)) \leq CK l_{q_1}(\gamma)$$

holds for any simple closed curve $\gamma \subset X$.

Proof. The result follows from the following observation. From (4), we see that both of the two constants C_1 and C_2 in Theorem 1 can be replaced by a constant $C = C(X, Y)$ depending only on X and Y, where $C = C(X, Y)$ is universal for all $q_1 \in Q_1(X)$ and $q_2 \in Q_1(Y)$.

From another viewpoint, the result also follows from Theorem A and Lemma 1. □

Since Lemma 2 improves Lemma 1, from Theorem A and Lemma 2 we obtain the following improvement of Theorem 2.

THEOREM 3. *For any $\epsilon > 0$, there exists a constant C_ϵ depending only on ϵ such that for any X, $Y \in M_\epsilon(S)$, any K-quasiconformal mapping $f : X \to Y$, and any two quadratic differentials $q_1 \in Q_1(X)$ and $q_2 \in Q_1(Y)$,*

$$\frac{l_{q_1}(\gamma)}{C_\epsilon K} \leq l_{q_2}(f(\gamma)) \leq C_\epsilon K l_{q_1}(\gamma)$$

holds for any simple closed curve $\gamma \subset X$.

ACKNOWLEDGEMENTS. The author is very grateful to the referee of the previous version of this paper for their suggestions, which significantly improved it and lead to the current version. The author also thanks the present referee for helpful suggestions which made the paper more precise and readable and for their interest in this topic. The research was supported by NNSF for Young Scientists of China (11101290) and NNSF of China (11071179, 10871211).

REFERENCES

1. L. Ahlfors, *Lectures on quasiconformal mappings* (Van Nostrand Reinhold Princeton, NJ, 1966).

2. M. Duchin, C. Leininger and K. Rafi, Length spectra and degeneration of flat metrics, *Invent. Math.* **182**(2) (2010), 231–277.

3. A. Fathi, F. Laudenbach and V. Poenaru, translated by D. Kim and D. Margalit, *Thurston's work on surfaces* (Mathematical Notes, 48, Princeton University Press, Princeton, 2012).

4. F. Gardiner, *Teichmüller theory and quadratic differentials* (John Wiley, New York, NY, 1987).

5. U. Hamenstädt, Closed Teichmüller geodesics in the thin part of moduli space, arXiv:math/0511349 (2009).

6. Y. Imayoshi and M. Taniguchi, *An introduction to Teichmüller space* (translated and revised from the Japanese by the authors) (Springer-Verlag, Tokyo, Japan, 1992).

7. S. Kerckhoff, The asymptotic geometry of Teichmüller space, *Topology* **19**(1) (1980), 23–41.

8. M. Kontsevich and A. Zorich, Connected components of the moduli spaces of Abelian differentials with prescribed singularities, *Invent. Math.* **153**(3) (2003), 631–678.

9. K. Strebel, *Quadratic differentials* (Springer-Verlag, New York, NY, 1984).

10. W. Thurston, On the geometry and dynamics of diffeomorphisms of surfaces, *Bull. Amer. Math. Soc.* **19(2)** (1988), 417–431.

11. S. Wolpert, The length spectra as moduli for compact Riemann surfaces, *Ann. Math.* **109(2)** (1979), 323–351.