BULL. AUSTRAL. MATH. SOC. 34C10
VOL. 10 (1974), 9-14.

Nonoscillation of third order retarded equations

Bhagat Singh and R.S. Dahiya

The third order delay equation

$$y'''(t) + a(t)y_\tau(t) = 0$$

is studied for its nonoscillatory nature under the general condition in which $a(t)$ has been allowed to oscillate. It is shown by way of a differential inequality that if $g(t)$ is a thrice differentiable and eventually positive function then

$$g'''(t) + t^2|a(t)|g(t) \le 0$$

is sufficient for this equation to have bounded nonoscillatory solutions.

1. Introduction

In this paper, we study the third order retarded equation

$$y'''(t) + a(t)y_\tau(t) = 0 , \tag{1}$$

where $y_\tau(t) \equiv y\big(t-\tau(t)\big)$ and

(i) $a : (-\infty, \infty) \to (-\infty, \infty)$ is a continuous function,

(ii) τ is a nonnegative, real valued, continuous and bounded function of $t \in (-\infty, \infty)$.

Our aim here is to prove a theorem that establishes the existence of a bounded nonoscillatory solution of equation (1) on some half line $[t_0, \infty)$. Equations of the type

Received 31 July 1973.

(2) $$y'''(t) + p(t)y(t) = 0 , \quad p(t) \geq 0$$

have been studied by Onose [2], but the assumption that p be nonnegative is restrictive. For this reason, the study of equation (1) under oscillating a becomes all the more interesting.

Call a function $g \in C[t_0, \infty)$ oscillatory if it has arbitrarily large zeros on $[t_0, \infty)$. Otherwise call it non-oscillatory.

In what follows, the term "solution" will be applied only to continuous and extendable solutions of equation (1) over some half line $[t_0, \infty)$, $t_0 > 0$.

The second order retarded equation

(3) $$y''(t) + a(t)y_\tau(t) = 0$$

when a is oscillating has been studied by Wong [4]. The most interesting part of our theorem is the differential inequality

(4) $$g'''(t) + t^2|a(t)|g(t) \leq 0 , \quad g(t) > 0$$

as being sufficient for the existence of bounded non-oscillatory solutions of equation (1).

2. Main result

THEOREM 1. *Let g be a thrice differentiable function on some half line $[T, \infty)$, $T \geq t_0 > 0$ such that*

(5) $$\liminf_{t\to\infty} g(t) > 0 ,$$

and

$$g'''(t) + t^2|a(t)|g(t) \leq 0$$

eventually. Then equation (1) *has bounded non-oscillatory solutions.*

Proof. Let T be large enough so that $g(t) > 0$ in $[T, \infty)$. Then inequality (4) implies that there exists some $T_1 > T$ such that

$$g'''(t) \leq 0 , \quad g''(t) > 0 , \quad g(t) > 0 , \quad t \geq T_1 .$$

https://doi.org/10.1017/S0004972700040569 Published online by Cambridge University Press

This means that g' is monotonic, and hence two cases arise.

Case 1. $g'(t) \geq 0$, $t \geq T_1$. Dividing (4) by $g(t)$ and integrating between $[T_1, t]$ we have

$$(6)\qquad \frac{g''(t)}{g(t)} - \frac{g''(T_1)}{g(T_1)} + \int_{T_1}^{t} \frac{g''(s)g'(s)}{g^2(s)}\, ds + \int_{T_1}^{t} s^2|a(s)|\, ds \leq 0 \;;$$

(6) immediately implies

$$(7)\qquad \lim_{t\to\infty} \int_{T_1}^{t} s^2|a(s)|\, ds < \infty \;.$$

Case 2. $g'(t) \leq 0$, $t \geq T_1$. In this case also we shall prove that (7) holds. Suppose to the contrary

$$(8)\qquad \int_{T_1}^{\infty} t^2|a(t)|\, dt = \infty \;.$$

Then from inequality (6) and (8) it follows that

$$(9)\qquad \lim_{t\to\infty} \int_{T_1}^{t} \frac{g''(s)g'(s)}{g^2(s)}\, ds = -\infty \;.$$

Now g'' is decreasing and nonnegative. Therefore

$$\lim_{t\to\infty} \int_{T_1}^{t} \frac{g''(s)g'(s)}{g^2(s)}\, ds \geq \lim_{t\to\infty} \left[g''(T_1) \int_{T_1}^{t} \frac{g'(s)}{g^2(s)}\, ds \right]$$
$$= \lim_{t\to\infty} \left\{ g''(T_1) \left[-\frac{1}{g(t)} + \frac{1}{g(T_1)} \right] \right\} > -\infty$$

by Condition (5). This is the required contradiction. Thus conclusion (7) holds as a result of inequality (4) and condition (5).

Now we employ a process of approximation as used in Singh [3].

Define the approximations

$$(10)\qquad y_0(t) \equiv 1 \;, \quad y_0'(t) \equiv 0 \;, \quad y_0''(t) \equiv 0 \;,$$

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(11) $y_n(t) = \frac{1}{2} - \int_t^\infty \frac{(s-t)^2}{2} a(s) y_{n-1}(s-\tau(s)) ds$, $n = 1, 2, 3, \ldots, \infty$.

Let $T_2 > T_1$ be so large that for $t \geq T_2$,

(12) $$\int_t^\infty s^2 |a(s)| ds \leq \frac{1}{2} .$$

From (10), (11) and (12) it follows that if $t \geq T_2$,

$$\begin{aligned} |y_1(t)| &\leq \frac{1}{2} + \int_t^\infty \frac{(s-t)^2}{2} |a(s)| |y_0(s-\tau(s))| \\ &\leq \frac{1}{2} + \frac{1}{2} = 1 \end{aligned}$$

and

$$|y_1'(t)| \leq \frac{1}{2} .$$

Similarly for any positive integer n , boundedness of τ implies the existence of a large positive number $M \geq T_2$ such that for $t \geq M$,

(13) $$|y_n(t)| \leq 1 , \quad |y_n(t-\tau(t))| \leq 1$$

and

(14) $$|y_n'(t)| \leq \frac{1}{2} .$$

Thus $\{y_n(t)\}_{n=1}^\infty$ is a uniformly bounded equicontinuous sequence of functions on some positive half line $t \geq M$. By Arzela's Theorem it has a uniformly convergent subsequence

$$\left\{y_{n_k}(t)\right\}_{k=1}^\infty$$

which converges to a solution of the integral equation

(15) $$y(t) = \frac{1}{2} - \int_t^\infty \frac{(s-t)^2}{2} a(s) y(s-\tau(s)) ds .$$

This solution in turn is the required bounded solution of equation (1). This solution is also nonoscillatory.

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REMARK. It is interesting to note that the delay term $\tau(t)$ does not play any role in inequality (4).

Consider

$$g(t) = t^{3/2} ,$$

$$a(t) = \frac{3}{8} \frac{\sin t}{1+t^7} ,$$

$$g'''(t) = \frac{3}{2} \cdot \frac{1}{2} \cdot \left(-\frac{1}{2}\right) \cdot \frac{1}{t^{3/2}} ;$$

then $\lim_{t\to\infty} g(t) = \infty$,

$$\begin{aligned} g'''(t) + t^2|a(t)|g(t) &= -\frac{3}{8} \cdot \frac{1}{t^{3/2}} + \frac{3}{8} \frac{t^2 \cdot t^{3/2}}{(1+t)^7} \cdot |\sin t| \\ &= -\frac{3}{8}\left[\frac{1}{t^{3/2}} - \frac{t^{7/2}}{(1+t)^7}|\sin t|\right] \\ &= -\frac{3}{8}\left[t^{-7/2}\left\{t^2 - |\sin t| \frac{t^7}{(1+t^7)}\right\}\right] \\ &< 0 \quad \text{for large } t . \end{aligned}$$

Thus the equation

$$y'''(t) + \frac{3}{8} \frac{\sin t}{(1+t)^7} y_\tau(t) = 0$$

has a bounded nonoscillatory solution.

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Department of Mathematics,
University of Wisconsin,
Manitowoc Center,
Manitowoc,
Wisconsin,
USA;

Department of Mathematics,
Iowa State University,
Ames,
Iowa,
USA.