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ON THE HAHN-BANACH EXTENSION PROPERTY

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1. **Introduction.** In this paper, we consider real linear spaces. By $(V: \| \|)$ we mean a normed (real) linear space V with norm $\| \|$. By the statement "V has the (Y, X) norm preserving (Hahn-Banach) extension property" we mean the following: Y is a subspace of the normed linear space X, V is a normed linear space, and any bounded linear function $f: Y \to V$ has a linear extension $F: X \to V$ such that $\|F\| = \|f\|$. By the statement "V has the unrestricted norm preserving (Hahn-Banach) extension property" we mean that V has the (Y, X) norm preserving extension property for all Y and X with $Y \subset X$. By $(V: \leq)$ we mean a partial ordered linear space (OLS) with the vector ordering $\leq$ which is not necessarily antisymmetric. By the statement "V has the (Y, X) dominated (Hahn-Banach) extension property" we mean the following: $(V: \leq)$ is an OLS, Y is a subspace of the linear space X, and for any sublinear (i.e., subadditive and positively homogeneous) function $p: X \to V$, any linear function $f: Y \to V$ such that $f(y) \leq p(y)$ for all $y \in Y$ has a linear extension $F: X \to V$ such that $F(x) \leq p(x)$ for all $x \in X$. By the statement "V has the unrestricted dominated (Hahn-Banach) extension property" we mean that V has the (Y, X) dominated-extension property for all Y and X with $Y \subset X$.

The classical Hahn-Banach theorem asserts that the real number field R has the unrestricted norm preserving extension property and also the unrestricted dominated extension property. In [1], G. Elliott and I. Halperin proved that for all finite-dimensional normed linear spaces V there is a single pair (Y_0, X_0) such that when V has the (Y_0, X_0) norm preserving extension property then V must have the unrestricted norm preserving extension property. This result is stated precisely as follows:

THEOREM I. *Let $X_0 = C(3)$, the normed linear space with sup norm of all continuous functionals on the discrete topological space of three elements, and let Y_0 be a subspace of X_0 generated by* $(0, 1, 1)$ *and* $(1, 0, 1)$.

If a finite-dimensional normed linear space V has the (Y_0, X_0) norm preserving extension property then V has the unrestricted norm preserving extension property.

The question arises: Is there a corresponding result to the above theorem for dominated extensions in ordered linear spaces? The answer is "yes". We shall show in the main theorem (§3) that there exists a class $\mathfrak{A}$ of OLS's which includes

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the finite dimensional OLS's such that there is a single pair (Y_0, X_0) of linear spaces such that if $V \in \mathfrak{A}$ has the (Y_0, X_0) dominated extension property then V must have the unrestricted dominated extension property.

2. **Preliminaries.** Besides Theorem I, some results proved in [2], [3], [4] and [6] will be used in the proof of our main theorem. Let V be a linear space. A non-empty subset C of V is said to be a wedge if $u, v \in C$ and $t \in R$, $t \geq 0$, imply that $u+v$ and tu are in C. A wedge C is said to be sharp if $u \in C$ and $-u \in C$ imply that $u = 0$, the zero element of V. If $(V: \leq)$ is an OLS, then the set $C = \{v: v \geq 0\}$ is a wedge and is called the positive wedge of $(V: \leq)$. Conversely, a wedge C in a linear space V determines a vector ordering $\leq$ by taking $v \geq 0$ iff $v \in C$. Therefore, a wedge C in V uniquely determines and is determined by a vector ordering $\leq$. The positive wedge C corresponding to the vector ordering $\leq$ is sharp iff the vector ordering is antisymmetric. For convenience, if C is the positive wedge corresponding to the ordering $\leq$, we sometimes write $(V: C)$ instead of $(V: \leq)$. A wedge C in an OLS $(V: C)$ is said to be reproducing if V is the linear hull of C. An OLS $(V: \leq)$ is said to have the least upper bound property if every set of elements with an upper bound has a least upper bound (not necessarily unique). The least upper bound is unique if the ordering $\leq$ is antisymmetric (or, equivalently, the positive wedge C is sharp). If $(V: C)$ has the least upper bound property and if C is reproducing and sharp, we call $(V: C)$ a boundedly complete vector lattice. A point e of an OLS $(V: \leq)$ is said to be an order unit of V if $e > 0$ is such that, given any $v \in V$ we have $-\lambda e \leq v \leq \lambda e$ for some $\lambda \in R$. A point u of a wedge C in V is said to be a core point of C if C contains a line segment through u in each direction. A wedge C in a linear space V is said to be lineally closed if every line intersects C in a set which is closed in the natural topology of the line, or equivalently, if $v_1 \in C$, $v \in C$ and $tv_1 - v \in C$ for some real $t \geq 0$ implies that $t_v v_1 - v \in C$; where $t_v = \inf\{t \in R: tv_1 - v \in C, t \geq 0\}$. We state without proof some results which will be used in the sequel.

THEOREM II ([6]). *If an OLS $(V: \leq)$ has the least upper bound property, then V has the unrestricted dominated extension property.*

THEOREM III (W. E. Bonnice and R. J. Silverman [3], [4]). *If a finite-dimensional OLS $(V: C)$ has the unrestricted dominated extension property, then C is lineally closed.*

THEOREM IV. (1) *A point u is a core point of the wedge C in $(V: C)$ iff u is an order unit of $(V: \leq)$ where $\leq$ is the vector ordering corresponding to C.* (2) *If C is a finite dimensional wedge (i.e., the linear hull of C is finite dimensional), then C has a core point relative to its linear hull.*

THEOREM V ([3] p. 211). *Let $(V: C)$ be an OLS and let V_1 be the linear hull of C. Then $(V: C)$ has the (Y, X) dominated extension property (unrestricted dominated extension property) iff $(V_1: C)$ has the (Y, X) dominated extension property (unrestricted dominated extension property).*

Let $(V: \| \|)$ be a normed linear space, e be a vector of norm one in V and let $P=\{v \in V: v=\lambda(u+e),\ \lambda \in R,\ \lambda \geq 0,\ \|u\| \leq 1\}$. Then P is a wedge and $(V: P)$ is an OLS with e as an order unit. We shall call this OLS $(V: P)$ the OLS deduced from $(V: \| \|)$ and e. Conversely, if $(V: C)$ is an OLS with an order unit e and is such that C is sharp and lineally closed then the function $\| \|: V \to R$ defined by $\|x\| = \inf\{\lambda \in R: -\lambda e \leq x \leq \lambda e,\ \lambda \geq 0\}$ for all $x \in V$, is a norm. We shall call this normed linear space $(V: \| \|)$ the normed linear space deduced from $(V: C)$ and the order unit e. Moreover, it is easy to verify that the set

$$P = \{x \in V: x = \lambda(u+e),\ \lambda \in R,\ \lambda \geq 0,\ \|u\| \leq 1\}$$

coincides with C and the OLS $(V: P)$ deduced from $(V: \| \|)$ coincides with the original OLS $(V: C)$. From this remark and Theorem 1, Theorem 2 of [2], we have the following theorem:

THEOREM VI (Nachbin [2]). *Let $(V: C)$ be an OLS with an order unit e such that C is sharp and lineally closed. Let $(V: \| \|)$ be the normed linear space deduced from $(V: C)$ and the order unit e. Then $(V: \| \|)$ has the unrestricted norm preserving extension property iff $(V: C)$ is a boundedly complete vector lattice.*

3. **The main theorem.** We begin with the following lemmas:

LEMMA 1. *Let $(V: \leq)$ be an OLS with an order unit e such that the positive wedge C corresponding to $\leq$ is lineally closed and sharp, and let $(V: \| \|)$ be the normed linear space deduced from $(V: \leq)$ and the order unit e. If $(V: \leq)$ has the (Y, X) dominated extension property, then $(V: \| \|)$ has the (Y, X) norm preserving extension property when X is normed by any normed linear space norm.*

Proof. Assume that $(V: \leq)$ has the (Y, X) dominated extension property and that $\| \|$ is a normed linear space norm on X. Let $f: Y \to V$ be a bounded linear function and define a function $p: X \to V$ by $p(x) = \|f\|\,\|x\|e$ for all $x \in X$. Then p is sublinear. Moreover, from the assumption that $(V: \| \|)$ is deduced from $(V: \leq)$ and the order unit e of $(V: \leq)$, and that C is lineally closed, we have

$$p(y) = \|f\|\,\|y\|e \geq \|f(y)\|e \geq f(y) \quad \text{for all } y \in Y.$$

Therefore, by the (Y, X) dominated extension property of $(V: \leq)$ there exists a linear extension F of f on the whole space X into V such that $\|f\|\,\|x\|e = p(x) \geq F(x)$ for all $x \in X$. Since $\|v\| = \inf\{\lambda \geq 0: -\lambda e \leq v \leq \lambda e\}$ for all $v \in V$, $\|f\|\,\|x\| \geq \|F(x)\|$ for all $x \in X$. This implies that $\|F\| \leq \|f\|$. On the other hand,

$$\begin{aligned} \|F\| &= \sup\{\|F(x)\|: \|x\| \leq 1, x \in X\} \\ &\geq \sup\{\|F(y)\|: \|y\| \leq 1, y \in Y\} \\ &= \|f\|. \end{aligned}$$

Thus, $\|F\| = \|f\|$. This shows that $(V: \| \|)$ has the (Y, X) norm preserving extension property.

LEMMA 2. *Let* $R_2=\{(a, b)\colon a, b \in R\}$ *and let* $R_3=\{(a, b, c)\colon a, b, c \in R\}$. *If a finite dimensional OLS* $(V\colon C)$ *has the* (R_2, R_3) *dominated extension property, then the positive wedge* C *is lineally closed.*

Proof. We remark that if $(V\colon C)$ has the (R_2, R_3) dominated extension property then it has the (R, R_2) dominated extension property. With this remark, it is easily seen from the proof of Theorem III ([4] pp. 844–849, [5]) that if the unrestricted dominated extension property of $(V\colon C)$ is replaced by the (R_2, R_3) dominated extension property the result is still valid and hence the lemma follows.

LEMMA 3. *If* $(V\colon \leq)$ *is a finite dimensional OLS such that the ordering* $\leq$ *is antisymmetric and if* $(V\colon \leq)$ *has the* (R_2, R_3) *dominated extension property, then* $(V\colon \leq)$ *has the unrestricted dominated extension property.*

Proof. Let C be the positive wedge of V corresponding to $\leq$. By Theorem V, we may assume that V is the linear hull of C without loss of generality. Then by Theorem IV, since V is finite dimensional, C has a core point e which is an order unit of $(V\colon \leq)$. Define $\| \; \|\colon V \to R$ by $\|v\|=\inf\{\lambda \in R\colon \lambda e \geq v \geq -\lambda e, \lambda \geq 0\}$ for all $v \in V$. Since $\leq$ is antisymmetric, C is sharp. Furthermore, since $(V\colon \leq)$ has the (R_2, R_3) dominated extension property, then, by Lemma 2, C is lineally closed. Hence $(V\colon \| \; \|)$ is the normed linear space deduced from $(V\colon \leq)$ and the order unit e. Thus, by Lemma 1, $(V\colon \| \; \|)$ has the (R_2, R_3) norm preserving extension property when R_3 is normed by any normed linear space norm. Let $X_0=C(3)$ and let $Y_0=\{\lambda(0, 1, 1)+u(1, 0, 1)\colon \lambda, u \in R\} \subset C(3)$. Since $C(3)$ is a three-dimensional normed real linear space and Y_0 is a two-dimensional subspace of $C(3)$, $(V\colon \| \; \|)$ has the (Y_0, X_0) norm preserving extension property. Thus, by Theorem I, $(V\colon \| \; \|)$ has the unrestricted norm-preserving extension property. Then, by Theorem VI, the original ordered linear space $(V\colon \leq)$ is a boundedly complete vector lattice, and by Theorem II, $(V\colon \leq)$ has the unrestricted dominated extension property.

THEOREM. *Let* $(V\colon \leq)$ *be an OLS and let* $V_0=\{v \in V\colon v \geq 0$ *and* $-v \geq 0\}$. *If the quotient linear space* V/V_0 *is finite dimensional, then the following two statements are equivalent:*

(1) *V has the (R_2, R_3) dominated extension property.*
(2) *V has the unrestricted dominated extension property.*

Proof. Clearly (2) implies (1). To see (1) implies (2), assume that $(V\colon \leq)$ has the (R_2, R_3) dominated extension property. By Theorem V, we may assume that the positive wedge C of $(V, \leq)$ is reproducing without loss of generality. Let V_1 be a subspace of V such that $V_1 \cong V/V_0$. Then $V=V_1 \oplus V_0$, the algebraic direct sum of the subspaces V_1 and V_0. We show that $(V_1\colon \leq)$ has the (R_2, R_3) dominated extension property. Let $p\colon R_3 \to V_1$ be a sublinear function and let $f\colon R_2 \to V_1$ be a linear function such that $f(y) \leq p(y)$ for all $y \in R_2$. Since $(V, \leq)$ has the $R_2, R_3)$

dominated extension property, f has a linear extension $F: R_3 \to V$ such that $F(x) \le p(x)$ for all $x \in R_3$. Let $F(x) = F_1(x) + F_0(x)$, where $F_1(x) \in V_1$ and $F_0(x) \in V_0$ for all $x \in R_3$. Since $F(y) = f(y) \in V_1$ for all $y \in R_2$, $F_0(y) = 0$ and hence $F_1(y) = f(y)$ for all $y \in R_2$. Also, the linearity of F implies the linearity of F_1: thus F_1 is a linear extension of f on R_3 into V_1. Furthermore, F_1 is dominated by p. Indeed, since $F_0(x) \in V_0$,

$$p(x) - F_1(x) = p(x) - F(x) + F_0(x) \ge 0 \quad \text{for all } x \in R_3.$$

This shows that $(V_1: \le)$ has the (R_2, R_3) dominated extension property. By our assumption $(V_1: \le)$ is finite dimensional, and since the set $\{v \in V_1 : v \ge 0 \text{ and } -v \ge 0\} = \{0\}$, the ordering $\le$ is antisymmetric on V_1. Therefore, by the proof of Lemma 3 $(V_1: \le)$ is a boundedly complete vector lattice. It follows that V has the least upper bound property and hence, by Theorem II, V has the unrestricted dominated extension property.

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