Canad. Math. Bull. Vol. **66** (1), 2023, pp. 269–285
http://dx.doi.org/10.4153/S0008439522000273

Spectrality of a class of Moran measures on $\mathbb{R}^n$ with consecutive digit sets

Si Chen and Qian Li

Abstract. Let $\{R_k\}_{k=1}^{\infty}$ be a sequence of expanding integer matrices in $M_n(\mathbb{Z})$, and let $\{D_k\}_{k=1}^{\infty}$ be a sequence of finite digit sets with integer vectors in $\mathbb{Z}^n$. In this paper, we prove that under certain conditions in terms of (R_k, D_k) for $k \geq 1$, the Moran measure

$$\mu_{\{R_k\},\{D_k\}} := \delta_{R_1^{-1}D_1} * \delta_{R_1^{-1}R_2^{-1}D_2} * \cdots$$

is a spectral measure. For the converse, we get a necessity condition for the admissible pair (R, D).

1 Introduction

Let μ be a Borel probability measure with compact support on $\mathbb{R}^n$. A fundamental problem in harmonic analysis associated with μ is whether there exists a set $\Lambda \subseteq \mathbb{R}^n$ such that $E_\Lambda := \left\{e^{2\pi i\langle\lambda,x\rangle} : \lambda \in \Lambda\right\}$ is an orthonormal basis for $L^2(\mu)$. If so, we say μ is a *spectral measure* and Λ a *spectrum* of μ. In particular, if μ is the Lebesgue measure restricted on a Borel set Ω, then the set Ω is called a *spectral set*. The study of spectral measures dates back to the work of Fuglede [14] in 1974, who conjectured that Ω is a spectral set if and only if the set Ω tiles $\mathbb{R}^n$ by translations. Although the conjecture has been proved to be false in dimension $n \geq 3$ [20, 21, 30], it is still open in Dimensions 1 and 2.

The studies of spectral measures entered into the realm of fractals when Jorgensen and Pedersen [18] gave the first example of a singular, nonatomic, fractal spectral measure. Their construction is based on a scale- 4 Cantor set, where the first and third intervals are kept and the other two are discarded. The appropriate measure for this set is the Bernoulli convolution μ_4, which is the invariant measure of the iterated function system $\{\tau_0(x) = x/4,\ \tau_2(x) = (x+2)/4\}$. They proved that this measure is a spectral measure with spectrum $\Lambda := \left\{\sum_{k=0}^{n} 4^k d_k : d_k \in \{0,1\}, n \in \mathbb{N}\right\}$. Jorgensen and Pedersen opened up a new field in researching the orthogonal harmonic analysis of fractal measures including self-similar measures/self-affine measures and generally Moran measures (see [19, 27–29]). Later on, in $\mathbb{R}$, a large class of self-similar measures have been proved to be spectral measures by Łaba and Wang [22]. Let $b \geq 2$ be an

Received by the editors July 18, 2021; revised April 6, 2022, accepted April 7, 2022.
Published online on Cambridge Core April 13, 2022.
This work was supported by the National Natural Science Foundation of China 11971194.
AMS subject classification: **28A80**, 42C05, 46C05.
Keywords: Admissible, Hadamard triple, Moran measure, infinite convolution, orthonormal basis of exponential, spectral measures.

integer, and let $D \subseteq \mathbb{Z}$ be a finite digit set with $0 \in D$. Łaba and Wang [22] proved that the self-similar measure $\mu_{b,D}$, which is generated by the iterated function system $\{\tau_d(x) = \frac{x+d}{b}\}_{d\in D}$, is a spectral measure if (b, D) is admissible (see Definition1.1). Recently, Dutkay, Hausserman, and Lai [10] generalized the result to higher dimensions, thus settling a long-standing conjecture proposed by Jorgensen and Pedersen. Meanwhile, many interesting spectral measures have been found (see, e.g., [1–7, 9, 12, 13, 15, 16, 23–26, 29] and the references therein for recent advances), but there are only a few classes.

In the last decade, many researchers studied the spectrality of the Moran measures, which are the nonself-similar generalization of the Cantor measures through the infinite convolution. Note that most results of the known cases are concentrated on one-dimensional Moran measures (see, e.g., [1, 2, 12, 13, 15] and the references therein). There are few studies involving Moran measures of higher dimension other than [8, 26, 29]. In this paper, we focus on the Moran measures on $\mathbb{R}^n$. Let $\{R_k\}_{k=1}^\infty$ be a sequence of expanding matrices with integer entries, and let $\{D_k\}_{k=1}^\infty$ be a sequence of finite sets in $\mathbb{Z}^n$. A *Moran measure* is defined by the following infinite convolution of finite measures:

$$\mu_{\{R_k\},\{D_k\}} = \delta_{R_1^{-1}D_1} * \delta_{R_1^{-1}R_2^{-1}D_2} * \cdots, \tag{1.1}$$

assuming the infinite convolution is weakly convergent to a Borel probability measure (see [11] for an equivalent definition). Here, $\delta_E = \frac{1}{\#E}\sum_{e\in E}\delta_e$, where $\#E$ is the cardinality of a finite set E and δ_e is the Dirac measure at the point $e \in E$. In 2000, Stricharz [29] first considered the conditions under which the infinite convolution is convergent to a Borel probability measure with compact support and the associated measure is a spectral measure. Later on, An and He [2] investigated the spectral property of infinite convolution with consecutive digits in $\mathbb{R}$. More precisely, let $\{b_k\}_{k=1}^\infty$ be a sequence of integers with all $b_k \ge 2$, and let $\{D_k\}_{k=1}^\infty$ be a sequence of finite digit sets, where $D_k = \{0, 1, \dots, q_k - 1\}$ is a digit set of integers with $\sup\{x : x \in b_k^{-1}D_k, k \ge 1\} < \infty$. They proved the following theorem.

Theorem 1.1 *Suppose that $q_k | b_k$ for $k \ge 1$. Then, the Moran measure $\mu_{\{b_k\},\{D_k\}}$ is a spectral measure.*

Recently, Dutkay, Emami, and Lai [8] studied the general Moran measure $\mu_{\{R_k\},\{D_k\}}$ on $\mathbb{R}^n$, and they investigated the spectrality and its more general frame spectrality using the idea of frame towers and Riesz-sequence towers.

In this paper, we continue to investigate the spectral property of Moran measures on $\mathbb{R}^n$. Let $\{R_k\}_{k=1}^\infty$ be a sequence of expanding matrices with integer entries, and let $D_k = \{0, 1, \dots, q_k - 1\}\boldsymbol{v}$ where the integer $q_k \ge 2$ and $\boldsymbol{v} \in \mathbb{Z}^n$ for $k \ge 1$.

First of all, we need a decomposition of integer matrices, which has been proved in [25]. Let $R \in M_n(\mathbb{Z})$, and let $\{\boldsymbol{v}, R\boldsymbol{v}, \dots, R^{n-1}\boldsymbol{v}\}$ be a set of vectors in $\mathbb{Z}^n$ with rank $r \le n$ and $\boldsymbol{v} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$. According to [25], there exists a unimodular matrix $B \in M_n(\mathbb{Z})$ such that $B^{-1}\boldsymbol{v} = (\boldsymbol{v}_r^T, 0, \dots, 0)^T$ with $\boldsymbol{v}_r \in \mathbb{Z}^r$ and

$$\widetilde{R} := B^{-1}RB = \begin{pmatrix} M_1 & C \\ 0 & M_2 \end{pmatrix}, \tag{1.2}$$

https://doi.org/10.4153/S0008439522000273 Published online by Cambridge University Press

where $M_1 \in M_r(\mathbb{Z})$, $M_2 \in M_{n-r}(\mathbb{Z})$, and $C \in M_{r,n-r}(\mathbb{Z})$. Here, we use A^T to denote the transposition of a vector or matrix A.

Similar to the known theorem in [29], we have the following theorem.

Theorem 1.2 *Let $R_k = b_k R^{m_k}$ for $k \geq 1$, where $R \in M_n(\mathbb{Z})$ is an expanding matrix and the sequences $\{b_k\}_{k=1}^{\infty}$ and $\{m_k\}_{k=1}^{\infty}$ are of positive integers. Let $D_k = \{0, 1, \ldots, q_k - 1\}\boldsymbol{v}$ be a digit set with $\boldsymbol{v} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ and the integer $q_k \geq 2$ for each $k \geq 1$. Suppose that $\sup_{k\geq 1} |\frac{q_k}{b_k}| < \infty$. Then, the sequence of measures*

$$\mu_n := \delta_{R_1^{-1}D_1} * \delta_{R_1^{-1}R_2^{-1}D_2} * \cdots * \delta_{R_1^{-1}\cdots R_n^{-1}D_n}$$

converges to a Borel probability measure $\mu_{\{R_k\},\{D_k\}}$ with compact support in a weak sense.

The measure $\mu_{\{R_k\},\{D_k\}}$ in Theorem 1.2 is the Moran measure on which we focus in the following of the paper. Now, we introduce the main theorem of the paper.

Theorem 1.3 *Let D_k, R_k be given as in Theorem 1.2, and let $\boldsymbol{v} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ be an eigenvector of R with respect to an eigenvalue λ. If λb_k is divisible by q_k for $k \geq 1$, then $\mu_{\{R_k\},\{D_k\}}$ is a spectral measure.*

In fact, we have the following more general conclusion. We will prove it in Section 3 so as to prove Theorem 1.3.

Theorem 1.4 *Let D_k, R_k be given as in Theorem 1.2. Furthermore, we replace m_k by $m_k r$ for $k \geq 2$ where r is the rank of vectors $\{\boldsymbol{v}, R\boldsymbol{v}, \ldots, R^{n-1}\boldsymbol{v}\}$. Suppose that the characteristic polynomial of M_1 see (1.2) is $f(x) = x^r + c$. If $q_k | b_k \det(M_1)$ for $k \geq 1$, then the measure $\mu_{\{R_k\},\{D_k\}}$ is a spectral measure.*

It is clear that the condition $\sup_{k\geq 1} |\frac{q_k}{b_k}| < \infty$ is satisfied in Theorems 1.3 and 1.4. Furthermore, if we replace $b_k \in \mathbb{N} \setminus \{0\}$ by $b_k \in \mathbb{Z} \setminus \{0\}$ in the above three theorems, the same results hold.

Next, we consider the converse of Theorem 1.4. However, it is too complicated for us to draw a necessity condition for the spectral measure $\mu_{\{R_k\},\{D_k\}}$. We simplify it to the case that $R_k = R$ and $D_k = D$ for all $k \geq 1$ and obtain the following result. Before introducing it, we need a standard notation usually used in this setting.

Definition 1.1 Let $R \in M_n(\mathbb{Z})$ be an expanding matrix (i.e., all its eigenvalues have modulus strictly greater than 1), and let D be a finite subset of $\mathbb{Z}^n$. We say that (R, D) is admissible if there exists a finite subset $L \subseteq \mathbb{Z}^n$ with $\#D = \#L = q$ such that the matrix

$$H = \frac{1}{\sqrt{q}} \left(e^{-2\pi i \langle R^{-1}d, l \rangle} \right)_{d\in D, l\in L}$$

is unitary, i.e., $H^*H = I$, where H^* denotes the transposed conjugate of H. At this time, (R, D, L) is also called a *Hadamard triple*, or $(R^{-1}D, L)$ is called a *compatible pair*.

https://doi.org/10.4153/S0008439522000273 Published online by Cambridge University Press

By making use of the matrix decomposition as in (1.2), we obtain the following conclusion.

Theorem 1.5 *Let $R \in M_n(\mathbb{Z})$ be an expanding matrix, and let $D = \{0, 1, \ldots, q-1\}\boldsymbol{v}$ be a digit set with $\boldsymbol{v} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ and the integer $q \geq 2$. If (R, D) is admissible, then $q \mid \det(M_1)$.*

In general, the converse of Theorem 1.5 is not true for $n \geq 2$ (see Example 5.2). However, in $\mathbb{R}$, assuming $\boldsymbol{v} = 1$, then q is a factor of $\det(M_1)$ if and only if (R, D) is admissible (see [2]).

The paper is organized as follows. In Section 2, we introduce some basic definitions and properties of spectral measures. In Section 3, we will give the proofs of Theorems 1.3 and 1.4. Moreover, we devote Section 4 to proving Theorem 1.5. In Section 5, we will give some examples to illustrate the theories.

2 Preliminaries

Let μ be a Borel probability measure with compact support in $\mathbb{R}^n$. The Fourier transform of μ is defined as usual

$$\hat{\mu}(\xi) = \int e^{-2\pi i \langle \xi, x \rangle} \mathrm{d}\mu(x)$$

for any $\xi \in \mathbb{R}^n$. Let $\mathcal{Z}(\hat{\mu}) = \{\xi \in \mathbb{R}^n : \hat{\mu}(\xi) = 0\}$ be the zero set of $\hat{\mu}$. Then, for a discrete set $\Lambda \subset \mathbb{R}^n$, $E(\Lambda) = \{e^{-2\pi i \langle \lambda, x \rangle} : \lambda \in \Lambda\}$ is an orthogonal set of $L^2(\mu)$ if and only if $\hat{\mu}(\lambda - \lambda') = 0$ for $\lambda \neq \lambda' \in \Lambda$, which is equivalent to

$$(\Lambda - \Lambda) \setminus \{\mathbf{0}\} \subseteq \mathcal{Z}(\hat{\mu}). \tag{2.1}$$

In this case, we call Λ an *orthogonal set (resp. spectrum)* of μ if E_Λ is an orthonormal family (resp. basis) for $L^2(\mu)$. Since orthogonal sets (or spectra) are invariant under translation, without loss of generality, we always assume that $\mathbf{0} \in \Lambda$ for any orthogonal set Λ of μ.

For any $\xi \in \mathbb{R}^n$, define

$$Q_\Lambda^{(\mu)}(\xi) = \sum_{\lambda \in \Lambda} | \hat{\mu}(\xi + \lambda) |^2 . \tag{2.2}$$

The following lemma is a basic criterion for the spectrality of measure μ, which was proved in [18].

Lemma 2.1 *Let μ be a Borel probability measure with compact support in $\mathbb{R}^n$, and let $\Lambda \subseteq \mathbb{R}^n$ be a countable subset. Then,*

(i) *Λ is an orthogonal set of μ if and only if $Q_\Lambda^{(\mu)}(\xi) \leq 1$ for $\xi \in \mathbb{R}^n$.*

(ii) *Λ is a spectrum of μ if and only if $Q_\Lambda^{(\mu)}(\xi) \equiv 1$ for $\xi \in \mathbb{R}^n$.*

(iii) *$Q_\Lambda^{(\mu)}(x)$ has an entire analytic extension to $\mathbb{C}^2$ if Λ is an orthogonal set of μ.*

Definition 2.1 Let R and $\widetilde{R}$ be $n \times n$ integer matrices, and let the finite sets $D, L, \widetilde{D}, \widetilde{L}$ be in $\mathbb{R}^n$. We say that two triples (R, D, L) and $(\widetilde{R}, \widetilde{D}, \widetilde{L})$ are *conjugate* (through the

https://doi.org/10.4153/S0008439522000273 Published online by Cambridge University Press

matrix B) if there exists an integer invertible matrix B such that $\widetilde{R} = B^{-1}RB$, $\widetilde{D} = B^{-1}D$, and $\widetilde{L} = B^*L$, where B^* denotes the transposed conjugate of B, in fact, $B^* = B^T$.

We have the following conclusion for the conjugate relationship.

Lemma 2.2 *Suppose that* (R_k, D_k, L_k) *and* $(\widetilde{R}_k, \widetilde{D}_k, \widetilde{L}_k)$ *are conjugate triples, through the same matrix B, for any* $k \geq 1$. *Then,*

(i) $(\widetilde{R}_k, \widetilde{D}_k, \widetilde{L}_k)$ *is a Hadamard triple if* (R_k, D_k, L_k) *is a Hadamard triple.*
(ii) Λ *is a spectrum of* $\mu_{\{R_k\},\{D_k\}}$ *if and only if* $B^*\Lambda$ *is a spectrum of* $\mu_{\{\widetilde{R}_k\},\{\widetilde{D}_k\}}$.

Proof (i) Note that $\widetilde{R}_k = B^{-1}R_kB$, $\widetilde{D}_k = B^{-1}D_k$, and $\widetilde{L}_k = B^*L_k$. Then,

$$\left(e^{-2\pi i\langle \widetilde{R}_k^{-1}d,l\rangle}\right)_{d\in\widetilde{D}_k,l\in\widetilde{L}_k} = \left(e^{-2\pi i\langle B^{-1}\widetilde{R}_k^{-1}BB^{-1}d,B^*l\rangle}\right)_{d\in D_k,l\in L_k} = \left(e^{-2\pi i\langle \widetilde{R}_k^{-1}d,l\rangle}\right)_{d\in D_k,l\in L_k}.$$

Furthermore, $\widetilde{L}_k \subset \mathbb{Z}^2$ since $L_k \subset \mathbb{Z}^2$ and B is an integer matrix. Hence, the conclusion follows directly from Definition 1.1.

As for (ii), we recall

$$\mu_{\{R_k\},\{D_k\}} = \delta_{R_1^{-1}D_1} * \delta_{R_1^{-1}R_2^{-1}D_2} * \cdots.$$

By the definition of Fourier transform of $\mu_{\{R_k\},\{D_k\}}$, we have

$$\hat{\mu}_{\{R_k\},\{D_k\}}(\xi) = \prod_{k=1}^{+\infty}\hat{\delta}_{R_1^{-1}R_2^{-1}\cdots R_k^{-1}D_k}(\xi) = \prod_{k=1}^{+\infty}\frac{1}{q_k}\sum_{d\in D_k}e^{-2\pi i\langle R_1^{-1}\cdots R_k^{-1}d,\xi\rangle},$$

where $q_k := \#D_k = \#\widetilde{D}_k$. Then, for any $\lambda \in \Lambda$ and $\xi \in \mathbb{R}^n$,

$$\begin{aligned}\hat{\mu}_{\{R_k\},\{D_k\}}(\lambda+\xi) &= \prod_{k=1}^{+\infty}\frac{1}{q_k}\sum_{d\in D_k}e^{-2\pi i\langle R_1^{-1}\cdots R_k^{-1}d,(\lambda+\xi)\rangle}\\ &= \prod_{k=1}^{+\infty}\frac{1}{q_k}\sum_{d\in D_k}e^{-2\pi i\langle B^{-1}R_1^{-1}B\cdots B^{-1}R_k^{-1}BB^{-1}d,B^*(\lambda+\xi)\rangle}\\ &= \prod_{k=1}^{+\infty}\frac{1}{q_k}\sum_{d\in\widetilde{D}_k}e^{-2\pi i\langle \widetilde{R}_1^{-1}\cdots\widetilde{R}_k^{-1}d,B^*(\lambda+\xi)\rangle}\\ &= \hat{\mu}_{\{\widetilde{R}_k\},\{\widetilde{D}_k\}}(B^*(\lambda+\xi)).\end{aligned} \tag{2.3}$$

Hence, $Q_\Lambda^{(\mu_{\{R_k\},\{D_k\}})}(\xi) = Q_{B^*\Lambda}^{(\mu_{\{\widetilde{R}_k\},\{\widetilde{D}_k\}})}(B^*\xi)$. Furthermore, (ii) follows from Lemma 2.1. ∎

In Sections 3 and 4, we employ the following lemma several times which was proved in [25].

Lemma 2.3 *Let* $\boldsymbol{v} \in \mathbb{Z}^n\backslash\{\mathbf{0}\}$, *and let* $R \in M_n(\mathbb{Z})$. *If* $\{\boldsymbol{v}, R\boldsymbol{v}, \ldots, R^{n-1}\boldsymbol{v}\}$ *is linearly dependent with rank* $r < n$, *then there exists a unimodular matrix* $B \in M_n(\mathbb{Z})$ *such that*

$B^{-1}\boldsymbol{v} = (\boldsymbol{v}_r^T, 0, \ldots, 0)^T \in \mathbb{Z}^n$ and

$$\widetilde{R} := B^{-1}RB = \begin{pmatrix} M_1 & C \\ 0 & M_2 \end{pmatrix},$$

where $\boldsymbol{v}_r \in \mathbb{Z}^r$, $M_1 \in M_r(\mathbb{Z})$, $M_2 \in M_{n-r}(\mathbb{Z})$, and $C \in M_{r,n-r}(\mathbb{Z})$.

3 Proofs of Theorems 1.3 and 1.4

In this section, we will focus on the Moran measure $\mu_{\{R_k\},\{D_k\}}$ defined in Theorem 1.2. For the sake of convenience, we introduce some notations from symbolic dynamical system. Denote $\Theta^0 = \{\vartheta\}$ and

$$\Theta^n = \{\theta_1 \cdots \theta_n : \theta_k \in D_k, 1 \le k \le n\}$$

for $n \ge 1$. Then, the collection of all finite words is

$$\Theta^* = \bigcup_{n=0}^{\infty} \Theta^n,$$

and the set of all infinite words is denoted by

$$\Theta^\infty = \{\theta_1\theta_2 \cdots : \theta_k \in D_k, k \ge 1\}.$$

First, we give the proof of Theorem 1.2, which is the preparation for the proof of Theorem 1.3.

Proof of Theorem 1.2 Let $B(0,r)$ be the open ball centered at the origin with radius r on $\mathbb{R}^n$. Denote

$$f_{k,d}(x) = R_k^{-1}(x + d)$$

with $d \in D_k$ and $k \ge 1$. Then, there exists $n \ge 1$ and $\theta_k \in D_k$ for $1 \le k \le n$ such that

$$f_{1,\theta_1} \circ f_{2,\theta_2} \circ \cdots \circ f_{n,\theta_n}(B(0,r)) \subseteq B(0,r).$$

For any $\theta = \theta_1 \cdots \theta_n \in \Theta^n$, we write

$$f_\theta(x) = f_{1,\theta_1} \circ f_{2,\theta_2} \circ \cdots \circ f_{n,\theta_n}.$$

Denote

$$T(\{R_k\},\{D_k\}) = \left\{ \sum_{k=1}^{\infty} (R_k \ldots R_1)^{-1} d_k \ : \ d_k \in D_k \right\} := \sum_{k=1}^{\infty} (R_k \ldots R_1)^{-1} D_k.$$

Then, it is easy to check that

$$T(\{R_k\},\{D_k\}) = \bigcap_{n=1}^{\infty} \bigcup_{\theta \in \Theta^n} f_\theta(B(0,r)).$$

Thus, it is a compact set.

We now define two bounded linear operators T_1 and T_2 as follows. $T_1 : \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$T_1 x = B^* x,$$

where the unimodular matrix B satisfies $B^{-1}RB = \begin{pmatrix} M_1 & C \\ 0 & M_2 \end{pmatrix}$ and $B^{-1}\boldsymbol{v} = (\boldsymbol{v_r}^T, 0, \ldots, 0)^T$ (see Lemma 2.3). $T_2 : \mathbb{R}^n \to \mathbb{R}^r$ is given by

$$T_2\big((x_1, x_2, \ldots, x_r, \ldots, x_n)^T\big) = (x_1, \ldots, x_r)^T.$$

For any $n \geq 1$, we denote

$$\mu_n = \delta_{R_1^{-1}D_1} * \delta_{R_1^{-1}R_2^{-1}D_2} * \cdots * \delta_{R_1^{-1}\cdots R_n^{-1}D_n}.$$

Fix a compact set $I \subseteq \mathbb{R}^n$. For each $\xi \in I$, we get

$$\hat{\mu}_n(\xi) = \hat{\delta}_{R_1^{-1}D_1}(\xi)\hat{\delta}_{R_1^{-1}R_2^{-1}D_2}(\xi)\cdots\hat{\delta}_{R_1^{-1}\cdots R_n^{-1}D_n}(\xi),$$

and

$$\begin{aligned} \left|\hat{\delta}_{R_1^{-1}\cdots R_k^{-1}D_k}(\xi) - 1\right| &= \left|\frac{1}{q_k}\sum_{d \in D_k}\left(e^{-2\pi i\langle R_1^{-1}\cdots R_k^{-1}d, \xi\rangle} - 1\right)\right| \\ &= \left|\frac{1}{q_k}\sum_{l=0}^{q_k-1}\left(e^{-2l\pi i\langle (b_1\cdots b_k)^{-1}B^{-1}R^{-m_1}B\cdots B^{-1}R^{-m_k}BB^{-1}\boldsymbol{v}, B^*\xi\rangle} - 1\right)\right| \\ &= \left|\frac{1}{q_k}\sum_{l=0}^{q_k-1}\left(e^{-2l\pi i\langle (b_1\cdots b_k)^{-1}M_1^{-\eta_k}\boldsymbol{v}_r, T_2T_1\xi\rangle} - 1\right)\right|, \end{aligned}$$

where $\eta_k := m_1 + m_2 + \cdots + m_k$ for $1 \leq k \leq n$. Moreover, we have

$$\begin{aligned} &\left|e^{-2l\pi i\langle (b_1\cdots b_k)^{-1}M_1^{-\eta_k}\boldsymbol{v}_r, T_2T_1\xi\rangle} - 1\right| \\ &\leq \left|2l\pi\langle (b_1\cdots b_k)^{-1}M_1^{-\eta_k}\boldsymbol{v_r}, T_2T_1\xi\rangle\right| \\ &\leq 2l\pi(b_1\cdots b_k)^{-1}\|M_1^{-\eta_k}\| \cdot \|\boldsymbol{v_r}\| \cdot \|T_2T_1\xi\|, \end{aligned}$$

where the first inequality follows from the fact that $|e^{ix} - 1| \leq |x|$, and the second inequality follows from the Schwartz inequality. Hence,

$$\begin{aligned} \sum_{k=1}^{\infty}\left|\hat{\delta}_{R_1^{-1}\cdots R_k^{-1}D_k}(\xi) - 1\right| &\leq \sum_{k=1}^{\infty}\frac{1}{q_k}\sum_{l=0}^{q_k-1}2l\pi(b_1\cdots b_k)^{-1}\|M_1^{-\eta_k}\| \cdot \|\boldsymbol{v}_r\| \cdot \|T_2T_1\xi\| \\ &\leq \pi\sum_{k=1}^{\infty}\frac{q_k - 1}{b_k}\|M_1^{-\eta_k}\| \cdot \|\boldsymbol{v}_r\| \cdot \|T_2T_1\xi\| \qquad (3.1) \\ &\leq C\|\boldsymbol{v}_r\| \cdot \|T_2\| \cdot \|T_1\| \cdot \|\xi\|\sum_{k=1}^{\infty}\|M_1^{-\eta_k}\|, \end{aligned}$$

where $C = \pi\sup_{k\geq 1}\frac{q_k}{b_k}$. We can easily obtain $\sum_{k=1}^{\infty}\|M_1^{-\eta_k}\| < \infty$ from the fact that R is an expanding integer matrix. As T_1, T_2 are bounded, it follows from (3.1) that $\hat{\mu}_n(\xi) = \prod_{k=1}^{n}\hat{\delta}_{R_1^{-1}\cdots R_k^{-1}D_k}(\xi)$ converges uniformly on each compact set to an entire function $f(\xi) = \prod_{k=1}^{\infty}\hat{\delta}_{R_1^{-1}\cdots R_k^{-1}D_k}(\xi)$. By Levy's continuity theorem [17, p. 167], there exists a probability measure μ such that $\hat{\mu}(x) = f(x)$ and μ_n converges weakly to μ. Moreover, the support of μ is compact. ■

https://doi.org/10.4153/S0008439522000273 Published online by Cambridge University Press

In the following of this section, we define

$$R_k^{(r)} = \begin{cases} b_1 M_1^{m_1}, & k = 1, \\ b_k M_1^{m_k r}, & k > 1, \end{cases} \qquad D_k^{(r)} = \{0, 1, \dots, q_k - 1\}\boldsymbol{v}_r, \tag{3.2}$$

where $M_1, \boldsymbol{v}_r$ are the same ones as in Lemma 2.3. We will prove Theorem 1.4 first. Before doing this, we introduce a needed lemma.

Lemma 3.1 *Let R be an expanding integer matrix, and let R_k, D_k be given as in Theorem 1.4. Then, $\mu_{\{R_k\},\{D_k\}}$ is a spectral measure if and only if $\mu_{\{R_k^{(r)}\},\{D_k^{(r)}\}}$ is a spectral measure.*

Proof According to Lemma 2.3, there exists a unimodular matrix $B \in M_n(\mathbb{Z})$ such that

$$\widetilde{R} := B^{-1}RB = \begin{pmatrix} M_1 & C \\ 0 & M_2 \end{pmatrix}, \quad \widetilde{\boldsymbol{v}} = (\boldsymbol{v}_r{}^T, 0, \dots, 0)^T, \tag{3.3}$$

$$\widetilde{D}_k := B^{-1}D_k = \{0, 1, \dots, q_k - 1\}(\boldsymbol{v}_r{}^T, 0, \dots, 0)^T, \tag{3.4}$$

where $M_1 \in M_r(\mathbb{Z})$, $M_2 \in M_{n-r}(\mathbb{Z})$, $C \in M_{r,n-r}(\mathbb{Z})$, and $\boldsymbol{v}_r \in \mathbb{Z}^r$. Denote $\widetilde{R}_k = B^{-1}R_k B$. Then, Lemma 2.2 implies that $\mu_{\{R_k\},\{D_k\}}$ is a spectral measure if and only if $\mu_{\{\widetilde{R}_k\},\{\widetilde{D}_k\}}$ is a spectral measure. Note that

$$\widetilde{R}_1^{-1} = B^{-1}R^{-m_1}B = \frac{1}{b_1}\begin{pmatrix} M_1^{-m_1} & \times \\ \mathbf{0} & M_2^{-m_1} \end{pmatrix}. \tag{3.5}$$

For any $\xi = (\xi_1, \dots, \xi_n)^T \in \mathbb{R}^n$, we denote its first r terms by $\xi^{(r)} = (\xi_1, \dots, \xi_r)^T \in \mathbb{R}^r$. Then,

$$\begin{aligned} \hat{\delta}_{\widetilde{R}_1^{-1}\cdots\widetilde{R}_k^{-1}\widetilde{D}_k}(\xi) &= \frac{1}{q_k}\sum_{d\in\widetilde{D}_k} e^{-2\pi i\langle \widetilde{R}_1^{-1}\cdots\widetilde{R}_k^{-1}d, \xi\rangle} \\ &= \frac{1}{q_k}\sum_{j=0}^{q_k-1} e^{-2\pi i j\langle \widetilde{R}_1^{-1}\cdots\widetilde{R}_k^{-1}(\boldsymbol{v}_r{}^T, 0, \dots, 0)^T, \xi\rangle} \\ &= \frac{1}{q_k}\sum_{j=0}^{q_k-1} e^{-\frac{2\pi i j}{b_1\cdots b_k}\langle M_1^{-m_1}\cdots M_k^{-m_k r}\boldsymbol{v}_r{}^T, \xi^{(r)}\rangle} = \hat{\delta}_{(R_k^{(r)}\cdots R_1^{(r)})^{-1}D_k^{(r)}}(\xi^{(r)}) \end{aligned} \tag{3.6}$$

for any $k \geq 1$. It follows that

(3.7)

$$\hat{\mu}_{\{\widetilde{R}_k\},\{\widetilde{D}_k\}}(\xi) = \prod_{k=1}^{\infty}\hat{\delta}_{\widetilde{R}_1^{-1}\cdots\widetilde{R}_k^{-1}\widetilde{D}_k}(\xi) = \prod_{k=1}^{\infty}\hat{\delta}_{(R_k^{(r)}\cdots R_1^{(r)})^{-1}D_k^{(r)}}(\xi^{(r)}) = \hat{\mu}_{\{R_k^{(r)}\},\{D_k^{(r)}\}}(\xi^{(r)}).$$

Now, we define a bounded linear operator $T : \mathbb{R}^n \to \mathbb{R}^r$ given by

$$T\big((\xi_1, \dots, \xi_r, \dots, \xi_n)^T\big) = (\xi_1, \dots, \xi_r)^T.$$

If $\mu_{\{\widetilde{R}_k\},\{\widetilde{D}_k\}}$ is a spectral measure with a spectrum Λ, we set

$$\Lambda' = \left\{\lambda^{(r)} : T(\lambda) = \lambda^{(r)}, \lambda \in \Lambda\right\}.$$

Then, (3.7) implies

$$\sum_{\lambda\in\Lambda} |\hat{\mu}_{\{\widetilde{R}_k\},\{\widetilde{D}_k\}}(\xi+\lambda)|^2 = \sum_{\lambda^{(r)}\in\Lambda'} |\hat{\mu}_{\{R_k^{(r)}\},\{D_k^{(r)}\}}(\xi^{(r)}+\lambda^{(r)})|^2. \tag{3.8}$$

It follows from Lemma 2.1 that $\mu_{\{R_k^{(r)}\},\{D_k^{(r)}\}}$ is a spectral measure with a spectrum Λ'. Conversely, if $\mu_{\{R_k^{(r)}\},\{D_k^{(r)}\}}$ is a spectral measure with a spectrum Λ', we let $\Lambda = \left\{(\lambda^T, 0, \dots, 0)^T : \lambda \in \Lambda'\right\}$. Proceeding as in (3.8), we obtain that Λ is a spectrum of $\mu_{\{\widetilde{R}_k\},\{\widetilde{D}_k\}}$. Now, we complete the proof. ∎

The following lemma has been proved in [25].

Lemma 3.2 *Let $R \in M_n(\mathbb{Z})$ be a matrix with characteristic polynomial $f(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ and $\boldsymbol{v} = (x_1, x_2, \dots, x_n)^T \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$. If the set of vectors $\{\boldsymbol{v}, R\boldsymbol{v}, \dots, R^{n-1}\boldsymbol{v}\}$ is linearly independent, then there exists an integer matrix B such that*

$$\widetilde{R} := B^{-1}RB = \begin{pmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & 0 & \cdots & 1 \\ -a_n & 0 & 0 & \cdots & 0 \end{pmatrix}$$

and $\widetilde{\boldsymbol{v}} = B^{-1}\boldsymbol{v} = (0, 0, \dots, 0, 1)^T$.

Moreover, we can choose the matrix $B = (R^{n-1}\boldsymbol{v}, R^{n-2}\boldsymbol{v}, \dots, R\boldsymbol{v}, \boldsymbol{v})$ from the proof of Lemma 3.2 in [25].

Proof of Theorem 1.4 By Lemma 3.1, we just need to prove $\mu_{\{R_k^{(r)}\},\{D_k^{(r)}\}}$ is a spectral measure. As $\{\boldsymbol{v}, R\boldsymbol{v}, \dots, R^{n-1}\boldsymbol{v}\}$ is linearly dependent with rank r, we know that

$$\{B^{-1}\boldsymbol{v}, B^{-1}R\boldsymbol{v}, \dots, B^{-1}R^{n-1}\boldsymbol{v}\}$$

is also linearly dependent with rank r, where B is the same one as in (3.3). And thus, $\{\widetilde{\boldsymbol{v}}, \widetilde{R}\widetilde{\boldsymbol{v}}, \dots, \widetilde{R}^{n-1}\widetilde{\boldsymbol{v}}\}$ is linearly dependent with rank r, where $\widetilde{R}$ and $\widetilde{\boldsymbol{v}}$ are the same ones as in (3.3) and (3.4). Now, we claim that $\{\widetilde{\boldsymbol{v}}, \widetilde{R}\widetilde{\boldsymbol{v}}, \dots, \widetilde{R}^{r-1}\widetilde{\boldsymbol{v}}\}$ is linearly independent. In fact, if $\{\widetilde{\boldsymbol{v}}, \widetilde{R}\widetilde{\boldsymbol{v}}, \dots, \widetilde{R}^{r-1}\widetilde{\boldsymbol{v}}\}$ is dependent, then there exist $\{l_i\}_{i=0}^{r-1} \subset \mathbb{Z}$ such that

$$l_0\boldsymbol{v} + l_1\widetilde{R}\widetilde{\boldsymbol{v}} + \cdots + l_{r-1}\widetilde{R}^{r-1}\widetilde{\boldsymbol{v}} = 0. \tag{3.9}$$

Denote s as the first index such that $l_s \neq 0$. Then, (3.9) implies that

$$l_s\widetilde{R}^s\widetilde{\boldsymbol{v}} + \cdots + l_{r-1}\widetilde{R}^{r-1}\widetilde{\boldsymbol{v}} = 0.$$

https://doi.org/10.4153/S0008439522000273 Published online by Cambridge University Press

It follows that

$$\widetilde{\boldsymbol{v}} = -\frac{1}{l_s}(l_{s+1}\widetilde{R}\widetilde{\boldsymbol{v}} + \cdots + l_{r-1}\widetilde{R}^{r-1-s}\widetilde{\boldsymbol{v}}). \tag{3.10}$$

Then, we know that $\{0, \widetilde{R}\widetilde{\boldsymbol{v}}, \ldots, \widetilde{R}^{n-1}\widetilde{\boldsymbol{v}}\}$ is also linearly dependent with rank r. Note that (3.10) is equivalent to the following equality:

$$\widetilde{R}\widetilde{\boldsymbol{v}} = -\frac{1}{l_s}(l_{s+1}\widetilde{R}^2\widetilde{\boldsymbol{v}} + \cdots + l_{r-1}\widetilde{R}^{r-s}\widetilde{\boldsymbol{v}}).$$

And thus, $\{0, 0, \widetilde{R}^2\widetilde{\boldsymbol{v}}, \ldots, \widetilde{R}^{n-1}\widetilde{\boldsymbol{v}}\}$ is linearly dependent with rank r. Proceeding inductively for finite steps, we know that $\{0, 0, \ldots, 0, \widetilde{R}^{s+1}\widetilde{\boldsymbol{v}}, \ldots, \widetilde{R}^{r-1}\widetilde{\boldsymbol{v}}\}$ is linearly dependent with rank r, which is impossible. Therefore, the claim follows. Then, we know from the claim and (3.3) that $\{\boldsymbol{v}_r, M_1\boldsymbol{v}_r, \ldots, M_1^{r-1}\boldsymbol{v}_r\}$ is linearly independent. Combining Lemmas 2.2 and 3.2, we only need to show $\mu_{\{\widetilde{R}_k^{(r)}\},\{\widetilde{D}_k^{(r)}\}}$ is a spectral measure where

$$\widetilde{R}_k^{(r)} = \begin{cases} b_1\widetilde{M}_1^{m_1}, & k = 1, \\ b_k\widetilde{M}_1^{m_k r}, & k > 1, \end{cases} \quad m_1 = m_1'r + l, \text{ with } m_1' \geq 0\,, 1 \leq l \leq r,$$

$$\widetilde{D}_k^{(r)} = \{0, 1, \ldots, q_k - 1\}\widetilde{\boldsymbol{v}}_r, \;\; \widetilde{\boldsymbol{v}}_r = (0, \ldots, 0, 1)^T,$$

and

$$\widetilde{M}_1 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -c & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

It follows from $q_k | b_k \det(M_1)$ that $q_k \,| b_k c$. For the sake of brevity, we denote $B_k = \widetilde{R}_k^{(r)}$ from now on. As

$$\widetilde{M}_1^{-(m_1+\sum_{i=2}^k m_i r)} = \left(-\frac{1}{c}\right)^{m_1'+\sum_{i=2}^k m_k} \begin{pmatrix} 0 & -\frac{1}{c}E_{l\times l} \\ E_{(r-l)\times(r-l)} & 0 \end{pmatrix},$$

then for any $\xi = (\xi_1, \ldots, \xi_r)^T \in \mathbb{R}^r$, we have

$$\langle B_1^{-1}\cdots B_k^{-1}\widetilde{\boldsymbol{v}}_r, \xi\rangle = \frac{\xi_l}{b_1\cdots b_k(-c)^{m_1'+m_2+\cdots+m_k+1}},$$

where $l = m_1 - m_1'r \in \{1, 2, \ldots, r\}$. Now, we set

$$\mathfrak{b}_k = \begin{cases} b_1|c|^{m_1'+1}, & k = 1, \\ b_k|c|^{m_k}, & k > 1, \end{cases} \quad \text{and} \quad \mathfrak{D}_k = \{0, 1, \ldots, q_k - 1\}.$$

Applying Theorem 1.1, we know that there exists a set $\Lambda \subseteq \mathbb{R}$ satisfying $Q_\Lambda^{(\mu_{\{\mathfrak{b}_k\},\{\mathfrak{D}_k\}})}(\xi) \equiv 1$ for any $\xi \in \mathbb{R}$. Denote

$$\Lambda' = \{(0, \ldots, 0, \lambda, 0, \ldots, 0)^T : \lambda \in \Lambda\},$$

https://doi.org/10.4153/S0008439522000273 Published online by Cambridge University Press

where λ is the lth coordinate of $(0,\dots,0,\lambda,0,\dots,0)^T$. Then, for any

$$\xi = (\xi_1,\dots,\xi_l,\dots,\xi_r)^T \in \mathbb{R}^r,$$

we have

$$\begin{aligned}
Q_{\Lambda'}^{(\mu_{\{\widetilde{R}_k^{(r)}\},\{\widetilde{D}_k^{(r)}\}})}(\xi) &= \sum_{\lambda'\in\Lambda'} |\hat{\mu}_{\{\widetilde{R}_k^{(r)}\},\{\widetilde{D}_k^{(r)}\}}(\xi+\lambda')|^2 \\
&= \sum_{\lambda'\in\Lambda'} \prod_{k=1}^{\infty} \left| \frac{1}{q_k} \sum_{d\in\widetilde{D}_k^{(r)}} e^{-2\pi i \langle B_1^{-1}\cdots B_k^{-1} d,\ \xi+\lambda'\rangle} \right|^2 \\
&= \sum_{\lambda\in\Lambda} \prod_{k=1}^{\infty} \frac{1}{q_k} \left| \sum_{l=0}^{q_k-1} e^{-2\pi i \frac{l(\xi_l+\lambda)}{b_1\cdots b_k |c|^{m_1'+m_2+\cdots+m_k+1}}} \right|^2 \\
&= \sum_{\lambda\in\Lambda} |\hat{\mu}_{\{\mathfrak{b}_k\},\{\mathfrak{D}_k\}}(\xi_l+\lambda)|^2 \\
&= Q_{\Lambda}^{(\mu_{\{\mathfrak{b}_k\},\{\mathfrak{D}_k\}})}(\xi_l) \equiv 1.
\end{aligned}$$

Hence, $\mu_{\{\widetilde{R}_k^{(r)}\},\{\widetilde{D}_k^{(r)}\}}$ is a spectral measure with the spectrum Λ'. ∎

As an application of Theorem 1.4, the proof of Theorem 1.3 is apparent.

Proof of Theorem 1.3 Since $R\boldsymbol{v} = \lambda\boldsymbol{v}$, we have

$$\{\boldsymbol{v}, R\boldsymbol{v},\dots,R^{n-1}\boldsymbol{v}\} = \{\boldsymbol{v},\lambda\boldsymbol{v},\dots,\lambda^{n-1}\boldsymbol{v}\}.$$

It follows that the rank of vectors $\{\boldsymbol{v}, R\boldsymbol{v},\dots,R^{n-1}\boldsymbol{v}\}$ is 1, i.e., $r = 1$. From Lemma 2.3, we know that there exists a unimodular matrix $B \in M_n(\mathbb{Z})$ such that

$$B^{-1}RBB^{-1}\boldsymbol{v} = \lambda B^{-1}\boldsymbol{v}, \tag{3.11}$$

i.e.,

$$\begin{pmatrix} M_1 & C \\ 0 & M_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{v}_r \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} \boldsymbol{v}_r \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where $\boldsymbol{v}_r \in \mathbb{Z}^r$, $M_1 \in M_r(\mathbb{Z})$, $M_2 \in M_{n-r}(\mathbb{Z})$, and $C \in M_{r,n-r}(\mathbb{Z})$. Hence, $M_1\boldsymbol{v}_r = \lambda\boldsymbol{v}_r$. That is, λ is an eigenvalue of M_1. And thus, $\lambda = \det(M_1)$ as $r = 1$. Then, the characteristic polynomial of M_1 is $x - M_1$. Applying Theorem 1.4, we know that $\mu_{\{R_k\},\{D_k\}}$ is a spectral measure. ∎

4 Proof of Theorem 1.5

In this section, we will prove Theorem 1.5. According to the dependence of the set of vectors $\{\boldsymbol{v}, R\boldsymbol{v},\dots,R^{n-1}\boldsymbol{v}\}$, we distinguish the following two cases: $r = n$ (Theorem 4.1) and $r < n$ (Theorem 4.3).

https://doi.org/10.4153/S0008439522000273 Published online by Cambridge University Press

Theorem 4.1 *Let $R \in M_n(\mathbb{Z})$ be an expanding matrix, and let $D = \{0, 1, \dots, q-1\}\boldsymbol{v}$, where the integer $q \geqslant 2$ and $\boldsymbol{v} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$. Suppose $\{\boldsymbol{v}, R\boldsymbol{v}, \dots, R^{n-1}\boldsymbol{v}\}$ is linearly independent. If (R, D) is admissible, then $q \,|\, \det(R)$.*

Proof Let $f(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$ be the characteristic polynomial of R. According to Lemma 3.2, there exists an integer matrix $B = (R^{n-1}\boldsymbol{v}, R^{n-2}\boldsymbol{v}, \dots, R\boldsymbol{v}, \boldsymbol{v})$ such that

$$\widetilde{R} = B^{-1}RB = \begin{pmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & 0 & \cdots & 1 \\ -a_n & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{4.1}$$

$$\widetilde{D} = B^{-1}D = \{0, 1, \dots, q-1\}(0, 0, \dots, 0, 1)^T, \tag{4.2}$$

$$\widetilde{\boldsymbol{v}} = B^{-1}\boldsymbol{v} = (0, 0, \dots, 0, 1)^T. \tag{4.3}$$

Denote

$$d = \gcd(a_n, q), \quad q = dq', \quad |a_n| = da_n', \tag{4.4}$$

where q' and a_n' are positive integers with $\gcd(q', a_n') = 1$. As (R, D) is admissible, we know from Lemma 2.2 that $(\widetilde{R}, \widetilde{D})$ is admissible. Then, there exists

$$C = \left\{ \boldsymbol{x}^{(j)} : \boldsymbol{x}^{(j)} = \left(x_1^{(j)}, x_2^{(j)}, \dots, x_n^{(j)}\right)^T \in \mathbb{Z}^n, j \in \{0, 1, 2, \dots, q-1\} \right\}$$

with $\boldsymbol{x}^{(0)} = \mathbf{0}$ such that $(\widetilde{R}, \widetilde{D}, C)$ is a Hadamard triple. Note

$$\widetilde{R}^{-1}\widetilde{\boldsymbol{v}} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -\frac{1}{a_n} \\ 1 & 0 & \cdots & 0 & -\frac{a_1}{a_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -\frac{a_{n-2}}{a_n} \\ 0 & 0 & \cdots & 1 & -\frac{a_{n-1}}{a_n} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{a_n} \\ -\frac{a_1}{a_n} \\ \vdots \\ -\frac{a_{n-2}}{a_n} \\ -\frac{a_{n-1}}{a_n} \end{pmatrix}.$$

Denote $\widetilde{D} = \{\widetilde{d}_k\}_{k=0}^{q-1}$. Then, for any $m \in \mathbb{Z}$,

$$\begin{aligned} \hat{\delta}_{\widetilde{R}^{-1}\widetilde{D}}(\boldsymbol{x}^{(j)}) &= \frac{1}{q}\sum_{k=0}^{q-1} e^{-2\pi i \langle \widetilde{R}^{-1}\widetilde{d}_k, \boldsymbol{x}^{(j)} \rangle} \\ &= \frac{1}{q}\sum_{k=0}^{q-1} e^{2\pi i k \frac{x_1^{(j)} + a_1 x_2^{(j)} + \cdots + a_{n-2}x_{n-1}^{(j)} + a_{n-1}x_n^{(j)}}{a_n}} \\ &= \frac{1}{q}\sum_{k=0}^{q-1} e^{2\pi i k \frac{(x_1^{(j)} + ma_n) + a_1 x_2^{(j)} + \cdots + a_{n-2}x_{n-1}^{(j)} + a_{n-1}x_n^{(j)}}{a_n}}. \end{aligned} \tag{4.5}$$

Note that there exists $m_j \in \mathbb{Z}$ so that $0 \le L_j := (x_1^{(j)} + m_j a_n) + a_1 x_2^{(j)} + \cdots + a_{n-2}x_{n-1}^{(j)} + a_{n-1}x_n^{(j)} \le |a_n| - 1$ for each $0 \le j \le q-1$. Since $(\widetilde{R}, \widetilde{D}, C)$ is a Hadamard triple, it follows from (4.5) that $L_i \ne L_j$ for any $0 \le i \ne j \le q-1$. Then, we may as well suppose that $0 = L_0 < L_1 < \cdots < L_{q-1} < |a_n|$. Set

$$\widetilde{C} = \left\{\widetilde{\boldsymbol{x}}^{(j)} : \widetilde{\boldsymbol{x}}^{(j)} = \boldsymbol{x}^{(j)} + (m_j a_n, 0, \dots, 0)^T, \boldsymbol{x}^{(j)} \in C\right\}$$

with $m_0 = 0$. Obviously, $(\widetilde{R}, \widetilde{D}, \widetilde{C})$ is a Hadamard triple. This together with (4.5) implies that for each $1 \le j \le q-1$,

$$\left|\frac{1}{q}\sum_{\widetilde{d}_k \in \widetilde{D}} e^{-2\pi i \langle \widetilde{R}^{-1}\widetilde{d}_k, \widetilde{x}^{(j)} - \mathbf{0}\rangle}\right| = \frac{|\sin(a_n^{-1} q L_j \pi)|}{|q \sin(a_n^{-1} L_j \pi)|} = 0.$$

It follows that $L_j = \alpha_j a_n' (0 \le j \le q-1)$ where all α_j are integers with $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{q-1}$. Hence, $\alpha_{q-1} \ge q-1$. And thus, $|a_n| > L_{q-1} = \alpha_{q-1}a_n' \ge (q-1)a_n'$. This together with (4.4) implies that $d > q-1$. On the other hand, $d = \gcd(q, a_n)$. Therefore, $d = q$. Notice that $\det(R) = (-1)^n a_n$. Then, the assertion follows. ■

Especially, we have the following corollary.

Corollary 4.2 *Let*

$$R = \begin{pmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & 0 & \cdots & 1 \\ -a_n & 0 & 0 & \cdots & 0 \end{pmatrix}$$

be an expanding integer matrix, and let $D = \{0, 1, \dots, q-1\}\boldsymbol{v}$, where $\boldsymbol{v} = (0, 0, \dots, 0, 1)^T$. Then, (R, D) is admissible if and only if $q | \det(R)$.

Proof The necessity is proved in Theorem 4.1. Conversely, Set

$$L = \{0, 1, \dots, q-1\}\left(\frac{a_n}{q}, 0, \dots, 0\right)^T.$$

As $\det(R) = (-1)^n a_n$ and $q | \det(R)$, we have $L \subset \mathbb{Z}^n$. Note that

$$R^{-1}\boldsymbol{v} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -\frac{1}{a_n} \\ 1 & 0 & \cdots & 0 & -\frac{a_1}{a_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -\frac{a_{n-2}}{a_n} \\ 0 & 0 & \cdots & 1 & -\frac{a_{n-1}}{a_n} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{a_n} \\ -\frac{a_1}{a_n} \\ \vdots \\ -\frac{a_{n-2}}{a_n} \\ -\frac{a_{n-1}}{a_n} \end{pmatrix}.$$

https://doi.org/10.4153/S0008439522000273 Published online by Cambridge University Press

Then, for any distinct $l_1, l_2 \in \{0,1,\ldots,q-1\}$,

$$\hat{\delta}_{R^{-1}D}\left((l_1-l_2)\Big(\frac{ca_n}{q},0,\ldots,0\Big)^T\right) = \frac{1}{q}\sum_{d\in D} e^{-2\pi i\langle R^{-1}d,(l_1-l_2)(\frac{ca_n}{q},0,\ldots,0)^T\rangle}$$
$$= \frac{1}{q}\sum_{k=0}^{q-1} e^{-2\pi i\langle kR^{-1}(0,0,\ldots,0,1)^T,(l_1-l_2)(\frac{ca_n}{q},0,\ldots,0)^T\rangle} = \frac{1}{q}\sum_{k=0}^{q-1} e^{-2\pi i\frac{kc(l_2-l_1)}{q}} = 0.$$

Hence, (R,D,L) is a Hadamard triple. ∎

Theorem 4.3 *Let $R \in M_n(\mathbb{Z})$ be an expanding matrix, and let $D = \{0,1,\ldots,q-1\}\boldsymbol{v}$, where the integer $q \geqslant 2$ and $\boldsymbol{v} \in \mathbb{Z}^n \smallsetminus \{\mathbf{0}\}$. Suppose that $\{\boldsymbol{v}, R\boldsymbol{v},\ldots,R^{n-1}\boldsymbol{v}\}$ is linearly dependent with rank $r < n$. If (R,D) is admissible, then $q|\det(M_1)$, where M_1 is expressed as in (1.2).*

Proof According to Lemma 2.3, there exists a unimodular matrix $B \in M_n(\mathbb{Z})$ such that

$$(4.6)\qquad \widetilde{R} := B^{-1}RB = \begin{pmatrix} M_1 & C \\ 0 & M_2 \end{pmatrix} \in M_n(\mathbb{Z}),\ \ \widetilde{\boldsymbol{v}} := B^{-1}\boldsymbol{v} = (\boldsymbol{v}_r^T,0,\ldots,0)^T$$

and

$$(4.7)\qquad \widetilde{D} := B^{-1}D = \{0,1,\ldots,q-1\}(\boldsymbol{v}_r^T,0,\ldots,0)^T,$$

where $\boldsymbol{v}_r \in \mathbb{Z}^r$, $M_1 \in M_r(\mathbb{Z})$, $M_2 \in M_{n-r}(\mathbb{Z})$, and $C \in M_{r,n-r}(\mathbb{Z})$. By Lemma 2.2, we know that $(\widetilde{R},\widetilde{D})$ is admissible since (R,D) is admissible. As $\{\boldsymbol{v},R\boldsymbol{v},\ldots,R^{n-1}\boldsymbol{v}\}$ is linearly dependent with rank r, we know that $\{\widetilde{\boldsymbol{v}},\widetilde{R}\widetilde{\boldsymbol{v}},\ldots,\widetilde{R}^{n-1}\widetilde{\boldsymbol{v}}\}$ is also linearly dependent with rank r. Similar to the proof in Theorem 1.4, we know that $\{\widetilde{\boldsymbol{v}},\widetilde{R}\widetilde{\boldsymbol{v}},\ldots,\widetilde{R}^{r-1}\widetilde{\boldsymbol{v}}\}$ is linearly independent. Then, we know from (4.6) that $\{\boldsymbol{v}_r, M_1\boldsymbol{v}_r,\ldots,M_1^{r-1}\boldsymbol{v}_r\}$ is linearly independent. Now, we define a bounded linear operator $T:\mathbb{R}^n \to \mathbb{R}^r$ given by

$$T\big((x_1,\ldots,x_r,\ldots,x_n)^T\big) = (x_1,\ldots,x_r)^T$$

for any $(x_1,\ldots,x_n)^T \in \mathbb{R}^n$. Note that

$$\widetilde{R}^{-1}\widetilde{\boldsymbol{v}} = \begin{pmatrix} M_1^{-1} & \times \\ \mathbf{0} & M_2^{-1} \end{pmatrix}\begin{pmatrix} \boldsymbol{v}_r \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} M_1^{-1}\boldsymbol{v}_r \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since $(\widetilde{R},\widetilde{D})$ is admissible, there exists $\widetilde{L} \subseteq \mathbb{Z}^n$ with $\#\widetilde{L} = \#\widetilde{D} = q$ such that

$$H = \frac{1}{\sqrt{q}}\left(e^{-2\pi i\langle R^{-1}d,l\rangle}\right)_{d\in\widetilde{D},l\in\widetilde{L}} = \frac{1}{\sqrt{q}}\left(e^{-2\pi ik\langle M_1^{-1}\boldsymbol{v}_r,l\rangle}\right)_{k\in\{0,1,\ldots,q-1\},l\in T(\widetilde{L})}$$

is unitary. It follows that $(M_1, T(\widetilde{D}), T(\widetilde{L}))$ is a Hadamard triple. Combining with Theorem 4.1, we have $q|\det(M_1)$. ∎

Now, we have all ingredients to prove Theorem 1.5.

https://doi.org/10.4153/S0008439522000273 Published online by Cambridge University Press

Proof of Theorem 1.5 The conclusion follows directly from Theorems 4.1 and 4.3. ∎

5 Some examples

In this section, we give some examples to illustrate our theory. The first example is an application of Theorem 1.4.

Example 5.1 Let $R_k = b_k \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}^{m_k}$ and $D_k = \{0, 1, \ldots, q_k - 1\}\boldsymbol{v}$, where $b_k, m_k \in \mathbb{Z}^+$ and $\boldsymbol{v} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ for $k \geq 1$. Then, $\mu_{\{R_k\},\{D_k\}}$ is a spectral measure if $q_k | 4b_k$.

Proof Let

$$R = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}, \; \boldsymbol{v} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

Thus,

$$R\boldsymbol{v} = 4\boldsymbol{v} \;\text{ and }\; B^{-1}RB = \begin{pmatrix} 4 & -3 & 3 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

According to Theorem 1.4, $\mu_{\{R_k\},\{D_k\}}$ is a spectral measure if $q_k | 4b_k$. ∎

The following example is given to explain that the condition $q | \det(M_1)$ in Theorem 1.5 is not sufficient.

Example 5.2 Let $R = \begin{pmatrix} 2k+1 & 0 \\ 2c & 2c \end{pmatrix} \in M_2(\mathbb{Z})$ and $D = \{0, 1\}\boldsymbol{v}$, where $\boldsymbol{v} = (1, 0)^T$ and k, c are nonzero integers. Then, (R, D) is not admissible.

Proof By a direct calculation, we have

$$\mathcal{Z}(\hat{\delta}_D) = \frac{1}{2}(1, 0)^T + (\mathbb{Z}, \mathbb{R})^T, \tag{5.1}$$

where $(\mathbb{Z}, \mathbb{R}) = \{(x_1, x_2) : x_1 \in \mathbb{Z}, x_2 \in \mathbb{R}\}$. If (R, D) is admissible, then there exists $C_1 \subseteq \mathbb{Z}^2$ such that $(R^{-1}D, C_1)$ is a compatible pair with $\mathbf{0} \in C_1$. Let $C = R^{*-1}C_1$. Then, we know that $C \subseteq \mathcal{Z}(\hat{\delta}_D)$ and there exist $n \in \mathbb{Z}$ and $m \in \mathbb{R}$ such that

$$C = \left\{\mathbf{0}, (\frac{1}{2} + n, m)^T\right\}.$$

https://doi.org/10.4153/S0008439522000273 Published online by Cambridge University Press

Since $C_1 = R^* C \subseteq \mathbb{Z}^2$, we have

$$R^* \begin{pmatrix} \frac{1}{2} + n \\ m \end{pmatrix} = \begin{pmatrix} \left(\frac{1}{2} + n\right)(2k+1) + 2mc \\ 2mc \end{pmatrix} \in \mathbb{Z}^2,$$

i.e.,

$$\begin{cases} (2k+1)(1+2n) + 4mc \in 2\mathbb{Z}, \\ 2cm \in \mathbb{Z}. \end{cases} \tag{5.2}$$

This is a contradiction. Hence, (R, D) is not admissible. ■

In Example 5.2, we notice that $\{\boldsymbol{v}, R\boldsymbol{v}\}$ is linearly independent and $2|\det(R)$, but (R, D) is not admissible. Hence, the condition $q|\det(M_1)$ in Theorem 1.5 is not sufficient.

References

[1] L.-X. An, X.-Y. Fu, and C.-K. Lai, *On spectral Cantor–Moran measures and a variant of Bourgain's sum of sine problem*. Adv. Math. **349**(2019), 84–124.
[2] L.-X. An and X.-G. He, *A class of spectral Moran measures*. J. Funct. Anal. **266**(2014), 343–354.
[3] X.-R. Dai, *When does a Bernoulli convolution admit a spectrum?* Adv. Math. **231**(2012), 1681–1693.
[4] X.-R. Dai, X.-Y. Fu, and Z.-H. Yan, *Spectrality of self-affine Sierpinski-type measures on* $\mathbb{R}^2$. Appl. Comput. Harmon. Anal. **52**(2021), 63–81.
[5] X.-R. Dai, X.-G. He, and C.-K. Lai, *Spectral property of Cantor measures with consecutive digits*. Adv. Math. **242**(2013), 187–208.
[6] X.-R. Dai, X.-G. He, and K.-S. Lau, *On spectral N-Bernoulli measures*. Adv. Math. **259**(2014), 511–531.
[7] Q.-R. Deng and K.-S. Lau, *Sierpinski-type spectral self-similar measures*. J. Funct. Anal. **268**(2015), 1310–1326.
[8] D. Dutkay, S. Emami, and C.-K. Lai, *Existence and exactness of exponential Riesz sequences and frames for fractal measures*. J. Anal. Math. **143**(2021), 289–311.
[9] D. Dutkay, D. Han, and Q. Sun, *On the spectra of a Cantor measure*. Adv. Math. **221**(2009), 251–276.
[10] D. Dutkay, J. Haussermann, and C.-K. Lai, *Hadamard triples generate self-affine spectral measuresąč*. Trans. Amer. Math. Soc. **371**(2019), 1439–1481.
[11] K. J. Falconer, *Fractal geometry: mathematical foundations and applications*, Wiley, New York, 1990.
[12] Y.-S. Fu and Z.-X. Wen, *Spectrality property of a class of Moran measures on* $\mathbb{R}$. J. Math. Anal. Appl. **430**(2015), 572–584.
[13] Y.-S. Fu and Z.-X. Wen, *Spectrality of infinite convolutions with three-element digit sets*. Monatsh. Math. **183**(2017), 465–485.
[14] B. Fuglede, *Commuting self-adjoint partial differential operators and a group theoretic problem*. J. Funct. Anal. **16**(1974), 101–121.
[15] L. He and X.-G. He, *On the Fourier orthonormal basis of Cantor–Moran measure*. J. Funct. Anal. **272**(2017), 1980–2004.
[16] T. Y. Hu and K. S. Lau, *Spectral property of the Bernoulli convolution*. Adv. Math. **219**(2008), 554–567.
[17] J. Jacod and P. Protter, *Probability essentials*. 2nd ed., Universitext, Springer, Berlin, 2003.
[18] P. Jorgensen and S. Pedersen, *Dense analytic subspaces in fractal* L^2 *spaces*. J. Anal. Math. 75(1998), 185–228.
[19] P. Jorgensen and S. Pedersen, *Orthogonal harmonic analysis of fractal measures*. Electron. Res. Announc. Amer. Math. Soc. **4**(1998), 35–42.
[20] M. Kolountzakis and M. Matolcsi, *Complex Hadamard matrices and the spectral set conjecture*. Collect. Math. **Extra**(2006), 281–291.

[21] M. Kolountzakis and M. Matolcsi, *Tiles with no spectra*. Forum Math. **18**(2006), 519–528.
[22] I. Łaba and Y. Wang, *On spectral Cantor measures*. J. Funct. Anal. **193**(2002), 409–420.
[23] J.-L. Li, $\mu_{M,D}$*−orthogonality and compatible pair*. J. Funct. Anal. **244**(2007), 628–638.
[24] J.-L. Li, *Spectral of a class self-affine measures*. J. Funct. Anal. **260**(2011), 1086–1095.
[25] J. Liu and J. Luo, *Spectral property of self-affine measures on* $\mathbb{R}^n$. J. Funct. Anal. **272**(2017), 599–612.
[26] Z.-S. Liu and X.-H. Dong, *Spectra of a class of Moran measures (Chinese)*. Adv. Math. (China) **47**(2018), 441–447.
[27] R. Stricharz, *Fourier asymptotics of fractal measures*. J. Funct. Anal. **89**(1990), 154–187.
[28] R. Stricharz, *Self-similarity in harmonic analysis*. J. Fourier Anal. Appl. **1**(1994), 1–37.
[29] R. Stricharz, *Mock Fourier series and transforms associated with certain Cantor measures*. J. Anal. Math. **81**(2000), 209–238.
[30] T. Tao, *Fuglede's conjecture is false in 5 or higher dimensions*. Math. Res. Lett. **11**(2004), 251–258.

College of Mathematics and Computational Science, Hunan First Normal University, Changsha 410205, China
e-mail: csjl0432@163.com

School of Mathematics and Statistics, Central China Normal University, Wuhan 430079, China
e-mail: liqian303606@163.com

https://doi.org/10.4153/S0008439522000273 Published online by Cambridge University Press