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ON QUOTIENT LOOPS OF NORMAL SUBLOOPS

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1. **Introduction.** The following result is due to Wielandt [1, Lemma 2.9]: Let A, B, K be N-submodules of some N-module, where N is a zero symmetric near-ring. Then the N-module, $\Gamma := (A+K)\cap(B+K) \mid (A\cap B)+K$ is commutative. Using this result Wielandt obtained density theorem for 2-primitive near-rings with identity. Betsch [1] used Wielandt's result to obtain the density theorem for 0-primitive near-rings. The purpose of this paper is to extend this result for loops.

2. **Result.** We prove the result for additive loops. For the definitions of loops, normal subloops see [2]. If G is any additive loop and H is a normal subloop of G, the quotient loop G modulo H is denoted by G/H in which addition is defined as $(x+H)+(y+H)=(x+y)+H$ for all $x+H$, $y+H$ in G/H [2, p. 61]; further, $x\in H$ iff $x+H=H$. Let G be any additive loop. For any $a\in G$, we shall denote the unique left and right additive inverses of a by a_1 and a_r respectively.

Proposition 2.1. *Let G be an additive loop and A, B, K be normal subloops of G, then the additive loop $\bar{G}=(A+K)\cap(B+K)\mid(A\cap B)+K$ is an additive abelian group.*

Proof. Let $E=(A\cap B)+K$ and $H=(A+K)\cap(B+K)$ and let 0 be the identity of the loop G. It is enough to show that for all $x, y, z\in H$; $(x+(y+z))+E=((x+y)+z)+E$ and $(x+y)+E=(y+x)+E$. Let $x, y, z\in H$. Then, $x\in A+K$; hence $x=a+p$, for some $a\in A$ and $p\in K$. Since, $y, z\in H$; $y, z\in B+K$, hence, $y=b+q$ and $z=c+r$, where $b, c\in B$ and $q, r\in K$. We wish to show that $x+E=a+E$, $y+E=b+E$ and $z+E=c+E$. Since E is a normal subloop of G [2, iv, Theorem 1.2 and Theorem 1.4], $(x+E)=(a+p)+E=a+(p+E)$. Since $p\in K$, $p\in E$ and hence $p+E=E$. Therefore, $x+E=a+E$. By a similar argument we get, $y+E=b+E$ and $z+E=c+E$. Since A is a normal subloop of G and since $a\in A$, $A+((a+b)+c)=(A+(a+b))+c=((A+a)+b)+c=(A+b)+c=A+(b+c)$ and $A+(a+(b+c))=(A+a)+(b+c)=A+(b+c)$. Therefore, $A+((a+b)+c)=A+(a+(b+c))$. Since, $(a+b)+c\in A+((a+b)+c)$, we have $(a+b)+c\in A+(a+(b+c))$; since A is a normal subloop of G, we have,

$$((a+b)+c)+(a+(b+c))_r\in\{A+(a+(b+c))\}+(a+(b+c))_r$$
$$=A+\{(a+(b+c))+(a+(b+c))_r\}=A+0=A.$$

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By a similar argument, we get, $B+((a+b)+c)=B+(a+(b+c))$ and consequently $((a+b)+c)+(a+(b+c))_r \in B$. Therefore, $((a+b)+c)+(a+(b+c))_r \in A \cap B \subseteq E$. Hence,

$$\{((a+b)+c)+(a+(b+c))_r\}+E=E=0+E=\{(a+(b+c))+(a+(b+c))_r\}+E.$$

Since G/E is a loop and since cancellation laws hold in a loop, we get, $(a+(b+c))+E=((a+b)+c)+E$. Therefore,

$$(x+(y+z))+E=(a+(b+c))+E=((a+b)+c)+E=((x+y)+z)+E.$$

Hence, the loop $\bar{G}$ is associative and consequently $\bar{G}$ is a group. Now we show that $\bar{G}$ is abelian. Since A is a normal subloop of G and $a \in A$ we have, $A+(a+b)=(A+a)+b=A+b$ and $A+(b+a)=(b+a)+A=b+(a+A)=b+A=A+b$. Therefore, $A+(a+b)=A+(b+a)$ and hence, $(a+b)+(b+a)_r \in A$. By a similar argument we get, $(a+b)+(b+a)_r \in B$. Therefore $(a+b)+(b+a)_r \in A \cap B \subseteq E$. Hence, $(a+b)+E=(b+a)+E$. Therefore, $(x+y)+E=(a+b)+E=(b+a)+E=(y+x)+E$. Hence, $\bar{G}$ is abelian. Hence the result.

Corollary 2.2. *Let G be an additive loop and let no nonzero epimorphic image of any normal subloop of G be an abelian group. Then the lattice of normal subloops of G is distributive.*

Proof. Let A, B, K be normal subloops of G and let $H=(A+K)\cap(B+K)$, $E=(A\cap B)+K$ and $\bar{G}=H/E$. Now, H is a normal subloop of G [2, iv, Theorems 1.2; 1.4], $\bar{G}$ is an epimorphic image of H; but $\bar{G}$ is an abelian group (prop. 2.1), hence we must have $H=E$, that is, $(A+K)\cap(B+K)=(A\cap B)+K$. Therefore, the lattice of normal subloops of G is distributive.

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