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LARGE ABELIAN SUBGROUPS OF CHEVALLEY GROUPS

MICHAEL J. J. BARRY

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Abstract

For any group S let $\mathrm{Ab}(S) = \{A \mid A$ is an abelian subgroup of S of maximal order$\}$. Let G be a Chevalley group of type A_n, B_n, C_n or D_n over a finite field of characteristic p and let $U \in \mathrm{Syl}_p(G)$. In this paper $\mathrm{Ab}(U)$ is determined for all such groups.

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Introduction

Let $q = p^k$, p a prime number. For an odd prime r different from p, a theorem of Alperin (1965) shows that an r-Sylow subgroup of $\mathrm{GL}(n,q)$ has a unique largest normal abelian subgroup and that no other abelian subgroup has order as great. Goozeff (1970) considered a p-Sylow subgroup U of $\mathrm{GL}(n,q)$ where q is odd. He bounded the order of an abelian subgroup of U and showed that this bound is always attained. Goozeff also pointed out that, if n is even, U has a unique largest abelian subgroup. Thwaites (1972) considered a p-Sylow subgroup U of $\mathrm{GL}(n,p)$. He showed that if n is even, U contains precisely one abelian subgroup of maximal rank, while if n is odd and $n \geqslant 5$, U contains precisely two abelian subgroups of maximal rank.

Theorem 2.1 of this paper identifies $\mathrm{Ab}(U)$ where U is a p-Sylow subgroup of $\mathrm{SL}(n,q)$ and hence of $\mathrm{GL}(n,q)$ with no restriction on whether q is even or odd. If n is even $|\mathrm{Ab}(U)| = 1$; while if n is odd and $n \geqslant 5$, $|\mathrm{Ab}(U)| = 2$. Finally if $n = 3$ then $|\mathrm{Ab}(U)| = q+1$. In all cases $\mathrm{Ab}(U)$ contains groups which are elementary abelian and so the abelian subgroups of maximal rank can be read from the list of groups in $\mathrm{Ab}(U)$. Hence Theorem 2.1 generalizes the results of Goozeff and Thwaites. In Sections 2–5 solutions to the problem for groups of type B_n, C_n and D_n are also

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presented and the results are arranged in order of difficulty. In Section 6 we calculate the Thompson subgroup $J(U)$ of U and we note that $J(U) = \langle A \mid A \leqslant U, A$ abelian of maximal rank$\rangle$.

Most of the results of this paper were contained in the author's doctoral thesis which was written under the direction of Professor Warren J. Wong at the University of Notre Dame. The author wishes to thank Professor Wong for posing this problem in the first place, and for his patience and encouragement during its solution. Thanks too are owed to the Department of Mathematics at Notre Dame for its generous support throughout.

1. Notation and terminology

The current standard references for the theory of Chevalley groups are Steinberg (1968) and Carter (1972). In this section we will fix our notation and dwell a little on those aspects of Chevalley groups which we will need.

F will always denote a finite field of $q = p^k$ elements where p is a prime number. Let Φ be a root system for a simple finite-dimensional Lie algebra g over $\mathbf{C}$. Then Φ^+ will denote the set of positive roots and $\Pi = \{r_1, r_2 \ldots, r_n\}$ the fundamental set relative to some ordering. The universal Chevalley group of type g over F, denoted by $g(q)$, is obtained from a particular representation of g over $\mathbf{C}$ by choosing an admissible lattice and 'going mod-p'. Thus $A_n(q)$ will mean the universal Chevalley group of type A_n over F and the meaning of $B_n(q)$, $C_n(q)$ and $D_n(q)$ is now clear.

Now

$$A_n(q) \cong \mathrm{SL}(n+1, q), \quad C_n(q) \cong \mathrm{Sp}(2n, q),$$

$$B_n(q) = \mathrm{Spin}(2n+1, q) \quad \text{and} \quad D_n(q) \cong \mathrm{Spin}(2n, q)$$

where $\mathrm{Spin}(2n+1, q)$ (respectively $\mathrm{Spin}(2n, q)$), is the universal central extension of Ω $(2n+1, q)$ (respectively Ω $(2n, q)$), the commutator subgroup of $O(2n+1, q)$ (respectively $O^+(2n, q)$) using the notation of Carter (1972), p.6.

Let $G = g(q)$. Then $G = \langle X(r) \mid r \in \Phi \rangle$ where the root subgroup

$$X(r) = \langle x(r, t) \mid t \in F \rangle \cong F$$

as an additive group. We define the subgroup $U = \langle X(r) \mid r \in \Phi^+ \rangle$. Now U is a p-Sylow subgroup of G. We shall be concerned with $\mathrm{Ab}(U)$.

Let B be the normalizer of U in G. Then B is a semi-direct product of U with H where H is an abelian p'-group. The parabolic subgroups of G are those subgroups which contain B and there is a natural bijection between the family of subsets of Π and the parabolic subgroups of G.

THEOREM 1.1. *Suppose G is a Chevalley group amd P_J the parabolic subgroup naturally associated with $J \subseteq \Pi$. Then $P_J = L_J U_J$, a semidirect product with $U_J \lhd P_J$, is* *known as the Levi decomposition of* P_J. *Here* $U_J = \langle X(r) | r \in \Phi^+ - \Phi_J \rangle$, $L_J = \langle H, S_J \rangle$, $S_J = \langle X(r) | r \in \Phi_J \rangle$ and Φ_J *is the root system with J as the fundamental set.*

We next define a partial order on Φ as follows: if $r, s \in \Phi$ then $r \geqslant s$ if $r - s$ is a non-negative linear integral combination of elements of Π.

EXAMPLES. In Theorem 1.1, if $J = \Pi - \{s_1, \ldots, s_r\}$ then $U_J = \langle X(r) | r \geqslant s_j$ for some j such that $1 \leqslant j \leqslant r \rangle$.

Throughout this paper, the Dynkin diagrams of the indecomposable roots systems of type A_n, B_n, C_n, and D_n will be labelled as in Humphreys (1972), p. 58.

The root system Φ can be thought of as a subset of $\mathbf{R}^n$ where $\Pi = \{r_1, r_2, \ldots, r_n\}$ is a basis for $\mathbf{R}^n$. If $\mathbf{R}^n$ is equipped with an inner product $(,)$ we define a new form $\langle , \rangle$ on $\mathbf{R}^n$ as follows:

$$\langle r, s \rangle = 2(r, s)/(s, s)$$

for all $r, s \in \mathbf{R}^n$, $s \neq 0$. A vector $\lambda \in \mathbf{R}^n$ is called an *abstract weight* provided that $\langle \lambda, r \rangle$ is integral for all $r \in \Phi$. These vectors form a lattice Λ which has a basis of fundamental dominant weights $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ for which $\langle \lambda_i, r_j \rangle = \delta_{ij}$.

A *dominant weight* is any non-negative linear integral combination of the λ_i, $1 \leqslant i \leqslant n$. Denote by Λ^+ the set of all dominant weights. For each $\lambda \in \Lambda^+$ there is (to within isomorphism) exactly one irreducible g-module $V(\lambda)$ over $\mathbf{C}$ whose highest weight is λ and this weight occurs with multiplicity 1. By choosing an admissible lattice and 'going mod-p' we construct a $g(q)$-module over F which we denote by $V(g, n, \lambda)$ where n is the rank of g.

2. The solution for $A_n(q)$, any q, and $C_n(q)$, q odd

THEOREM 2.1.
(a) *Let* $G = A_{2n+1}(q)$ *and* $B = \langle X(r) | r \geqslant r_{n+1} \rangle$ *Then* $\mathrm{Ab}(U) = \{B\}$ *and* $|B| = q^{(n+1)^2}$.
(b) *Let* $G = A_{2n}(q)$, $B(1) = \langle X(r) | r \geqslant r_n \rangle$ *and* $B(2) = \langle X(r) | r \geqslant r_{n+1} \rangle$. *Then*
(i) *if* $n > 1$, $\mathrm{Ab}(U) = \{B(1), B(2)\}$;
(ii) *if* $n = 1$, $\mathrm{Ab}(U) = \{B(1), B(2), B(a) | a \in F^*\}$,
where $B(a) = \langle x(r_1, t)\, x(r_2, at),\ X(r_1 + r_2) | t \in F \rangle$. *Further, any element of* $\mathrm{Ab}(U)$ *has order* $q^{n(n+1)}$.

PROOF. We will prove only (*b*) from which the method of proving (*a*) will be quite apparent. We proceed by induction on n.

Consider $n = 1$. Then $U = \langle X(r_1), X(r_2), X(r_1 + r_2) \rangle$. If $A \in \mathrm{Ab}(U)$ then clearly $Z(U) = X(r_1 + r_2) \leqslant A$. Now

$$C_U(x(r_1, t)) = \langle X(r_1), X(r_1 + r_2) \rangle$$

and

$$C_U(x(r_1, at)\,x(r_2, t)) = \langle x(r_1, as)\,x(r_2, s),\ X(r_1+r_2) | s \in F \rangle \text{ if } t \neq 0.$$

Gathering these pieces of information together and noting that these centralizers are abelian we get part (ii) of (b).

We now assume the result is true for any integer r where $1 \leqslant r < n$. Let P_J be the parabolic subgroup of $G = A_{2n}(q)$ associated with $J = \Pi - \{r_1, r_{2n}\}$. Then

$$S_J = \langle X(r) | r \in \Phi_J \rangle \cong A_{2n-2}(q)$$

and

$$U_J = \langle X(r) | r \geqslant r_1 \text{ or } r \geqslant r_{2n} \rangle.$$

Further, $[U_J, U_J] = X(r_0) = Z(U_J)$ where $r_0 = r_1 + r_2 + \ldots + r_{2n}$ is the highest root. From now on let $V = U_J$. Let $U_1 = \langle X(r) | r \in \Phi_J^+ \rangle$. Then $U_1 \in \mathrm{Syl}_p(S_J)$ and by induction.

$$\begin{aligned} \mathrm{Ab}(U_1) &= \{B_1(1), B_1(2)\} \quad \text{if } n > 2 \\ &= \{B_1(1), B_1(2), B_1(a) | a \in F^*\} \quad \text{if } n = 2, \end{aligned}$$

where

$$B_1(1) = \langle X(r) | r \in \Phi_J^+,\ r \geqslant r_n \rangle,$$

$$B_2(2) = \langle X(r) | r \in \Phi_J^+,\ r \geqslant r_{n+1} \rangle$$

and

$$B_2(a) = \langle x(r_2, t)\,x(r_3, at),\ X(r_2 + r_3) | t \in F \rangle.$$

LEMMA 2.2. *If* $C \in \mathrm{Ab}(V)$, *then* $|C| = q^{2n}$.

PROOF. Let $K = \{r \in \Phi^+ | r \neq r_0, r \geqslant r_1 \text{ or } r \geqslant r_{2n}\}$ and let $'$ be the natural epimorphism of V on $V|Z(V) = V'$. Then V' is abelian since $Z(V) = [V, V]$. Now V' can be made into a vector space over F with basis $\{x'(r, 1) | r \in K\}$ by defining

$$tx'(r, 1) + ux'(s, 1) = x'(r, t)\,x'(a, u)$$

for all $r, s \in K$, $t, u \in F$.

Moreover it is even possible to equip V' with an alternating bilinear form (,) as follows:

$$x(r_0, (v_1', v_2')) = [v_1, v_2]$$

for all $v_1', v_2' \in V'$. Note this definition is independent of the choices of the preimages v_1, v_2 in V since the kernel of $' = Z(V) = [V, V]$. The form is non-degenerate since $(v_1', v') = 0$ for all $v' \in V'$ implies $[v_1, v] = 1$ for all $v \in V$ which implies $v_1 \in Z(V)$ and finally that $v_1' = 0$.

It is clear that a subgroup W of V is abelian if and only if $(W)'$ is contained in a totally isotropic subspace of V'. The maximal dimension of a totally isotropic subspace of $V' = \dim V'/2 = 2n-1$. Therefore W abelian implies that $|W'| \leqslant q^{2n-1}$ and hence $|W| \leqslant q^{2n}$. In fact $\langle X(r) | r \geqslant r_1 \rangle$ is an abelian subgroup of V of order q^{2n}. This completes the proof of Lemma 2.2.

Let

$$V_1(1) = \langle X(r) | r \geqslant r_1 + \ldots + r_n \text{ or } r \geqslant r_n + \ldots + r_{2n} \rangle$$

and

$$V_1(2) = \langle X(r) | r \geqslant r_1 + \ldots + r_{n+1} \text{ or } r \geqslant r_{n+1} + \ldots + r_{2n} \rangle.$$

LEMMA 2.3.
(a) $V_1(i) \lhd U$, $i = 1, 2$.
(b) $V_1(i) \in \mathrm{Ab}(V)$ *and hence* $C_V(V_1(i)) = V_1(i)$, $i = 1, 2$.
(c) $C_{V'}(B_1(i)) = (V_1(i))'$, $i = 1, 2$.
(d) $[B_1(i), V_1(i)] = 1$ *and* $B(i) = B_1(i)\, V_1(i)$, $i = 1, 2$.

PROOF.
(a) This follows from Chevalley's commutator formula.
(b) $V_1(i)$ is abelian, $|V_1(i)| = q^{2n}$ and so $V_1(i) \in \mathrm{Ab}(V)$ by Lemma 2.2. Hence $C_V(V_1(i)) = V_1(i)$, $i = 1, 2$.
(c) This follows by inspection.
(d) This follows from the commutator formula and the definitions of $B(i)$, $B_1(i)$ and $V_1(i)$, $i = 1, 2$.
This completes the proof of Lemma 2.3.

Let φ be the natural epimorphism of U on $U/V \cong U_1$, and let $A \in \mathrm{Ab}(U)$. Then $|A| \geqslant q^{n(n+1)} = |B(i)|$, $i = 1, 2$. Further, $(A)\varphi = A_1 \leqslant U_1$ and $|A| = |A_1|\,|A \cap V|$. Therefore

$$|B(i)| \leqslant |A| = |A_1|\,|A \cap V| \leqslant |B_1(i)| q^{2n} = |B_1(i)|\,|V_1(i)| = |B(i)|, \quad i = 1, 2,$$

making use of Lemmas 2.2 and 2.3. Hence $A_1 \in \mathrm{Ab}(U_1)$ and $A \cap V \in \mathrm{Ab}(V)$. Calling on our induction hypothesis $A_1 \in \mathrm{Ab}(U_1)$ means that $A_1 = B_1(1)$ or $B_1(2)$ if $n > 2$ and $A_1 = B_1(1)$ or $B_1(2)$ or $B_1(a)$ if $n = 2$.

We consider the case $n = 2$ and $A_1 = B_2(a)$ for some $a \in F^*$. Clearly

$$(A \cap V)' \leqslant C_V{}'(B_1(a)).$$

An easy calculation gives $C_{V'}(B_1(a)) = \langle X'(r_1 + r_2 + r_3), X'(r_2 + r_3 + r_4) \rangle$. This implies $|(A \cap V)'| \leqslant q^2$ and hence $|A \cap V| \leqslant q^3$. This contradicts $A \cap V \in \mathrm{Ab}(V)$ and so we have ruled out the case $A_1 = B_1(a)$ for some $a \in F.^*$ Therefore $A_1 = B_1(1)$ or $B_1(2)$ in all cases.

We may suppose that $A_1 = B_1(1)$. Then $A \leqslant B_1(1)\, V$. Note that $B_1(1) \cap V = 1$.

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Let $v \in A \cap V$, $a \in A$. Then $a = bv_1$ where $b \in B_1(1)$ and $v_1 \in V$. Now

$$1 = [a, v] = [b, v]^{v_1} [v_1, v].$$

Hence $[b, v]^{v_1} \in Z(V) = [V, V]$ and hence $[b, v] \in Z(V)$.

Now since $A_1 = B_1(1)$, given any $b \in B_1(1)$ there exists $a \in A$ such that $a = bw$ for some $w \in V$. Therefore $[B_1(1), v] \leqslant Z(V)$ and so $v' \in C_{V'}(B_1(1))$. By Lemma 2.3(c) $v \in V_1(1)$ and so $A \cap V \leqslant V_1(1)$. Since $|A \cap V| = |V_1(1)|$ we get equality.

Again let $a \in A$, $a = bv_1$ where $b \in B_1(1)$ and $v_1 \in V$ and let $v \in A \cap V = V_1(1)$. Then

$$1 = [a, v] = [b, v]^{v_1} [v_1, v] = [v_1, v]$$

since $[B_1(1), V_1(1)] = 1$ by Lemma 2.3(d). Therefore $v_1 \in C_V(V_1(1)) = V_1(1)$ by Lemma 2.3(b). We have proved $A \leqslant B_1(1)\, V_1(1) = B(1)$. $|A| \geqslant |B(1)|$ gives equality. This completes the proof of Theorem 2.1.

NOTE 2.4. Two techniques in this proof will be used again and again. Firstly J will always be chosen so that $[U_J, U_J] = X(r_0) \leqslant Z(U_J)$ where r_0 is the highest root and $Z(U_J)$ is the direct product of root subgroups. In this way by putting a non-degenerate alternating form on $U_J/Z(U_J)$ we will always be able to get an exact bound on the elements of $\mathrm{Ab}(U_J)$. Secondly, whenever we have a situtaion where $A \in \mathrm{Ab}(U)$, we have identified A_1 and $A \cap U_J$, and we can compute that $A \cap U_J \in \mathrm{Ab}(U_J)$, $[A_1, A \cap U_J] = 1$ and that the centralizer of A_1 in U_J' is $(A \cap U_J)'$ where $'$ maps U_J naturally onto $U_J/X(r_0)$, then we shall be able to prove that $A = A_1 \times (A \cap U_J)$ exactly as we did towards the end of the above proof.

THEOREM 2.5. *Let $G = C_n(q)$ where $n \geqslant 2$ and q is odd. Let $B = \langle X(r) | r \geqslant r_n \rangle$. Then* $\mathrm{Ab}(U) = \{B\}$ *and* $|B| = q^{n(n+1)/2}$.

PROOF. The proof is by induction on n. If $n = 2$,

$$B = \langle X(r_2),\ X(r_1 + r_2),\ X(2r_1 + r_2) \rangle$$

and $|B| = q^3$. In Wong (1969) it is established that in $PSp(4, q)$, q odd, and hence in $C_2(q)$, U contains a unique largest abelian subgroup of order q^3. Hence B is this subgroup and the result is true for $n = 2$.

We now assume the result for any integer r where $2 \leqslant r < n$. Let P_J be the parabolic subgroup of $G = C_n(q)$ associated with $J = \Pi - \{r_1\}$. Then $S_J = \langle X(r) | r \in \Phi_J \rangle \cong C_{n-1}(q)$, and $U_J = \langle X(r) | r \geqslant r_1 \rangle$. Further, $[U_J, U_J] = X(r_0) = Z(U_J)$ where $r_0 = 2r_1 + 2r_2 + \ldots + 2r_{n-1} + r_n$ is the highest root. Note that if q is even $[U_J, U_J] = 1$. From now on let $V = U_J$.

Let $U_1 = \langle X(r) | r \in \Phi_J^+ \rangle$. Then $U_1 \in \mathrm{Syl}_p(S_J)$ and by induction $\mathrm{Ab}(U_1) = \{B_1\}$ where $B_1 = \langle X(r) | r \in \Phi_J^+,\ r \geqslant r_n \rangle$. Finally let $V_1 = \langle X(r)\ r \geqslant r_1 + r_2 + \ldots + r_n \rangle$. Then using the techniques of Theorem 2.1 one forces, for any $A \in \mathrm{Ab}(U)$, that $A_1 = B_1$ and $A \cap V = V_1$ and finally that $A = B_1 V_1 = B$. Hence Theorem 2.5.

What about $C_n(q)$, q even? This will be attended to in Section 4.

3. The solution for $D_n(q)$

THEOREM 3.1. *Let $G = D_4(q)$. Then $A \in \mathrm{Ab}(U)$ implies $|A| = q^6$. Further,*

(a) *if q is odd, $\mathrm{Ab}(U) = \{B_1, B_2, B_3\}$ where $B_1 = \langle X(r) | r \geq r_1 \rangle$, $B_2 = \langle X(r) | r \geq r_3 \rangle$ and $B_3 = \langle X(r) | r \geq r_4 \rangle$;*

(b) *if q is even and $q \geq 4$, $\mathrm{Ab}(U) = \{B(1,a), B(2,b), B(3,c) | a, b, c \in F\}$ where*

$$B(1,a) = \langle x(r_1, t_1)\, x(r_3, at_1),\ x(r_1 + r_2, {}_2)\, x(r_2 + r_3, at_2),\ x(r_1 + r_2 + r_4, t_3)$$

$$x(r_2 + r_3 + r_4, at_3),\ X(r) | r \geq r_1 + r_2 + r_3,\ t_i \in F \rangle$$

and $B(2,c)$ and $B(3,c)$ are defined in a similar fashion;

(c) *if $q = 2$, $\mathrm{Ab}(U)$ is as in (b) with one additional element, namely A^* which is defined in Lemma 3.10.*

THEOREM 3.2. *Let $G = D_n(q)$ where $n \geq 5$. Then $A \in \mathrm{Ab}(U)$ implies $|A| = q^{n(n-1)/2}$. Further,*

(a) *if q is odd, $\mathrm{Ab}(U) = \{B_1, B_2\}$ where $B_1 = \langle X(r) | r \geq r_{n-1} \rangle$ and $B_2 = \langle X(r) | r \geq r_n \rangle$*

(b) *if q is even, $\mathrm{Ab}(U) = \{B_2, B(1,a) | a \in F\}$ where B_2 is as in (a) and*

$$B(1,a) = \langle x(r,t)\, x(\bar{r}, at), X(s) | r \geq r_{n-1}, r \not\geq r_{n-2} + r_{n-1} + r_n, s \geq r_{n-2} + r_{n-1} + r_n, t \in F \rangle,$$

where $\bar{r}_1 = r_1$, $\bar{r}_2 = r_2, \ldots, \bar{r}_{n-2} = r_{n-2}$, $\bar{r}_{n-1} = r_n, \bar{r}_n = r_{n-1}$ and $-$ is extended by linearity to Φ.

Before we attempt the proofs of Theorems 3.1 and 3.2 we establish some notation for this section and we prove some general results. Let P_J be the parabolic subgroup of G associated with $J = \Pi - \{r_2\}$. Then, if necessary, by Barry (1977), Theorem 3.2 we have $S_J = M \times N$ where $M = \langle X(r_1), X(-r_1) \rangle \cong A_1(q)$ and $N = \langle X(r) | r \in \Phi_K \rangle \cong D_{n-2}(q)$ where $K = \Pi - \{r_1, r_2\}$ Also $U_J = \langle X(r) | r \geq r_2 \rangle$ and $[U_J, U_J] = X(r_0) = Z(U_J)$ where $r_0 = r_1 + 2r_2 + \ldots + 2r_{n-2} + r_{n-1} + r_n$ is the highest root. From now on let $V = U_J$. $M \times N$ acts on V by right conjugation and by Barry (1977), Theorem 3.2 we have

$$V/[V, V] \cong V(A, 1, \lambda_1) \otimes W \quad \text{as an } M \times N\text{-module}$$

and

$$V/[V, V] \cong W \otimes W \quad \text{as an } N\text{-module},$$

where $W = V(D, n-2, \lambda_1)$ can be identified with $\langle X(r) | r \geq r_2, r \not\geq r_1 \rangle$.

Let $X = X(r_1)$ and $Y = \langle X(r) | r \in \Phi_K^+ \rangle$. Then $X \times Y \in \mathrm{Syl}_p(M \times N)$ and $U = XYV$. Let φ be the natural epimorphism of U on U/V and $'$ that of V on $V/[V, V]$.

LEMMA 3.3. *Let $x \in X$ and $y \in Y$. Let $\{e, f\}$ be a basis of $V(A, 1, \lambda_1)$ such that $ex = e$ and $fx = f + ae$. Then*

(1) *if $x \neq 1, a \neq 0$; further, $e \otimes u + f \otimes v \in C_{V'}(xy)$ if and only if $u \in \ker_W (y-1)^2$ and $v = -u(y-1)/a$ where $u, v \in W$ and so $\dim C_{V'}(xy) = \dim \ker_W (y-1)^2$.*

(2) *$e \otimes u + f \otimes v \in C_{V'}(y)$ if and only if $u, v \in C_W(y)$.*

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PROOF.

(1) $V(A,1,\lambda_1)$ is the natural representation for $A(q)\cong \mathrm{SL}(2,q)$ and hence is faithful. Therefore if $x\neq 1, a\neq 0$.

Let $e\otimes u+f\otimes v\in C_{V'}(xy)$ where $x\neq 1$. Then

$$\begin{aligned} e\otimes u+f\otimes v &= (e\otimes u+f\otimes v)\,xy. \\ &= e\otimes uy+(f+ae)\otimes vy \\ &= e\otimes (uy+avy)+f\otimes vy. \end{aligned}$$

Uniqueness of expression gives $vy=v$ and $u=uy+avy$. Therefore $v(y-1)=0$ and $v=-u(y-1)/a$. So $0=v(y-1)=-u(y-1)^2/a$ gives us that $u\in\ker_W(y-1)^2$. The result in the other direction is trivial. Since u determines v and $u\in\ker_W(y-1)^2$ we have $\dim C_{V'}(xy)=\dim\ker_W(y-1)^2$.

(2) Trivial.

We record

LEMMA 3.4. *If* $C\in\mathrm{Ab}(V)$ *then* $|C|=q^{2n-3}$.

REMARK 3.5. Note that $V=V_1V_2$ where $V_1=\langle X(r)\,|\,r\geqslant r_1+r_2\rangle\Delta U$ and $V_2=\langle X(r)\,|\,r\geqslant r_2,\ r\ngeqslant r_1\rangle$. Let $x\in X$, $y\in Y$ and $v_i\in V_i$, $i=1,2$. Then

$$\begin{aligned} [xy, v_1v_2] &= [x,v_2]^y\,[x,v_1]^{v_2y}\,[y,v_2]\,[y,v_1]^{v_2} \\ &= [x,v_2]^y\,[y,v_2]\,[y,v_1]^{v_2} \end{aligned}$$

since $[x,v_1]=1$. If $[xy,v_1v_2]\in X(r_0)$ then $[y,v_2]=1$ and hence

$$[xy,v_1v_2]=[x,v_2]\,[y,v_1]^{v_2}$$

since $x^y=x$ and $v_2^y=v_2$.

PROOF OF THEOREM. 3.1. This will follow in a series of lemmas.

LEMMA 3.6. *Let* $x=x(r_1,t_1)$, $y=x(r_3,at_1)$, $t_1\neq 0$. *Then*

$C_{V'}(xy)=$

$\langle x'(r_1+r_2,d_1)\,x'(r_2+r_3,\ -N(r_3,r_1+r_2)N(r_1,r_2+r_3)\,\mathrm{ad}_1),$

$x'(r_1+r_2+r_4,d_2)\,x'(r_2+r_3+r_4,\ -N(r_3,r_1+r_2+r_4)\,N(r_1,r_2+r_3+r_4)\,\mathrm{ad}_2),$

$X'(r_1+r_2+r_3),\ X'(r_1+r_2+r_3+r_4)\,|\,d_i\in F,\ i=1,2\rangle.$

PROOF. Let v_1v_2 be an element of the preimage of $C_{V'}(xy)$ in V where $v_i\in V_i$, $i=1,2$. Then $v_2=x(r_2+r_3,u_1)\,x(r_2+r_3+r_4,u_2)$ by Remark 3.5.

Suppose

$$v_1=x(r_1+r_2,d_1)\,x(r_1+r_2+r_3,d_2)\,x(r_1+r_2+r_4,d_3)\,x(r_1+r_2+r_3+r_4,d_4).$$

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We are ignoring any component of $Z(V)$ in v_1. Then

$$\begin{aligned}[x, v_2] &= [x(r_1, t_1),\ x(r_2+r_3, u_1)\ x(r_2+r_3+r_4, u_2)] \\ &= x(r_1+r_2+r_3+r_4,\ N(r_1, r_2+r_3+r_4)\, t_1 u_2) \\ &\quad \times x(r_1+r_2+r_3,\ N(r_1, r_2+r_3)\, t_1 u_1)\end{aligned}$$

using Chevalley's commutator formula. Also

$$\begin{aligned}[y, v_1] &= x(r_1+r_2+r_3+r_4,\ N(r_3, r_1+r_2+r_4)\, a t_1 d_3) \\ &\quad \times x(r_1+r_2+r_3,\ N(r_3, r_1+r_2)\, a t_1 d_1)\end{aligned}$$

again by Chevalley's formula.

Now $[y, v_1]^{v_2} = [y, v_1]$. Therefore

$$\begin{aligned}[xy, v_1 v_2] &= x(r_1+r_2+r_3,\ N(r_1, r_2+r_3)\, t_1 u_1 + N(r_3, r_1+r_2)\, a t_1 d_1) \\ &\quad \times x(r_1+r_2+r_3+r_4,\ N(r_1, r_2+r_3+r_4)\, t_1 u_2 + N(r_3, r_1+r_2+r_4)\, a t_1 d_3).\end{aligned}$$

Therefore $[xy, v_1 v_2] \in X(r_0)$ implies that

$$u_1 = -N(r_1, r_2+r_3)\, N(r_3, r_1+r_2)\, a d_1$$

and

$$u_2 = -N(r_1, r_2+r_3+r_4)\, N(r_3, r_1+r_2+r_4)\, a d_3$$

since $N(r,s) = \pm 1$ in these cases. This completes the proof of Lemma 3.6.

LEMMA 3.7. *Let* $x = x(r_1, t_1)$, $y = x(r_3, t_3)\, x(r_4, a t_3)$, t_1, t_3 *and* $a \neq 0$. *If* q *is odd then*

$$\begin{aligned}C_{V'}(xy) = \langle & x'(r_1+r_2+r_3, d_1)\, x'(r_1+r_2+r_4, d_2) \\ & x'(r_2+r_3+r_4,\ -(t_3 t_1^{-1})\, N(r_1, r_2+r_3+r_4)\, N(r_3, r_1+r_2+r_4)\, (d_2 + a d_1)), \\ & X'(r_1+r_2+r_3+r_4) \mid d_i \in F,\ i = 1, 2\rangle.\end{aligned}$$

If q *is even then*

$$\begin{aligned}C_{V'}(xy) = \langle & x'(r_1+r_2, d_1)\, x'(r_2+r_3,\ t_3 t_1^{-1} d_1)\, x'(r_2+r_4,\ a(t_3 t_1^{-1}) d_1) \\ & \times x'(r_2+r_3+r_4,\ a t_3^2 t_1^{-1} d_1), \\ & x'(r_1+r_2+r_3, d_2)\, x'(r_1+r_2+r_4, d_3)\, x'(r_2+r_3+r_4,\ (t_3 t_1^{-1})(d_3 + a d_2)), \\ & X'(r_1+r_2+r_3+r_4) \mid d_i \in F,\ i = 1, 2, 3\rangle.\end{aligned}$$

Here it should be noted that we are choosing $N(\bar{r}, \bar{s}) = N(r, s)$ where $\bar{r}_1 = r_1$, $\bar{r}_2 = r_2$, $\bar{r}_3 = r_4$ and $\bar{r}_4 = r_3$. That we can so choose follows from Steinberg (1959), Lemma 3.2.

PROOF. Let $v_1 v_2$ be an element of the preimage of $C_{V'}(xy)$ in V where $v_i \in V_i$, $i = 1, 2$. As before $[y, v_2] = 1$ and so $v_2 = x(r_2+r_3, u_1)\, x(r_2+r_4, -N(r_4, r_2+r_3) \times N(r_3, r_2+r_4)\, a u_1)\, x(r_2+r_3+r_4, u_2)$.

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Since we are choosing $N(\bar{r},\bar{s}) = N(r,s)$ we get

$$N(r_4, r_2+r_3)\,N(r_3, r_2+r_4) = 1$$

and so

$$v_2 = x(r_2+r_3, u_1)\,x(r_2+r_4, -au_1)\,x(r_2+r_3+r_4, u_2).$$

Computing we get

$$[x, v_2] = x(r_1+r_2+r_3+r_4,\ N(r_1, r_2+r_3+r_4)\,t_1\,u_2)\,x(r_1+r_2+r_4,\ N(r_1, r_2+r_4)(-at_1\,u_1))\ x(r_1+r_2+r_3,\ N(r_1, r_2+r_3)\,t_1\,u_1)\,z$$

where $z \in Z(V) = X(r_0)$. In similar fashion

$$[y, v_1] = x(r_1+r_2+r_3+r_4,\ N(r_3, r_1+r_2+r_4)\,t_3\,d_3 + N(r_3, r_1+r_2)\,N(r_1+r_2+r_3, r_4)\,at_3^2\,d_1 + N(r_4, r_1+r_2+r_3)\,at_3\,d_2) \times x(r_1+r_2+r_3,\ N(r_3, r_1+r_2)\,d_1\,t_3)\,x(r_1+r_2+r_4,\ N(r_4, r_1+r_2)\,d_1\,at_3).$$

Now $[y, v_1]^{v_2} = [y, v_1] \bmod Z(V)$. Therefore

$$[xy, v_1\,v_2] \bmod Z(V) = x(r_1+r_2+r_3,\ N(r_1, r_2+r_3)\,t_1\,u_1 + N(r_3, r_1+r_2)\,d_1\,t_3) \times x(r_1+r_2+r_4,\ N(r_1, r_2+r_4)(-at_1\,u_1) + N(r_4, r_1+r_2)\,d_1\,at_3) \times x(r_1+r_2+r_3+r_4,\ N(r_1, r_2+r_3+r_4)\,t_1\,u_2 + N(r_3, r_1+t_2+r_4)\,t_3\,d_3 + N(r_4, r_1+r_2+r_3)\,at_3\,d_2 + N(r_3, r_1+r_2)\,N(r_1+r_2+r_3, r_4)\,at_3^2 d_1).$$

Therefore $[xy, v_1\,v_2] \in Z(V)$ implies that

(a) $u_1 = -N(r_1, r_2+r_3)\,N(r_3, r_1+r_2)\,t_3\,t_1^{-1}\,d_1$,

(b) $u_1 = N(r_1, r_2+r_4)\,N(r_4, r_1+r_2)\,t_3\,t_1^{-1}\,d_1$ and

(c) $u_2 = -N(r_1, r_2+r_3+r_4)\,t_1^{-1}\,(N(r_3, r_1+r_2+r_4)\,t_3\,d_3 + N(r_4, r_1+r_2+r_3)\,at_3\,d_2 + N(r_3, r_1+r_2)\,N(r_1+r_2+r_3, r_4)\,at_3^2\,d_1)$.

If q is odd then (a) and (b) force $d_1 = u_1 = 0$ since $N(r_1, r_2+r_3) = N(r_1, r_2+r_4)$ and $N(r_3, r_1+r_2) = N(r_4, r_1+r_2)$. In this case

$$u_2 = -N(r_1, r_2+r_3+r_4)\,N(r_3, r_1+r_2+r_4)\,t_3\,t_1^{-1}(d_3 + ad_2).$$

If q is even, since $N(r,s) = 1$ in all cases, (a) and (b) are the same equation, namely $u_1 = t_3\,t_1^{-1}\,d_1$, and (c) becomes $u_2 = t_1^{-1}(t_3\,d_3 + at_3\,d_2 + at_3^2 d_1)$. This completes the proof of Lemma 3.7.

Let $A \in \mathrm{Ab}(U)$. Then $|A| \geqslant q^6$ since U contains abelian groups of this order. Also $(A)\varphi = A_1 \leqslant X \times Y$ and $|A| = |A_1|\,|A \cap V|$. Note that $A \cap V \geqslant Z(V) = X(r_0)$ by the maximality of A since $X(r_0) \leqslant Z(U)$. Suppose $a_1 \in A_1$ and $v \in A \cap V$. Then $a_1\,v_1 \in A$ for some $v_1 \in V$. Then

$$1 = [a_1\,v_1, v] = [a_1, v]^{v_1}\,[v_1, v].$$

Therefore $[a_1, v] \in Z(V) = [V, V]$ and so $[A_1, A \cap V] \leqslant Z(V)$. It follows that

$$(A \cap V)' \leqslant C_{V'}(A_1).$$

Now $C_{V'}(A_1)$ is a vector space over F and so $|C_{V'}(A_1)| = q^f$ for some integer $f \geqslant 0$.

LEMMA 3.8. *Let $x \in X$ and $y, y_1 \in Y$ with $x, y_1 \neq 1$. Then xy and y_1 cannot both be elements of A_1.*

PROOF. As observed

$$V/[V, V] = V' \cong V(A, 1, \lambda) \otimes W \quad \text{as an } M \times N\text{-module}$$
$$\cong W \otimes W \quad \text{as an } N\text{-module}.$$

Choose e, f as in Lemma 3.3. Now $e \otimes u + f \otimes v \in C_{V'}(xy)$ if and only if $u \in \ker_W (y-1)^2$ and $v = -u(y-1)/a$, and $e \otimes u + f \otimes v \in C_{V'}(y)$ if and only if $u, v \in C_W(y)$. Therefore $e \otimes u + f \otimes v \in C_{V'}(xy) \cap C_{V'}(y)$ implies that $u \in \ker(y-1)^2 \cap C_W(y)$ and $v = -u(y-1)/a$. Therefore xy and $y \in A_1$ implies $\dim C_{V'}(A_1) \leqslant \dim C_W(y)$. Identifying W with $\langle X(r) | r \geqslant r_2, r \ngeqslant r_1 \rangle$ we see easily that $y \neq 1$ gives $\dim C_W(y) = 2$. Therefore dim $C_{V'}(A_1) \leqslant 2$. Since $(A \cap V)' \leqslant C_{V'}(A_1)$ it follows that $|A \cap V| \leqslant q^3$. Now $|A| \geqslant q^6$ and so $|A_1| \geqslant q^3$. Recall that $Y = X(r_3) \times X(r_4)$ and so $|Y| = q^2$. Therefore $A = X \times Y$.

Now the preimage in V of $C_{V'}(X \times Y)$ is $\langle X(r_1 + r_2 + r_3 + r_4), X(r_0) \rangle$. Therefore $|A \cap V| \leqslant q^2$ and so $|A| = |A_1|\,|A \cap V| \leqslant q^5$ which is a contradiction. With this contradiction the proof of Lemma 3.8 is completed.

LEMMA 3.9. *If $q \geqslant 3$, $b = x(r_1, t_1) x(r_3, t_2) x(r_4, t_3)$, $t_1, t_2, t_3 \neq 0$ cannot be an element of A_1.*

PROOF. Suppose $b \in A_1$.

Observation (a). If $x(r_1, d_1) z(r_3, d_2) x(r_4, d_3) \in A_1$ where at least one of the d_i is zero then all of the d_i are zero or we get a contradiction by the symmetry of r_1, r_3 and r_4 and Lemma 3.8.

Observation (b). If $x(r_1, d_1) x(r_3, d_2) x(r_4, d_3) \in A_1$ such that $d_1 = t_1$ then either $d_2 = t_2$ and $d_3 = t_3$ or we get a contradiction again by Lemma 3.8. By the symmetry of r_1, r_3 and r_4 we could replace the condition $d_1 = t_1$ by $d_2 = t_2$ or $d_3 = t_3$. And so our conclusion now reads: if $d_i = t_i$ for any i then $d_i = t_i$ for all i, $1 \leqslant i \leqslant 3$.

Observations (a) and (b) force $|A_1| \leqslant q$ and force elements of $A_1 - \{1\}$ to be of the form $x(r_1, d) x(r_3, t) x(r_4, u)$ where $d, t, u \neq 0$. If $|A_1| < q$, then since by Lemma 3.4 $|A \cap V| \leqslant q^5$ we get $|A| < q^6$ which is a contradiction. Hence $|A_1| = q$.

If q is odd we are done since by Lemma 3.7 we get that $|C_{V'}(b)| = q^3$ which gives $|A \cap V| \leqslant q^4$ and so $|A| \leqslant q^5$—a contradiction. Suppose q is even and $q \geqslant 4$. Since $q \geqslant 3$, we can choose two distinct non-identity elements of A_1, x_1 and x_2 where $x_1 = x(r_1, c_1) x(r_3, c_2) x(r_4, ac_3)$ and $x_2 = x(r_1, b_1) x(r_3, b_3) x(r_4, a_1 b_3)$. If $C_{V'}(x_1) \neq$

$C_{V'}(x_2)$ then $|C_{V'}(A_1)| \leqslant q^3$ by Lemma 3.7 and so $|A \cap V| \leqslant q^4$ which gives the contradiction $|A| \leqslant q^5$. Hence $C_{V'}(x_1) = C_{V'}(x_2)$.

Using the notation of Lemma 3.7 this implies (i) $u_1 = (c_3 c_1^{-1}) d_1 = (b_3 b_1^{-1}) d_1$ for all $d_1 \in F$ and (ii) $u_2 = (c_3 c_1^{-1})(d_3 + ad_1 c_3 + ad_2) = (b_3 b_1^{-1})(d_3 + a_1 d_1 b_3 + a_1 d_2)$ for all $d_1, d_2, d_3 \in F$. (i) implies $c_3 c_1^{-1} = b_3 b_1^{-1}$ which in conjunction with (ii) implies $ad_1 c_3 + ad_2 = a_1 d_1 b_3 + a_1 d_2$ for all $d_1, d_2 \in F$. This is a contradiction which we see as follows. If $a = a_1$ take $d_1 = 1$ and $d_2 = 0$ to get $c_3 = b_3$ (recall $x_1 \neq x_2$ implies $c_3 \neq b_3$); if $a \neq a_1$ take $d_1 = 0$ and $d_2 = 1$ to get $a = a_1$. This completes the proof of Lemma 3.9.

LEMMA 3.10. *If* $|F| = 2$ *and* $x = x(r_1, 1)x(r_3, 1)x(r_4, 1) \in A_1$ *then the only possibility for* A *is*

$$A^* = \{xv, V_1 | v \in x(r_2 + r_3 + r_4, 1) V_1\}$$

where

$$\begin{aligned} V_1 = \langle & x(r_1 + r_2, 1)x(r_2 + r_3, 1)x(r_2 + r_4, 1)x(r_2 + r_3 + r_4, 1), \\ & x(r_1 + r_2 + r_3, d_1)x(r_1 + r_2 + r_4, d_2)x(r_2 + r_3 + r_4, d_1 + d_2), \\ & X(r_1 + r_2 + r_3 + r_4), X(r_0) | d_1, d_2 \in F\rangle. \end{aligned}$$

PROOF. As in Lemma 3.9 $|A_1| = q = 2$ in this case and so $A_1 = \{1, x\}$. Let

$$y_1 = x(r_1 + r_2, 1)x(r_2 + r_3, 1)x(r_2 + r_4, 1)x(r_2 + r_3 + r_4, 1)$$

and

$$y_2(d_1, d_2) = x(r_1 + r_2 + r_3, d_1)x(r_1 + r_2 + r_4, d_2)x(r_2 + r_3 + r_4, d_1 + d_2).$$

Using Lemma 3.7 one gets that the preimage V_1 of $C_{V'}(A_1)$ in V equals

$$\langle y_1, y_2(d_1, d_2), X(r_1 + r_2 + r_3 + r_4), X(r_0) | d_1, d_2 \in F\rangle.$$

One checks that V_1 is abelian.

Now $|A_1| = 2$ and $|A| \geqslant 2^6$ forces $|A \cap V| \geqslant 2^5$. But $A \cap V \leqslant V_1$ and so $A \cap V = V_1$. Now $[x, y_1] = x(r_4, 1)$, $[x, y_2(d_1, d_2)] = 1$ and $[x, W] = 1$, where

$$W = X(r_1 + r_2 + r_3 + r_4) \times X(r_0).$$

Let $a \in A - (A \cap V)$. Then $a = xv_1$ for some $v_1 \in V$. Now $[a, A \cap V] = 1$ forces $[v_1, y_1] = x(r_0, 1)$, $[v_1, y_2(d_1, d_2)] = 1$ and $[v_1, W] = 1$. Hence $v_1 \in C(\langle y_2(d_1, d_2), W | d_1, d_2 \in F\rangle)$ which turns out to be $\langle V_1, x(r_2 + r_3 + r_4, 1)\rangle$. Now $[v_1, y_1] = x(r_0, 1)$ gives

$$v_1 \in x(r_2 + r_3 + r_4, 1) V_1.$$

So we have that A is a subset of A^*. One computes that A^* is an abelian group of order 2^6 and so $A = A^*$. This completes the proof of Lemma 3.10.

LEMMA 3.11. *If* $x(r_1, t_1)x(r_3, at_1) \in A_1$, $t_1 \neq 0$ *then* $A_1 = \langle x(r_1, t)x(r_3, at) | t \in F\rangle$.

PROOF. (a) Suppose $a = 0$. Then by Lemma 3.8 and the symmetry of r_1, r_3 and r_4 we get that A_1 contains no non-identity elements of the form $x(r_3, b_1)x(r_4, b_2)$ or

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$x(r_1, c_1)\, x(r_4, c_2)$, $c_2 \neq 0$, or $x(r_1, d_1)\, x(r_3, d_2)\, x(r_4, d_3)$, $d_i \neq 0$, all i. (b) Suppose $a \neq 0$. Then A_1 contains no non-identity elements of the form

$$x(r_3, b_1)\, x(r_4, b_2) \quad \text{or} \quad x(r_1, c_1)\, x(r_4, c_2) \quad \text{or} \quad x(r_1, d_1)\, x(r_3, d_2)\, x(r_4, d_3),$$

again by Lemma 3.8 and the symmetry of r_1, r_3 and r_4. (a) and (b) force $A_1 \leqslant X(r_1) \times X(r_3)$ and so $|A_1| \leqslant q^2$. Note we know $|A_1| \geqslant q$ since $|A \cap V| \leqslant q^5$ and $|A| \geqslant q^6$.

Suppose $x_1 = x(r_1, t_1)\, x(r_3, at_1)$ and $x_2 = x(r_1, d_1)\, x(r_3, a_1 d_1)$ be non-identity elements of A_1 with $a \neq a_1$. Then Lemma 3.6 gives us that

$$\dim C_{V'}(A_1) \leqslant \dim C_{V'}(\langle x_1, x_2 \rangle) \leqslant 2.$$

Therefore $|A \cap V| \leqslant q^3$ and so $|A| \leqslant q^5$—a contradiction. Hence a must equal a_1 and this combined with $|A_1| \geqslant q$ forces the desired conclusion. This completes the proof of Lemma 3.11.

One notes that by the symmetry of r_1, r_3 and r_4 we might just as well consider elements of the form $x(r_3, t_1)\, x(r_4, at_1)$ and $x(r_4, t_1)\, x(r_1, at_1)$, $t_1 \neq 0$, in Lemma 3.11 and we would get the appropriate conclusion.

LEMMA 3.12. *$A \in \mathrm{Ab}(U)$ implies $|A| = q^6$.*

PROOF. Lemmas 3.10 and 3.11 tell us that the possibilities for A_1 are

$$\langle x(r_1, t)\, x(r_3, a_1 t) \mid t \in F \rangle, \quad \langle x(r_3, t)\, x(r_4, a_2 t) \mid t \in F \rangle, \quad \langle x(r_4, t)\, x(r_1, a_3 t) \mid t \in F \rangle,$$

where a_1, a_2 and a_3 range over F, and $\langle x(r_1, t)\, x(r_3, t)\, x(r_4, t) \mid t \in F \rangle$, this last one being possible only when $|F| = 2$. Now each of these possibilities has order q. Since $|A \cap V| \leqslant q^5$ it follows that $|A| \leqslant q^6$. Since we know $|A| \geqslant q^6$ we get that $|A| = q^6$. Hence Lemma 3.12.

LEMMA 3.13. *If q is odd the possibilities*

$$A_1 = \langle x(r_1, t)\, x(r_3, at) \mid t \in F \rangle \quad \textit{or} \quad \langle x(r_3, t)\, x(r_4, at) \mid t \in F \rangle \quad \textit{or}$$
$$\langle x(r_4, t)\, x(r_1, at) \mid t \in F \rangle$$

do not occur when $a \neq 0$.

PROOF. Suppose $A_1 = \langle x(r_1, t)\, x(r_3, at) \mid t \in F \rangle$ for some $a \neq 0$. Lemma 3.6 tells us that

$$\begin{aligned} C_{V'}(A_1) = \; & \langle x'(r_1 + r_2, d_1)\, x'(r_2 + r_3, \; -N(r_3, r_1 + r_2)\, N(r_1, r_2 + r_3)\, ad_1), \\ & x'(r_1 + r_2 + r_4, d_2)\, x'(r_2 + r_3 + r_4, \; -N(r_3, r_1 + r_2 + r_4)\, N(r_1, r_2 + r_3 + r_4)\, ad_2), \\ & X'(r_1 + r_2 + r_3),\, X'(r_1 + r_2 + r_3 + r_4) \mid d_i \in F \rangle. \end{aligned}$$

To simplify calculations choose $N(\bar{r},\bar{s}) = N(r,s)$ where now $\bar{r}_1 = r_3$, $\bar{r}_2 = r_2$, $\bar{r}_3 = r_1$ and $r_4 = \bar{r}_4$. Steinberg (1959), Lemma 3.2 again justifies the legitimacy of this procedure. Then

$$C_{V'}(A_1) = \langle x'(r_1+r_2,d_1)\,x'(r_2+r_3,-ad_1), x'(r_1+r_2+r_4,d_2)\,x'(r_2+r_3+r_4,-ad_2), X'(r_1+r_2+r_3), X'(r_1+r_2+r_3+r_4)\,|\,d_i \in F\rangle.$$

Now

$$\begin{aligned}&[x(r_1+r_2,d_1)\,x(r_2+r_3,-ad_1),\; x(r_1+r_2+r_4,d_2)\,x(r_2+r_3+r_4,-ad_2)]\\ &= [x(r_1+r_2,d_1),\; x(r_2+r_3+r_4,-ad_2)]\,[x(r_2+{}_3,-ad_1),\; x(r_1+r_2+r_4,d_2)]\\ &= x(r_0,\; [N(r_1+r_2,r_2+r_3+r_4)+N(r_2+r_3,r_1+r_2+r_4)](-ad_1\,d_2)).\end{aligned}$$

Since

$$N(r_1+r_2,r_2+r_3+r_4) = N(\overline{r_1+r_2},\overline{r_2+r_3+r_4}) = N(r_2+r_3,r_1+r_2+r_4)$$

these elements commute if and only if q is even.

Therefore, if q is odd, then $A \cap V$ is properly contained in the preimage of $C_{V'}(A_1)$ in V. This means that $|A \cap V| < q^5$ and as a consequence $|A| < q^6$—a contradiction. Thus $A_1 = \langle x(r_1,t)\,x(r_3,at)\,|\,t \in F\rangle$ is not a possibility if q is odd and $a \neq 0$. The two other configurations for A_1 are disposed of in like manner. This completes the proof of Lemma 3.13.

We are ready now to wind up the proof of Theorem 3.1. If $q = 2$ and $A_1 = \langle x(r_1,t)\,x(r_3,t)\,x(r_4,t)\,|\,t \in F\rangle$ then by Lemma 3.10 $A = A^*$. The other possibilities when q is even are $A_1 = \langle x(r_1,t) \times x(r_3,at)\,|\,t \in F\rangle$ for some $a \in F$ and the $r_3 - r_4$ and $r_4 - r_1$ mixtures. Suppose $A_1 = \langle x(r_1,t)\,x(r_3,at)\,|\,t \in F\rangle$. One checks that the preimage V_1 of $C_{V'}(A_1)$ in V is abelian of order q^5. Now $A \cap V \leqslant V_1$ and in fact $A \cap V = V_1$ by order considerations. One has that $A \cap V \in \mathrm{Ab}(V)$, $[A_1, A \cap V] = 1$ and so we prove that $A = A_1 \times (A \cap V)$ which turns out to be $B(1,a)$. The other remaining possibilities for A_1 are handled in similar fashion.

If q is odd A_1 is one of $X(r_1), X(r_3)$ or $X(r_4)$. If $A_1 = X(r_1)$ then $A = B_1$, if $A_1 = X(r_3)$ then $A = B_2$ finally if $A = X(r_4)$ then $A = B_3$. This completes the proof of Theorem 3.1.

PROOF OF THEOREM 3.2. We start with

LEMMA 3.14. *Let* $y \in Y$, $y \neq 1$. *Then* $\dim C_W(y) \leqslant 2n-6$.

PROOF. Recall $W = V(D, n-2, \lambda_1)$ as an N-module. Then N is represented on W as a subgroup of the orthogonal group $O^+(2n-4,q)$ using the notation of Carter (1972), p. 6, Y being represented faithfully. Since $N = [N,N]$, N maps into (actually onto) $\Omega(2n-4,q) = [O^+(2n-4,q),\; O^+(2n-4,q)]$. However, $\Omega(2n-4,q)$ contains no transvections and hence our lemma follows immediately.

The proof is by induction on n. Let $A \in \mathrm{Ab}(U)$. First we consider the case $n = 5$.

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Then $|A| \geqslant q^{10}$ since abelian subgroups of U of this order exist. As usual $(A)\varphi = A_1 \leqslant X \times Y$ and $|A| = |A_1|\,|A \cap V|$. Now $N \cong A_3(q)$ and so by Theorem 2.1(a) Y has a unique largest abelian subgroup $\langle X(r) | r \geqslant r_3,\ r \not\geqslant r_2 \rangle$ of order q^4. Hence $|A_1| \leqslant q^5$. Now by Lemma 3.4 we have $|A \cap V| \leqslant q^7$. Hence $|A| \leqslant q^{12}$.

LEMMA 3.15. *Let $x \in X$, $y \in Y$. Then no element of the form xy occurs in A_1 where $x \neq 1$. Hence $A_1 \leqslant Y$.*

PROOF. Suppose there exists $xy \in A_1$ with $x \neq 1$. Note that $|A_1| \geqslant q^3$ since $|A \cap V| \leqslant q^7$. We claim that there exists $y_1 \in Y \cap A$ with $y_1 \neq 1$. Suppose not. Then all elements of A_1 are of the form $x_2 y_2$ where $x_2 = 1$ implies $y_2 = 1$. If there exists $x_1 \in X$ such that $x_1 y_3$ and $x_1 y_4 \in A_1$ with $y_3 \neq y_4$ then $y_1 = y_3 y_4^{-1} \in A_1$ which is a contradiction. If there exists no such x_1 then $|A_1| \leqslant q$ which is another contradiction. Our claim holds.

So xy and $y_1 \in A_1$. Now $C_{V'}(A_1) \leqslant C_{V'}(xy) \cap C_{V'}(y_1)$. Choose $\{e, f\}$ as in Lemma 3.3. Then $e \otimes u + f \otimes v \in C_{V'}(xy) \cap C_{V'}(y_1)$ implies $u \in \ker_W (y-1)^2 \cap C_W(y_1)$ and $v = -u(y-1)/a$. Hence $\dim C_{V'}(A_1) \leqslant \dim C_W(y_1) \leqslant 4$ by Lemma 3.14. Therefore $|A \cap V| \leqslant q^5$. This forces $|A_1| \geqslant q^5$ which in turn forces A_1 to be the unique largest abelian subgroup of $X \times Y$ of order q^5, namely $X \times \langle X(r) | r \geqslant r_3, r \neq r_2 \rangle$. Easy calculation gives $C_{V'}(A_1) = X'(r_1 + r_2 + r_3 + r_4)$. Hence $|A \cap V| \leqslant q^2$ and $|A| \leqslant q^7$—a contradiction. This completes the proof of Lemma 3.15.

LEMMA 3.16. $|A_1| = q_3$. *If q is even then*

$$A_1 = \langle x(r_4, t)\, x(r_5, at),\ x(r_3 + r_4, d)\, x(r_3 + r_5, ad),\ X(r_3 + r_4 + r_5) | t, d \in F \rangle$$

for some fixed $a \in F$ or $A = \langle X(r_5), X(r_3 + r_5), X(r_3 + r_4 + r_5) \rangle$. If q is odd then A_1 is one of the two choices which remain when $a = 0$.

PROOF. $A_1 \leqslant Y$ by Theorem 3.15. This implies $|A_1| \leqslant q^4$ since Y has a unique largest abelian subgroup of order q^4.

Suppose $\dim C_W(A_1) \leqslant 2$. Then $\dim C_{V'}(A_1) = 2 \dim C_W(A_1) \leqslant 4$ which gives $|A \cap V| \leqslant q^6$ and $|A| \leqslant q^9$—a contradiction. Hence $\dim C_W(A) \geqslant 3$. Identifying W with $\langle X(r) | r \geqslant r_2,\ r \not\geqslant r_1 \rangle$ we get that $A_1 \times C_W(A_1)$ is an abelian subgroup contained in $P = \langle X(r) | r \in \Phi_L^+ \rangle$ where $L = \Pi - \{r_1\}$. Now $P \in \mathrm{Syl}_p(G_1)$ where $G_1 = \langle X(r) | r \in \Phi_L \rangle \cong D_4(q)$. Hence Theorem 3.1 implies $|A_1 \times C_W(A_1)| \leqslant q^6$.

Therefore $|A_1| \leqslant q^3$. But we know already that $|A_1| \geqslant q^3$. Hence $|A_1| = q^3$ and $|A_1 \times C_W(A_1)| = q^6$, that is, $A_1 \times C_W(A_1) \in \mathrm{Ab}(P)$. Checking through the elements of $\mathrm{Ab}(P)$ we find that A_1 is as desired. This concludes the proof of Lemma 3.16.

For all A_1 in Lemma 3.16 we find that the preimage D_1 of $C_{V'}(A_1)$ in V is an element of $\mathrm{Ab}(V)$ with $[A_1, D_1] = 1$. Order considerations force $A \cap V = D_1$ and we prove that $A = A_1 D_1$ in our usual fashion. This settles Theorem 3.2 for $n = 5$. Assume now that Theorem 3.2 is true for any integer k such that $5 \leqslant k < n$ and we deal with $U \in \mathrm{Syl}_p D_n(q)$. Let $A \in \mathrm{Ab}(U)$. Then $|A| \geqslant q^{n(n-1)/2}$ since abelian groups of this order exist in U. As usual $(A)\varphi = A_1 \leqslant X \times Y$ and $|A| = |A_1|\,|A \cap V|$. By Lemma 3.4 $|A \cap V| \leqslant q^{2n-3}$. If $L \in \mathrm{Ab}(Y)$ then $|L| = q^{(n-2)(n-3)/2}$ by our induction hypothesis and Theorem 3.1. Therefore $|A_1| \leqslant q^{(n-2)(n-3)/2+1}$ and $|A| \leqslant q^{(n-2)(n-3)/2+2n-2} = q^{n(n-1)/2+1}$.

LEMMA 3.17. *Let $x \in X, y \in Y$. Then no element of the form xy occurs in A_1 where $x \neq 1$. Hence $A_1 \leqslant Y$.*

PROOF. Suppose $xy \in A_1$ where $x \neq 1$. Exactly the same argument as was used in Lemma 3.15 forces $|A \cap V| \leqslant q^{2n-5}$. Hence $|A| \leqslant q^{(n-2)(n-3)/2+2n-4} = q^{n(n-1)/2-1}$—a contradiction. Hence Lemma 3.17.

LEMMA 3. 18. *$A_1 \in \mathrm{Ab}(Y)$. If q is odd then A_1 is one of C_1, C_2 where $C_1 = \langle X(r) | r \geqslant r_{n-1}, r \ngeqslant r_2 \rangle$ and $C_2 = \langle X(r) | r \geqslant r_n, r \neq r_2 \rangle$. If q is even then $A_1 = C_2$ or $A = C(1, a)$ for some $a \in F$ where*

$$C(1,a) = \langle x(r,t)\,x(\bar{r}, at),\ X(s) | r \geqslant r_{n-1},\ r \ngeqslant r_{n-2}+r_{n-1}+r_n,\ s \geqslant r_{n-2}+r_{n-1}+r_n,\ r, s \ngeqslant r_2,\ t \in F \rangle.$$

PROOF. Lemma 3.17 implies $A_1 \leqslant Y$ and so induction with Theorem 3.1 forces $|A_1| \leqslant q^{(n-2)(n-3)/2}$. Order considerations force equality and so $A_1 \in \mathrm{Ab}(Y)$. Induction again with Theorem 3.1 gives us $\mathrm{Ab}(Y)$. The list is not exclusive enough for our purposes and we eliminate the undesirable options by checking that in each case $\dim C_W(A_1) < n-2$. This implies that $|C_{V'}(A_1)| < q^{2n-4}$. Since $C_{V'}(A_1)$ is an F-vector space $|C_{V'}(A_1)| \leqslant q^{2n-5}$. Hence $|A \cap V| \leqslant q^{2n-4}$ and $|A| \leqslant q^{(n-2)(n-3)/2+2n-4} = q^{n(n-1)/2-1}$—a contradiction. This completes the proof of Lemma 3.18.

For all A_1 in the conclusion of Lemma 3.18 the preimage D_1 of $C_{V'}(A_1)$ in V is an element of $\mathrm{Ab}(V)$ with $[A_1, D_1] = 1$. Order considerations force $A \cap V = D_1$ and we prove that $A = A_1 D_1$ in our usual fashion. For $A_1 = C_2$ then $A = B_2$ and for $A_1 = C(1, a)$ then $A = B(1, a)$. This completes the proof of Theorem 3.2.

4. The solution for $B_n(q)$, q even

With this section the lacuna in Section 2 can be filled thanks to the fact that $B_n(q) \cong C_n(q)$ if q is even. Let $q = 2^m$, $m \geqslant 1$.

THEOREM 4.1. *Let $G = B_2(q)$. Let*

$$A_1 = \langle X(r_1),\ X(r_1+r_2),\ X(r_1+2r_2) \rangle,$$
$$A_2 = \langle X(r_2),\ X(r_1+r_2),\ X(r_1+2r_2) \rangle,$$
$$A_3 = \langle x(r_1,t)\,x(r_2,t),\ X(r_1+r_2),\ X(r_1+2r_2) \rangle.$$

Then
(a) *if* $q=2$, $\mathrm{Ab}(U)=\{A_1, A_2, A_3\}$,
(b) *if* $q>2$, $\mathrm{Ab}(U)=\{A_1, A_2\}$.

THEOREM. 4.2. *Let* $G=B_n(q)$ *with* $n\geqslant 3$ *and let* $B=\langle X(r)|r\geqslant r_n\rangle$. *Then* $\mathrm{Ab}(U)=\{B\}$ *and* $|B|=q^{n(n+1)/2}$.

COROLLARY 4.3. *Let* $G=C_n(q)$ *with* $n\geqslant 3$ *and let* $B=\langle X(r)|r\geqslant r_n\rangle$. *Then* $\mathrm{Ab}(U)=\{B\}$ *and* $|B|=q^{n(n+1)/2}$.

PROOF OF THEOREM. 4.1. Since q is even $Z(U)=\langle X(r_1+r_2), X(r_1+2r_2)\rangle$. Therefore $A\in\mathrm{Ab}(U)$ implies that $Z(U)\leqslant A$. It follows that in order to pin down the elements of $\mathrm{Ab}(U)$ we need only inspect $C_U(x)$ where $x=x(r_1,d_1)$, $x(r_2,d_2)$ or $x(r_1,d_1)\,x(r_2,d_2)$, $d_1,d_2\neq 0$. Now $C_U(x(r_1,d_1))=A_1$ and $C_U(x(r_2,d_2))=A_2$.

Finally let $x(r_1,e_1)\,x(r_2,e_2)\in C_U(x(r_1,d_1)\,x(r_2,d_2))$. Then

$$\begin{aligned}1&=[x(r_1,d_1)\,x(r_2,d_2),\ x(r_1,e_1)\,x(r_2,e_2)]\\&=x(r_1+r_2,d_1e_2+d_2e_1)\,x(r_1+2r_2,d_1e_2^2+e_1d_2^2)\end{aligned}$$

if and only if $d_1e_2=d_2e_1$ and $d_1e_2^2=e_1d_2^2$
if and only if $e_1=d_1d_2^{-1}e_2$ and $e_2^2=e_2d_2$,
if and only if $e_1=d_1$ and $e_2=d_2$ or $e_1=e_2=0$.

Gathering together these pieces of information on centralizers we have Theorem 4.1.

PROOF OF THEOREM. 4.2. The proof will be by induction on n. First we set up some notation. Let P_J be the parabolic subgroup of G associated with $J=\Pi-\{r_2\}$. Then, if necessary, by Barry (1977), Theorem 3.13 we have $S_J=M\times N$ where $M=\langle X(r_1), X(-r_1)\rangle\cong A_1(q)$ and $N=\langle X(r)|r\in\Phi_K\rangle\cong B_{n-2}(q)$ where $K=\Pi-\{r_1,r_2\}$.

Now $U_J=\langle X(r)|r\geqslant r_2\rangle$ and $[U_J,U_J]=X(r_0)$ where $r_0=r_1+2r_2+\ldots+2r_n$ is the highest root. However, since q is even

$$Z(U_J)=\langle X(r_2+r_3+\ldots+r_n),\ X(r_1+r_2+\ldots+r_n), X(r_0)\rangle.$$

Let $V=U_J$ from now on. $M\times N$ acts on V by right conjugation and by Barry (1977), Theorem 3.13 we have

$$\begin{aligned}V/[V,V]&\cong V(A,1,\lambda_1)\otimes W\quad\text{as an } M\times N\text{-module}\\&\cong W\otimes W\quad\text{as an } N\text{-module,}\end{aligned}$$

where $W=V(B,n-2,\lambda_1)$ can be identified with $\langle X(r)|r\geqslant r_2,\ r\not\geqslant r_1\rangle$.

Let $X=X(r_1)$ and $Y=\langle X(r)|r\in\Phi_K^+\rangle$. Then $X\times Y\in\mathrm{Syl}_2(M\times N)$ and $U=XYV$. Let φ be the natural epimorphism of U on U/V and $'$ that of V on $V/[V,V]$.

LEMMA 4.4. *Let* $C\in\mathrm{Ab}(V)$. *Then* $|C|=q^{2n-1}$.

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PROOF. Firstly we observe that $Z(V) \leqslant C$. Now $\bar{V} = V/Z(V)$ can be made into a vector space over F and indeed $\bar{V}$ can be equipped with a non-degenerate alternating bilinear form (,) as follows:

$$x(r_0, (\bar{v}_1, \bar{v}_2)) = [v_1, v_2] \quad \text{where } \bar{v}_1, \bar{v}_2 \in \bar{V}.$$

The non-degeneracy is assured by our factoring out by $Z(V)$.

Now it is clear, that for W a subgroup of V, that W is abelain if and only if W is contained in a totally isotropic subspace of $\bar{V}$. The maximum dimension of such a subspace is $2n-4$ since $|\bar{V}| = q^{4n-8}$. Therefore W abelian implies $|W| \leqslant q^{2n-4}$ and so $|W| \leqslant q^{2n-1}$. In fact abelian subgroups of V of this order exist, for example, $\langle X(r) | r \geqslant r_2 + \ldots + r_n \rangle$. Hence Lemma 4.4 holds.

One notes that Lemma 3.3 applies in the present context even though $Y, W, V/[V, V]$ et cetera have changed meaning.

Let $A \in \mathrm{Ab}(U)$. Then $|A| \geqslant q^6 = |B|$. Also $(A)\varphi = A_1 \leqslant X \times Y = X(r_1) \times X(r_3)$ and $|A| = |A_1|\,|A \cap V|$. Suppose $xy \in A_1$ with $x \in X, y \in Y$ and $x \neq 1$. Then by Lemma 3.3 we get $\dim C_{V'}(xy) \leqslant \dim W = 3$. Therefore $|A \cap V| \leqslant q^4$. Now $|A| \geqslant q^6$ forces $|A_1| = q^2$ and hence $A_1 = X \times Y$. But then $C_{V'}(A_1) = \langle X'(r_1 + r_2 + r_3), X'(r_1 + r_2 + 2r_3) \rangle$. This implies that $|A \cap V| \leqslant q^3$ and $|A| \leqslant q^5$ which is a contradiction. Therefore $A_1 \leqslant Y$ and order arguments force equality. The preimage $V_1 = \langle X(r) | r \geqslant r_2 + r_3 \rangle$ of $C_{V'}(A_1)$ in V is an element of $\mathrm{Ab}(V)$. As in previous sections we prove $A = V_1 Y = B$ to conclude the case of $n = 3$.

Assume now that Theorem 4.2 is true for any integer k such that $3 \leqslant k \leqslant n$ and we deal with $U \in \mathrm{Syl}_2(B_n(q))$. Let $A \in \mathrm{Ab}(U)$. Then $|A| \geqslant |B| = q^{n(n+)1/2}$ and $(A)\varphi = A_1 \leqslant X \times Y$. By induction Y has a bound $q^{(n-1)(n-2)/2}$ on abelian subgroups and so $|A_1| \leqslant q^{(n-1(n-2)/2+1}$.

Suppose $xy \in A_1$, with $x \in X, y \in Y$ and $x \neq 1$. Then by Lemma 3.3, $\dim C_{V'}(xy) \leqslant \dim W = 2n-3$ and so $|A \cap V| \leqslant q^{2n-2}$. Now

$$q^{n(n+1)/2} \leqslant |A| = |A_1\,|A \cap V| \leqslant q^{(n-1)(n-2)/2+1} . q^{2n-2} = q^{n(n+1)/2}$$

implies $|A_1| = q^{(n-2)(n-1)/2+1}$, which in turn implies $A_1 \in \mathrm{Ab}(X \times Y)$ and so $A_1 = X \times E$ where $E \in \mathrm{Ab}(Y)$. Induction and Theorem 4.1 provide the possibilities for E and by an easy calculation one finds that in each case $|C_{V'}(A_1)| \leqslant q^{n-1}$. This gives $|A \cap V| \leqslant q^n$ and so $|A| \leqslant q^{(n-1)(n-2)/2+1} . q^n < q^{n(n+1)/2}$ since $n \geqslant 3$. We have arrived at a contradiction and so $A_1 \leqslant Y$.

An order calculation forces $A_1 \in \mathrm{Ab}(U)$. Again by induction and Theorem 4.1 we have $A_1 = \langle X(r) | r \in \Phi_K^+, r \geqslant r_n \rangle$ if $n \geqslant 5$, A_1 is one of $E_i, i = 1, 2, 3$ if $n = 4$ and $q = 2$, while A_1 is one of $E_i, i = 1, 2$, if $n = 4$ and $q > 2$ where

$$E_1 = \langle X(r_4), X(r_3 + r_4), X(r_3 + 2r_4) \rangle,$$

$$E_2 = \langle X(r_3), X(r_3 + r_4), X(r_3 + 2r_4) \rangle$$

and

$$E_3 = \langle x(r_3, t)\, x(r_4, t), X(r_3 + r_4), X(r_3 + 2r_4) | t \in F \rangle.$$

One checks that $|C_{V'}(E_i)| \leqslant q^4$ if $n = 4$ and $i = 2$ or 3. Hence $|A \cap V| \leqslant q^5$ and $|A| \leqslant q^8$ which is a contradiction. In all cases then $A_1 = \langle X(r) | r \in \Phi_K^+, r \geqslant r_n \rangle$. The preimage V_1 in V of $C_{V'}(A_1) = \langle X(r) | r \geqslant r_2 + r_3 + \ldots + r_n \rangle$ is an element of $\mathrm{Ab}(U)$. In our usual fashion then we get $A = A_1 V_1 = B$. This completes the proof of Theorem 4.2.

Proof of Corollary 4.3. Since q is even, $C_n(q) \cong B_n(q)$ and so a 2-Sylow subgroup of $C_n(q)$ is isomorphic to a 2-Sylow subgroup of $B_n(q)$. If $n \geqslant 3$ Theorem 4.2 guarantees then a 2-Sylow subgroup of $B_n(q)$ and hence of $C_n(q)$ has a unique abelian subgroup of largest order $q^{n(n+1)/2}$. Now $B = \langle X(r) | r \geqslant r_n \rangle$ is an abelian subgroup of order $q^{n(n+1)/2}$ in the 2-Sylow subgroup U of $C_n(q)$. Hence $\mathrm{Ab}(U) = \{B\}$.

5. The solution for $B_n(q)$, q odd and $n \geqq 3$

Let $q = p^m$, p an odd prime and $m \geqslant 1$. Before we state the main results of this section we need to define some abelian subgroups of U. Firstly let $B = \langle X(r) | r \geqslant r_1 \rangle$.

Next for $(a_1, a_2, \ldots, a_n) \in F^n - \{0\}$ we define

$$B(a_1, a_2, \ldots, a_n) = \langle x(r_n, a_1 t)\, x(r_{n-1} + r_n, a_2 t) \ldots x(r_1 + r_2 + \ldots + r_n, a_n t),\; X(s) | s \geqslant r_{n-1} + 2r_n,\ t \in F \rangle.$$

For $(a_1, a_2, \ldots, a_{n-1}) \in F^{n-1} - \{0\}$ we define

$$C(a_1, a_2, \ldots, a_{n-1}) = \langle x(r_{n-1} + r_n, a_1 tf)\, x(r_{n-2} + r_{n-1} + r_n, a_2 t) \ldots \times x(r_1 + r_2 + \ldots + r_n, a_{n-1} t),\ X(r),\ X(s) | r \geqslant r_{n-1},\ r \ngeqslant r_n,\ s \geqslant r_{n-2} + 2r_{n-1} + 2r_n,\ t \in F \rangle.$$

Note that $B(a_1, a_2, \ldots, a_n) = B(b_1, b_2, \ldots, b_n)$ if and only if there exists $u \in F^*$ with $(a_1, a_2, \ldots, a_n) = u(b_1, b_2, \ldots, b_n)$. A similar remark holds for the groups $C(a_1, a_2, \ldots, a_{n-1})$.

Theorem 5.1. *Let $G = B_3(q)$. Then*

(a) *$\mathrm{Ab}(U) = \{B\}$ and $|B| = q^5$,*

(b) *if A is an abelian subgroup of U not contained in B, then $|A| \leqslant q^4$.*

Theorem 5.2. *Let $G = B_4(q)$. Then*

$$\mathrm{Ab}(U) = \{B,\ B(a_1, a_2, a_3, a_4),\ C(b_1, b_2, b_3)^{x(r_4, t)} | (a_1, a_2, a_3, a_4) \neq (0,0,0,0),\ (b_1, b_2, b_3) \neq (0,0,0),\ t \in F\}.$$

Hence any element of $\mathrm{Ab}(U)$ has order q^7.

Theorem 5.3. *Let $G = B_n(q)$ where $n \geqslant 5$. Then $\mathrm{Ab}(U) = \{B(a_1, a_2, \ldots, a_n), C(b_1, b_2, \ldots, b_{n-1})^{x(r_n, t)} | (a_1, a_2, \ldots, a_n) \in F^n - \{0\}, (b_1, b_2, \ldots, b_{n-1}) \in F^{n-1} - \{0\}, t \in F\}$. Hence any element of $\mathrm{Ab}(U)$ has order $q^{n(n-1)/2+1}$.*

The notation we set up in Section 4 at the beginning of the proof of Theorem 2.4 will apply in this section also. However here $[V, V] = Z(V) = X(r_0)$ since we are in odd characteristic. As a consequence of this we record

LEMMA 5.4. *If* $C \in \mathrm{Ab}(V)$, *then* $|C| = q^{2n-2}$.

LEMMA 5.5. *If* $y \in Y$, $y \neq 1$, *then* $\dim C_W(y) \leqslant \dim W - 2 = 2n-1$.

PROOF. This is exactly the same as the proof of Lemma 3.14.
One notes that Lemma 3.3 applies to this section as it did to section 4.

PROOF OF THEOREM 5.1. (a) Here $Y = X(r_3)$. Let $A \in \mathrm{Ab}(U)$. Then $|A| \geqslant |B| = q^5$. Since by Lemma 5.4 $|A \cap V| \leqslant q^4$ we get that $|A_1| \geqslant q^2$ where $A_1 = (A)\varphi$. Suppose $xy \in A_1$ where $x \in X$, $y \in Y$ and $x, y \neq 1$. Then by Lemma 3.3

$$\dim C_{V'}(xy) = \dim \ker_W (y-1)^2.$$

Now $y = x(r_3, t)$, $t \neq 0$, and $\ker_W (y-1)^2 \neq W$ since $x(r_2, 1)(y-1)^2 \neq 0$.

$$([[x(r_2, 1), x(r_3, t)], x(r_3, t)] = [x(r_2+r_3, \pm t)x(r_2+2r_3, \pm t^2),$$
$$x(r_3, t)] = x(r_2+2r_3, \pm 2t^2) \neq 1$$

since q is odd. The signs here depend only on structure constants). Therefore $\dim \ker_W (y-1)^2 \leqslant 2$. It follows that $|A \cap V| \leqslant q^3$. $|A| \geqslant q^5$ implies $A_1 = X \times Y$. But $C_{V'}(X \times Y) = X'(r_1+r_2+2r_3)$ giving $|A \cap V| \leqslant q^2$ and $|A| \leqslant q^4$—a contradiction. Therefore $A_1 \leqslant X$ or $A_1 \leqslant Y$. Suppose that $A_1 \leqslant Y$ and let $y \in Y$, $y \neq 1$. Then $C_W(y) = X(r_2+2r_3)$ and so $\dim C_{V'}(y) = 2 \dim C_W(y) = 2$. Hence $|A \cap V| \leqslant q^3$ and $|A| \leqslant q^4$—a contradiction. Hence $A_1 \leqslant X$ and now $|A_1| \geqslant q$ forces $A_1 = X$. We get $A = B$ in our usual fashion.

(b) Suppose A is an abelian subgroup of U such that $|A| > q^4$ which is not contained in B. Then by Lemma 5.4 $A \nleqslant V$. Therefore $(A)\varphi = A_1 \neq 1$. If $A_1 \nleqslant Y$ then $|A_1| \leqslant q$ and $|A \cap V| \leqslant q_3$ as in part (a) giving the contradiction $|A| \leqslant q^4$.

Suppose now that $a = x(r_1, t_1)x(r_3, t_3) \in A_1$, t_1, $t_3 \neq 0$. Then $\dim C_{V'}(a) = 2$ since $\dim \ker_W (x(r_3, t_3)-1)^2 = 2$. Therefore $|A \cap V| \leqslant q^3$. Now $A_1 \leqslant \langle x(r_1, t)x(r_3, ct) | t \in F, c = t_3 t_1^{-1} \rangle$ leads to $|A| \leqslant q^4$ and a contradiction. Suppose then that $a_1 = x(r_1, d)x(r_3 bd) \in A_1$ with $d \neq 0$, $b \neq c$. Then $C_{V'}(A_1) \leqslant C_{V'}(\mathrm{a}) \cap C_{V'}(a_1)$. But

$$C_{V'}(a) = \langle x'(r_1+r_2+r_3, t)x'(r_2+2r_3, \pm 2ct),\ X'(r_1+r_2+2r_3) | t \in F \rangle,$$

where the sign depends only on the structure constants. Therefore $\dim C_{V'}(A_1) \leqslant 1$ which implies $|A \cap V| \leqslant q^2$ and $|A| \leqslant q^4$—a contradiction. The supposition that $a_1 = x(r_3, d) \in A_1$ with $d \neq 0$ meets a similar fate. This leaves $A_1 \leqslant X$.

Now $C_{V'}(A_1) = \langle X'(r) | r \geqslant r_1 + r_2 \rangle$ and so $A \cap V \leqslant \langle X(r) | r \geqslant r_1 + r_2 \rangle$. $|A_1| \leqslant q$ and $|A| > q^4$ imply $|A \cap V| > q^3$. Let $a_1 \in A_1$. Then $a_1 v \in A$ for some $v \in V$. Let $v = v_1 w$

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where $v_1 \in \langle X(r) | r \geqslant r_2, r \ngeqslant r_1 \rangle$ and $w \in \langle X(r) | r \geqslant r_1 + r_2 \rangle$. Let $v_2 \in A \cap V$. Now$1 = [a_1 v_1 w, v_2] = [v_1, v_2]$ for all $v_2 \in A \cap V$. If $V_1 = \langle X(r) | r \geqslant r_1 + r_2 \rangle$ then V_1 is the natural vector space for $\langle X(r) | r \in \Phi_L \rangle \cong B_2(q)$ where $L = \{r_2, r_3\}$. Now $\dim C_{V_1}(v_1) > 3$ since $|A \cap V| > q^3$. By exactly the same argument as that of Lemma 5.5 $\dim C_{V_1}(v_1) > 3$ forces $v_1 = 1$. Hence $A \leqslant B$—a contradiction. This completes the proof of Theorem 5.1.

PROOF OF THEOREM 5.2. The proof will consist of a long series of lemmas. Firstly

$$y = \langle X(r_3), X(r_4), X(r_3 + r_4), X(r_3 + 2r_4) \rangle.$$

LEMMA. 5.6. *The representatives of the conjugacy classes of* Y *are as follows*:
(a) 1, *one class*,
(b) $x(r_3 + 2r_4, a)$, $a \neq 0$, $q-1$ *classes*,
(c) $x(r_3, b)$, $b \neq 0$, $q-1$ *classes*,
(d) $x(r_3 + r_4, c)$, $c \neq 0$, $q-1$ *classes*,
(e) $x(r_4, d)$, $d \neq 0$, $q-1$ *classes*,
(f) $x(r_3, b)\, x(r_3 + 2r_4, a)$, $a, b \neq 0$, $(q-1)^2$ *classes*,
(g) $x(r_3, b)\, x(r_4, d)$, $b, d \neq 1$, $(q-1)^2$ *classes*.

PROOF. This can be gleaned from a reading of Srinavasan (1968) or one can compute the result by hand.

Let $A \in \mathrm{Ab}(U)$. Then $|A| \geqslant |B| = q^7$. As usual $(A)\varphi = A_1 \leqslant X \times Y$ and $|A| = |A_1|\,|A \cap V|$. By Theorem 2.5 Y has a unique abelian subgroup of largest order q^3. Hence $|A_1| \leqslant q^4$. On the other hand, by Lemma 5.4 we have $|A \cap V| \leqslant q^6$ so that $|A_1| \geqslant q$.

LEMMA. 5.7. $A_1 \leqslant Y$ *or* $A_1 = X$.

PROOF. If $A_1 \leqslant X$ then $|A_1| \geqslant q$ forces $A_1 = X$. We will suppose that A_1 is neither contained in X nor in Y and obtain a contradiction. Then there exists $x, y \neq 1$ s.t. $x \in X$, $y \in Y$ and $xy \in A_1$.

Suppose $q^2 \geqslant |A_1| > q$. We claim that $|Y \cap A_1| > 1$. This we see by considering the projection epimorphism of $X \times Y$ onto X. Restricting to A_1, this has kernel $Y \cap A_1$, image contained in X and so $|Y \cap A_1|\,|X| \geqslant |A_1|$. Therefore $|Y \cap A_1| \geqslant |A_1|/|X| > 1$ since $|A_1| > q$. Let $y_1 \in Y \cap A_1$, $y_1 \neq 1$. Now

$$\dim C_{V'}(A_1) \leqslant \dim C_{V'}(\langle xy, y_1 \rangle) \leqslant \dim C_W(y_1) \leqslant \dim W - 2 = 3 \quad \text{in this case.}$$

Here we have used both Lemma 3.3 and Lemma 5.5. Thus $|A \cap V| \leqslant q^4$ and $|A| \leqslant q^6$ which is a contradiction.

Suppose now that $q^4 \geqslant |A_1| > q^2$. Using the projection of $X \times Y$ on X we get that $|A_1 \cap Y| > q$. Now $|(A_1 \cap Y) \times C_W(A_1 \cap Y)| \leqslant q^4$ by Theorem 5.1. $|A_1 \cap Y| > q$ and $C_W(A_1 \cap Y)$ a vector space over F force $|C_W(A_1 \cap Y)| \leqslant q^2$. Now $xy \in A_1$ and

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$A_1 \cap Y \leqslant A_1$ imply $\dim C_{V'}(A_1) \leqslant \dim C_W(A_1 \cap Y) \leqslant 2$. Hence $|A \cap V| \leqslant q^3$ and so $|A_1| = q^4$. This forces $A_1 = X \times E$ where E is the unique element of $\mathrm{Ab}(Y)$. One checks that $|C_{V'}(A_1)| = q$ in this case and this leads us to a contradiction.

We are left with $|A_1| = q$. The assumption is still that $xy \in A_1, x \in X, y \in Y, x, y \neq 1$. y cannot be conjugate to $x(r_3+r_4, c)$, $x(r_4, d)$ or $x(r_3, b)x(r_4, d)$, $b, c, d \neq 0$, since in each case $\dim \ker_W (y-1)^2 < \dim W = 5$. This implies $|C_{V'}(A_1)| \leqslant q^4$ which leads to $|A \cap V| \leqslant q^5$ and $|A| \leqslant q^6$—a contradiction.

Suppose y is conjugate in Y to $x(r_3, b), b \neq 0$. In fact we may syppose without loss of generality for what follows that $y = x(r_3, b)$. Recall that the long roots of a system of type B_n form a system of type D_n. Now the preimage V_1 in V of $C_{V'}(xy)$ is a direct product of $X(r_1+r_2+r_3+r_4)$ with D_4—contribution of order q^5 as in Lemma 3.6. This D_4—contribution was found to be non-abelian for odd q in the proof of Lemma 3.13. Therefore $A \cap V$ is properly contained in V and so $|A \cap V| < q^6$. Thus $|A| < q^7$—a contradiction. The same argument works for y conjugate to $x(r_3+2r_4, a)$, $a \neq 0$ or y conjugate to $x(r_3, b)\, x(r_3+2r_4, a)$, $a, b \neq 0$. This completes the proof of Lemma 5.7.

We suppose until further notice that $A_1 \leqslant Y$ and so $q \leqslant |A_1| \leqslant q^3$, $|A \cap V| \leqslant q^6$ and $q^7 \leqslant |A| \leqslant q^9$.

LEMMA 5.8. *$|A_1| = q$ or $|A_1| = q^2$ are the only possibilities.*

PROOF. Suppose $q^2 \geqslant |A_1| > q$. Then by Theorem 5.1(b) and the fact that $C_W(A_1)$ is a vector space over F, $|C_W(A_1)| \leqslant q^2$. This implies $|C_{V'}(A_1)| \leqslant q^4$ which results in $|A \cap V| \leqslant q^5$. Now $|A| \geqslant q^7$ forces $|A_1| = q^2$.

Suppose instead that $q^3 \geqslant |A_1| > q^2$. Again by Theorem 5.1(b) and the fact that $C_W(A_1)$ is a vector space over F we get $|C_W(A_1)| \leqslant q$, Thus $|A \cap V| \leqslant q^3$ and so $|A| \leqslant q^6$—a contradiction. Hence the lemma.

LEMMA 5.9. *Let $D_1 = \langle X(r) | r \geqslant r_3, r \ngeqslant r_2 \rangle$ and $D_2 = \langle X(r) | r \geqslant r_4, r \ngeqslant r_2 \rangle$. Then $A_1 \leqslant D_1$ or $A_1 \leqslant D_2$.*

PROOF. Suppose not, then there exists $y_1 \in A_1$ such that y_1 is conjugate $y = x(r_3, b)x(r_4, d)$, $b, d \neq 0$. Then $|C_W(y_1)| = |C_W(y)| = q$. This implies $|A \cap V| \leqslant q^2$ which leads to a contradiction. Hence the lemma.

LEMMA 5.10. *If $x(r_3, b) \in A_1$ with $b \neq 0$, then $X(r_3) \leqslant A_1$.*

PROOF. Now $x(r_3, b) \in D_1$ and so by Lemma 5.9 $A_1 \leqslant D_1$. One checks that if $y \in D_1$ then y commutes with the preimage in V of $C_{V'}(y)$. Since $C_{V'}(A_1) = \cap_{y \in A_1} C_{V'}(y)$ we have $[A_1, A \cap V] = 1$. But $|A| = |A_1|\,|A \cap V|$ implies that $A_1 \times (A \cap V) \in \mathrm{Ab}(U)$. If $x = x(r_3, t) \notin A$ for some $t \neq 0$, consider the goup $\langle x, A_1 \times (A \cap V) \rangle$. Then $[x, A_1] = 1$

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since D_1 is abelian and $[x, A\cap V] = 1$ since $[x(r_3, b), A\cap V] = 1$.Thus $\langle x, A_1\times(A\cap V)\rangle$ is an abelian group of larger order than A—contradiction. This completes the proof of Lemma 5.10.

LEMMA 5.11. *If* $|A_1| = q$ *then* A_1 *can only be one of the following*:
(a) $X(r_3+2r_4)$,
(b) $X(r_3)$,
(c) *a conjugate of* $X(r_3)$ *by* $x(r_4, t)$ *for* $t\in F^*$.

PROOF. If $y\in A_1$ and y conjugate to $x(r_4, d)$, $d\neq 0$, we can suppose without loss of generality for what follows that $y = x(r_4, d)$. Then the preimage V_1 in V of $C_{V'}(y) = \langle X(r_1+r_2), X(r_2), X(s)|s\geqslant r_2+r_3+2r_4\rangle$. Clearly $[V_1, V_1] = X(r_0)$ while $Z(V_1) = \langle X(r_1+r_2+r_3+2r_4), X(r_2+r_3+2r_4), X(r_0)\rangle$. Reasoning similar to that of Lemma 4.4 gives that $C\in \mathrm{Ab}(V_1)$ implies $|C| = q^5$. Therefore $|A\cap V|\leqslant q^5$ and so $|A|\leqslant q^6$—a contradiction. A similar argument rules out the case of $y\in A_1$ conjugate to $x(r_3+r_4, c)$, $c\neq 0$.

If $y\in A_1$ and y is conjugate to $x(r_3, b)\,x(r_3+2r_4, a)$, $a, b\neq 0$ we may assume without loss of generality that $y = x(r_3, b)\,x(r_3+2r_4, a)$. The preimage V_2 in V of

$$\begin{aligned}C_{V'}(y) = \langle &x(r_1+r_2+r_3, t_1)\,x(r_1+r_2+r_3+2r_4, \pm ct_1), X(r_1+r_2+r_3+r_4),\\ &x(r_2+r_3, t_2)\,x(r_2+r_3+2r_4, \pm ct_2), X(r_2+r_3+r_4),\\ &X(s)|s\geqslant r_2+2r_3+2r_4, t_1, t_2\in F, c = ab^{-1}\rangle,\end{aligned}$$

where the signs depend on the structure constants. Clearly $[V_2, V_2] = X(r_0)$. Further since the long roots of a system of type B_n form a system of type D_n, the proof of Lemma 3.13 guarantees that $x(r_1+r_2+r_3, t_1)\,x(r_1+r_2+r_3+2r_4, \pm ct_1)$ and $x(r_2+r_3, t_2)\,x(r_2+r_3+2r_4, \pm ct_2)$ do not commute thus ensuring that $Z(V_2) = \langle X(s)|s\geqslant r_2+2r_3+2r_4\rangle$. As for V_1, $C\in\mathrm{Ab}(V_2)$ implies $|C| = q^5$. This leads to $|A|\leqslant q^6$—a contradiction.

If $y\in A_1$ and y is conjugate to $x(r_3, b)$, $b\neq 0$, we may assume $y = x(r_3, b)$. Lemma 5.10 forces $X(r_3)\leqslant A_1$ and $|A_1| = q$ forces equality. Any element conjugate in Y to $x(r_3+2r_4, a)$ is of course equal to $x(r_3+2r_4, a)$ since $x(r_3+2r_4, a)\in Z(Y)$. If $x(r_3+2r_4, a)\in A_1$ for $a\neq 0$, then by an argument similar to that of Lemma 5.10 we get $X(r_3+2r_4)\leqslant A_1$ and $|A_1| = q$ gives equality. By Lemma 5.9 we have considered all the elements of Y to which an element of A_1 could be conjugate and so we have proved the lemma.

LEMMA 5.12. *If* $|A_1| = q^2$ *then* A_1 *can only be one of the following*:
(a) $\langle X(r_3+r_4), X(r_3+2r_4)\rangle$,
(b) $\langle x(r_4, t)\,x(r_3+r_4, at), X(r_3+2r_4)|t\in F\rangle$ *for some* $a\in F$,
(c) $\langle X(r_3), X(r_3+r_4)\rangle$,
(d) *a conjugate of* $\langle X(r_3), X(r_3+r_4)\rangle$ *by* $x(r_4, t)$ *for some* $t\in F^*$.

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PROOF. (a) and (b) list all abelian subgroups of order q^2 in D_2. So we concentrate on $A_1 \leqslant D_1$. Suppose $y = x(r_3, b) \in A_1$ for $b \neq 0$. Then by Lemma 5.10 we have $X(r_3) \leqslant A_1$. Suppose now that $y_1 = x(r_3 + 2r_4, a) \in A_1$, $a \neq 0$. The preimage V_1 in V of $C_{V'}(y) \cap C_{V'}(y) = \langle X(r_1 + r_2 + r_3 + r_4),\ X(r_2 + r_3 + r_4),\ X(s) | s \geqslant r_2 + 2r_3 + 2r_4 \rangle$. Clearly $[V_1, V_1] = X(r_0)$ while $Z(V_1) = \langle X(s) | s \geqslant r_2 + 2r_3 + 2r_4 \rangle$. Reasoning similar to that of Lemma 4.4 gives that $C \in \mathrm{Ab}(V_1)$ implies $|C| = q^4$. This is not sufficiently large since $|A_1| = q^2$ forces $|A \cap V| \geqslant q^5$. Thus $y \in A_1$ implies $y_1 \notin A_1$.

Suppose that y and $y_2 = x(r_3 + r_4, c)\, x(r_3 + 2r_4, a) \in A_1$, $a, c \neq 0$. Then the preimage V_2 in V of $C_{V'}(y) \cap C_{V'}(y_2)$ equals

$$\langle x(r_1 + r_2 + r_3, t_1)\, x(r_1 + r_2 + r_3 + r_4,\ \pm dt_1/2),\ Z(V_1),\\ x(r_2 + r_3, t_2)\, x(r_2 + r_3 + r_4,\ \pm dt_2/2) | t_1, t_2 \in F \rangle$$

where $d = ac^{-1}$ and the signs depend only on the structure constants. Now $Z(V_2) = Z(V_1)$ and $[V_2, V_2] = [V_1, V_1]$ imply as above that if $y \in A_1$ then $y_2 \notin A_1$. Therefore if $y \in A_1$ then $A_1 \leqslant \langle X(r_3),\ X(r_3 + r_4) \rangle$ and $|A_1| = q^2$ forces equality. If an element of A_1 is conjugate in Y to y then A_1 is conjugate to $\langle X(r_3),\ X(r_3 + r_4) \rangle$ by the same element. If an element of A_1 is conjugate in Y to $x(r_3 + 2r_4, a)$, $a \neq 0$, (and hence equal), a similar argument gives $A_1 = \langle X(r_3 + r_4),\ X(r_3 + 2r_4) \rangle$.

We are reduced to considering $A_1 \leqslant D_1$ containing no conjugates of $x(r_3, b), b \neq 0$, or of $x(r_3 + 2r_4, a), a \neq 0$. One notes since, for $a \neq 0, x(r_3 + 2r_4, a) \notin A_1$ and since $|A_1| = q^2$ then $A_1 \nleqslant \langle X(r_3 + r_4),\ X(r_3 + 2r_4) \rangle = Y_1$. Since all Y-conjugates of $x(r_3 + r_4, c), c \neq 0$, are contained in Y_1, there exists an element $y \in A_1$ with y conjugate to $x(r_3, b)\, x(r_3 + 2r_4, a)$, $a, b \neq 0$. As usual we may assume $y = x(r_3, b)\, x(r_3 + 2r_4 a)$, and then by an argument similar to that of Lemma 5.10 we get

$$\langle x(r_3, t)\, x(r_3 + 2r_4, ft) | f = ab^{-1},\ t \in F \rangle \leqslant A_1.$$

If $x(r_3, t_1)\, x(r_3 + 2r_4, t_2) \in A_1$ with $t_2 t_1^{-1} \neq f$ then $x(r_3, t_3) \in A_1$ for $t_3 \neq 0$ which is a contradiction. Therefore the rest of the elements of A_1 must be of the form $x = x(r_3, t_1)\, x(r_3 + r_4, t_2)\, x(r_3 + 2r_4, t_3)$ where $t_1, t_2 \neq 0$ and x is not a conjugate of $x(r_3, t_1)$ or x conjugate of $x(r_3 + r_4, t)$ for some $t \in F^*$. We claim that A_1 does in fact contain a conjugate of $x(r_3 + r_4, t)$ for some $t \in F^*$. Suppose not. Then $y_1 = x(r_3, t_1)\, x(r_3 + r_4, t_2)$ $x(r_3 + 2r_4, t_3) \in A_1$ for some $t_1, t_2 \neq 0$ for otherwise $|A_1| = q^2$ and $A_1 \leqslant \langle x(r_3, t)$ $x(r_3 + 2r_4, ft) | t \in F \rangle$—a contradiction. Since $x(r_3, t_1)\, x(r_3 + 2r_4, ft_1) \in A_1$ we get that $x(r_3 + r_4, t_2)\, x(r_3 + 2r_4, t_3 - ft_1) \in A_1$. This element is conjugate in Y to $x(r_3 + r_4, t_2)$—a contradiction. Hence our claim.

We now assume that $x(r_3 + r_4, t_2) \in A_1$ for $t_2 \neq 0$. As before $X(r_3 + r_4) \leqslant A_1$. We can no longer assume that $x(r_3, b)\, x(r_3 + 2r_4, a) \in A_1$ but only that some conjugate of it is in A_1. No element of the form $x(r_3 + r_4, t)\, x(r_3 + 2r_4, u) \in A_1$ where $u \neq 0$ since then we get $x(r_3 + 2r_4, u) \in A_1$ which is in contradiction to our assumption. Further $A_1 \nleqslant \langle X(r_3),\ X(r_3 + r_4) \rangle$ since A_1 contains no element conjugate to $x(r_3, b), b \neq 0$. Therefore A_1 contains an element of the form $y_3 = x(r_3, t)\, x(r_3 + r_4, u)\, x(r_3 + 2r_4, v)$

where $t, u \neq 0$ and y_3 is not a conjugate of $x(r_3, t)$. But then $y_2 = x(r_3, t)\, x(r_3+2r_4, v) \in A_1$. Let $y_1 = x(r_3+r_4, t_2)$. Then the preimage V_3 of $C_{V'}(y_1) \cap C_{V'}(y_2)$ in

$$V = \langle x(r_1+r_2+r_3, u_1)\, x(r_1+r_2+r_3+2r_4,\ \pm bu_1),\ X(s),$$
$$x(r_2+r_3, u_2)\ x(r_2+r_3+2r_4, \pm bu_2) | s \geqslant r_2+2r_3+2r_4,\ u_1, u_2 \in F,\ b = vt^{-1}\rangle,$$

where the signs depend only on the structure constants. Now q odd forces V_3 non-abelian as in the proof of Lemma 3.13. Hence $|A \cap V| < q^5$ which leads to $|A| < q^7$. With this final contradiction Lemma 5.12 is proved.

In Lemmas 5.7, 5.11 and 5.12 we have limited the possibilities for A_1. We will now see that all of these possiblities do occur. We ought perhaps, at this stage, to show that $A \in \mathrm{Ab}(U)$ implies $|A| = q^7$ by calculating and examining the preimage of $C_{V'}(A_1)$ for each of the possible choices for A_1 but this will become apparent anyway as we determine the possibilities for A.

If $A_1 = X$ then $A = B$ without further ado. Suppose then that $A_1 = X(r_3)$. Then the preimage V_1 of $C_{V'}(A_1)$ in V equals

$$\langle X(r) | r \geqslant r_2+r_3,\ r \neq r_1+r_2+r_3+2r_4,\ r_2+r_3+2r_4\rangle.$$

Now $Z(V_1) = \langle X(r_1+r_2+r_3),\ X(r_2+r_3),\ X(r) | r \geqslant r_2+2r_3+2r_4\rangle$. Therefore if $V_2 \in \mathrm{Ab}(V_1)$ then V_2 is of the form

$$\langle x(r_2+r_3+r_4, b_2 t)\, x(r_1+r_2+r_3+r_4, b_3 t),\ Z(V_1) | t \in F\rangle$$

for some $(b_2, b_3) \neq (0,0)$. Order considerations force $A \cap V = V_2$ for some $(b_2, b_3) \neq (0,0)$ and the usual argument gives $A = A_1 V_2$. A then turns out to be $C(0, b_2, b_3)$. If $A_1 = X(r_3)^{x(r_4,u)}$ then $A = C(0, b_2, b_3)^{x(r_4,u)}$.

If $A_1 = X(r_3+2r_4)$ we get in similar fashion that $A = B(0,0,a_3,a_4)$. Suppose now that $A_1 = \langle X(r_3),\ X(r_3+r_4)\rangle$. Then the preimage V_3 of $C_{V'}(A_1)$ in V equals

$$\langle X(r_1+r_2+r_3),\ X(r_2+r_3), X(r) | r \geqslant r_2+2r_3+2r_4\rangle.$$

V_3 is abelian and $|V_3| = q^5$. Now since $A \cap V \leqslant V_3$ and since $|A_1| = q^2$ forces $|A \cap V| \geqslant q^5$ we get equality.

Let $a \in A$ and $v \in A \cap V = V_3$. Then $a = a_1 v_1$ where $a_1 \in A$ and $v_1 \in V$. Then $1 = [a, v] = [a_1, v]^{v_1}\, [v_1, v] = [v_1, v]$ since $[A_1, V_3] = 1$. Thus

$$v_1 \in C_V(A \cap V) = \langle X(r) | r \geqslant r_2+r_3,\ r \neq r_1+r_2+r_3+2r_4,\ r_2+r_3+2r_4\rangle$$

which in turn implies

$$v_1 = x(r_2+r_3+r_4, t_1)\, x(r_1+r_3+r_3+r_4, t_2)\, v_2$$

where $v_2 \in A \cap V$. Let $b_1, b_2 \in A$ where

$$b_1 = x(r_3, t_1)\, x(r_3+r_4, t_2)\, x(r_2+r_3+t_4, a_1 t_2)\, x(r_1+r_2+r_3+r_4, a_2 t_2)\, w_1$$

and

$$b_2 = x(r_3, u_1)\, x(r_3+r_4, u_2)\, x(r_2+r_3+r_4, c_1 u_2)\, x(r_1+r_2+r_3+r_4, c_2 u_2)\, w_2$$

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where $w_1, w_2 \in A \cap V$ and $t_2, u_2 \neq 0$. Then since $[b_1, b_2] = 1$ we get $a_1 = c_1$ and $a_2 = c_2$. Again let $b_1, b_2 \in A$ where b_2 is as above and

$$b_1 = x(r_3, t_1)\,x(r_2+r_3+r_4, t_2)\,x(r_1+r_2+r_3+r_4, t_3)\,w_1,$$

where $w_1 \in A \cap V$. Since $[b_1, b_2] = 1$ we get $t_2 = t_3 = 0$. Therefore $A = C(1, a_1, a_2)$.

If $A_1 = \langle X(r_3),\ X(r_3+r_4)\rangle^{x(r_4,t)}$ then $A = C(1, a_1, a_2)^{x(r_4,t)}$. If $A_1 = \langle X(r_3+r_4), X(r_3+2r_4)\rangle$ then $A = B(0, 1, a_3, a_4)$ and finally if $A_1 = \langle x(r_4, t)\,x(r_3+r_4, at), X(r_3+2r_4) \mid t \in F\rangle$ then $A = B(1, a, a_3, a_4)$. This completes the proof of Theorem 5.2.

Proof of Theorem 5.3. We use induction on n. Consider firstly $n = 5$. Let $A \in \mathrm{Ab}(U)$. Then $|A| \geqslant |B(a_1, a_2, \ldots, a_5)| = q^{11}$. As usual $(A)\varphi = A_1 \leqslant X \times Y$ and $|A| = |A_1|\,|A \cap V|$. Since $Y \in \mathrm{Syl}_p(N)$ where $N \cong B_3(q)$ we get $|A_1| \leqslant q^6$ by Theorem 5.1. On the other hand by Lemma 5.4 $|A \cap V| \leqslant q^8$ and so $|A_1| \geqslant q^3$.

Lemma 5.13. $A_1 \leqslant Y$.

Proof. Suppose not. Then there exists $x \in X$, $y \in Y$ such that $x \neq 1$ and $xy \in A_1$. Consider the case $q^6 \geqslant |A_1| \geqslant q^5$. The restriction of the projection of $X \times Y$ on X to A_1 has kernel $A_1 \cap Y$, and image $\leqslant X$ and so $|A_1 \cap Y|\,|X| \geqslant |A_1|$ giving

$$|A_1 \cap Y| \geqslant |A_1|/|X| \geqslant q^4.$$

By Theorem 5.2$|(A_1 \cap Y) \times C_W(A_1 \cap Y| \leqslant q^7$ and so $|C_W(A_1 \cap Y)| \leqslant q^3$. Now $xy \in A_1$, $A_1 \cap Y \leqslant Y$ and Lemma 3.3 give $\dim C_{V'}(A_1) \leqslant \dim C_W(A_1 \cap Y) \leqslant 3$. Hence $|A \cap V| \leqslant q^4$ leading to $|A| \leqslant q^{10}$—a contradiction. A consideration of the cases $q^5 > |A_1| \geqslant q^4$ and $q^4 > |A_1| \geqslant q^3$ leads in each case to the contradiction $|A| < q^{10}$. Hence $A_1 \leqslant Y$.

Lemma 5.14. *The possibilities for A_1 are*:

(a) $B(a_1, a_2, a_3)$, $(a_1, a_2, a_3) \neq (0,0,0)$,
(b) $C(a_1, a_2)$, $(a_1, a_2) \neq (0, 0$, *or a conjugate by* $x(r_5, t)$,
(c) $B_1 = \langle X(r) \mid r \geqslant r_4 + 2r_5,\ r \ngeqslant r_2\rangle$,
(d) $C_1 = \langle X(r) \mid r = r_4,\ r_3 + r_4$ *or* $r_3 + 2r_4 + 2r_5\rangle$ *or a conjugate by* $x(r_5, t)$; *where*

$$B(a_1, a_2, a_3) = \langle x(r_5, a_1 t)\,x(r_4+r_5, a_2 t)\,x(r_3+r_4+r_5, a_3 t),$$
$$X(s) \mid s \geqslant r_4 + 2r_5,\ s \ngeqslant r_2,\ t \in F\rangle$$

and

$$C(a_1, a_2) = \langle x(r_4+r_5, a_1 t)\,x(r_3+r_4+r_5, a_2 t),\ X(s) \mid t \in F,$$
$$s = r_4, r_3+r_4 \text{ or } r_3+2r_4+2r_5\rangle$$

Proof. Since $A_1 \leqslant Y$ we have $q^5 \geqslant |A_1| \geqslant q^3$. Consider the case $q^5 \geqslant |A_1| > q_4$. By Theorem 5.2 $|A_1 \times C_W(A_1)| \leqslant q^7$. This implies $|C_W(A_1)| \leqslant q^2$ since $C_W(A_1)$ is a vector space over F. Hence $|C_{V'}(A_1)| \leqslant q^4$ and $|A \cap V| \leqslant q^5$ leading to the contradiction

$|A| \leqslant q^{10}$. The case $q^4 > |A_1| > q^3$ leads in similar fashion to the contradiction $|A| < q^{11}$. Hence $|A_1| = q^4$ or $|A_1| = q^3$.

Suppose $|A_1| = q^4$. Then $|C_W(A_1)| \leqslant q^3$. If $|C_W(A_1)| < q^3$ then $|C_W(A_1)| \leqslant q^2$ which implies $|A \cap V| \leqslant q^5$ and $|A| \leqslant q^9$—a contradiction. Hence $|C_W(A_1)| = q^3$ and $|A_1 \times C_W(A_1)| = q^7$. Theorem 5.2 now gives A_1 as (a) or (b).

Suppose $|A_1| = q^3$. Then as above $|C_W(A_1)| = q^4$ and $|A_1 \times C_W(A_1)| = q^7$. Theorem 5.2 again gives A_1 as (c) or (d). This completes the proof of Lemma 5.14.

If $A_1 = B(a_1, a_2, a_3)$, then $A = B(a_1, a_2, a_3, a_4, a_4)$ for some $a_4, a_5 \in F$; if $A_1 = C(a_1, a_2)^{x(r_5, t)}$, then $A = C(a_1, a_2, a_3, a_4)^{x(r_5, t)}$ for some $a_3, a_4 \in F$; if $A_1 = B_1$, then $A = B(0, 0, 0, a_4, a_5)$ where $(a_4, a_5) \neq (0, 0)$; if $A_1 = C_1^{x(r_5, t)}$, then $A = C(0, 0, a_3, a_4)^{x(r_5, t)}$ for $(a_3, a_4) \neq (0, 0)$. All these conclusions are arrived at by calculations similar to those at the end of Theorem 5.2.

Hence Theorem 5.3 is true for $n = 5$ and we now assume it true for any integer r where $5 \leqslant r < n$ and consider the case of $G = B_n(q)$. If $A \in \mathrm{Ab}(U)$ then $|A| \geqslant |B(a_1, a_2, \ldots, a_n)| = q^{n(n-1)/2+1}$, $(A)\varphi = A_1 \leqslant X \times Y$ and $|A| = |A_1|\,|A \cap V|$. By induction and Theorem 5.2 $C \in \mathrm{Ab}(Y)$ implies $|C| = q^{(n-2)(n-3)/2+1}$. Therefore $|A_1| \leqslant q^{(n-2)(n-3)/2+2}$. On the other hand by Lemma 5.4 $|A \cap V| \leqslant q^{2n-2}$ and so $|A_1| \geqslant q^{n(n-1)/2+1-(2n-2)} = q^{(n-2)(n-3)/2}$.

Lemma 5.15. $A_1 \leqslant Y$.

Proof. Suppose not. Then there exists $x \in X$, $y \in Y$ such that $x \neq 1$ and $xy \in A_1$. We claim that there exists $y_1 \in Y \cap A_1$, $y_1 \neq 1$. We have proved similar claims before and we take this claim as true without further ado. Now xy, $y_1 \in A_1$ implies $\dim C_{V'}(A_1) \leqslant \dim C_W(y_1) \leqslant \dim W - 2 = 2n - 5$ using Lemma 3.3 and Lemma 5.5 Thus $|A \cap V| \leqslant q^{2n-4}$.

Now $|A| \geqslant q^{n(n-1)/2+1}$ gives $|A_1| = q^{(n-2)(n-3)/2+2}$ which implies $A_1 = X \times E$ with $E \in \mathrm{Ab}(Y)$. For each possibility for E supplied by Theorem 5.2 and induction we find $\dim C_{V'}(X \times E) \leqslant n - 2$. This gives $|A \cap V| \leqslant q^{n-1}$ and so $|A| \leqslant q^{(n-2)(n-3)/2+2}.q^{n-1} = q^{(n^2-3n+8)/2} < q^{n(n-1)/2+1}$ since $n \geqslant 6$. With this contradiction $A_1 \leqslant Y$ is forced.

Now $A_1 \leqslant Y$ implies $q^{(n-2)(n-3)/2} \leqslant |A_1| \leqslant q^{(n-2)(n-3)/2+1}$.

Lemma. 5.16. *The possibilities for* A_1 *are*:

(a) $B(a_1, a_2, \ldots, a_{n-2})$ *for* $(a_1, a_2, \ldots, a_{n-2}) \neq (0, 0, \ldots, 0)$,

(b) $C(a_1, a_2, \ldots, a_{n-3})$ for $(a_1, a_2, \ldots, a_{n-3}) \neq (0, 0, \ldots, 0)$ *or a conjugate by* $x(r_n, t)$,

(c) $B_1 = \langle X(r) \mid r \geqslant r_{n-1} + 2r_n,\ r \not\geqslant r_2 \rangle$,

(d) $C_1 = \langle X(r), X(s) \mid r \geqslant r_{n-1},\ r \not\geqslant r_n,\ s \geqslant r_{n-2} + 2r_{n-1} + 2r_n,\ r, s \not\geqslant r_2 \rangle$ *or a conjugate by* $x(r_n, t)$ *where the definitions of* $B(a_1, a_2, \ldots, a_{n-2})$ *and* $C(a_1, a_2, \ldots, a_{n-2})$ *are obvious.*

PROOF. The case $q^{(n-2)(n-3)/2} < |A_1| < q^{(n-2)(n-3)/2+1}$ is ruled out in similar fashion to the case $q^3 < |A_1| < q^4$ in Lemma 5.14.

Suppose $|A_1| = q^{(n-2)(n-3)/2+1}$. Then by induction or Theorem 5.2, $|A_1 \times C_W(A_1)| \leqslant q^{(n-1)(n-2)/2+1}$ and so $|C_W(A_1)| \leqslant q^{n-2}$. $|C_W(A_1)| < q^{n-2}$ leads to a contradiction as in Lemma 5.14. Thus $|A_1 \times C_W(A_1)| = q^{(n-1)(n-2)/2+1}$. Induction or Theorem 5.2 gives A_1 is one of (a) or (b). In similar fashion $|A_1| = q^{(n-2)(n-3)/2}$ gives A_1 is one (c) or (d). This completes the proof of Lemma 5.16.

Exactly the same argument as in the case $n = 5$ completes the picture and proves Theorem 5.3.

6. The Thompson subgroup of U

We define the Thompson subgroup $J(U)$ of U as $\langle A \mid A \in \mathrm{Ab}(U)\rangle$.

THEOREM 6.1.

(a) *If* $G = A_{2n+1}(q)$ *then* $J(U) = \langle X(r) \mid r \geqslant {}_{n+1}\rangle$,
(b) *if* $G = A_{2n}(q)$ *then* $J(U) = \langle X(r) \mid r \geqslant r_n$ *or* $r \geqslant r_{n+1}\rangle$,
(c) *if* $G = B_2(2^m)$ *then* $J(U) = U$,
(d) *if* $G = B_2(q)$, q *odd, then* $J(U) = \langle X(r) \mid r \geqslant r_1\rangle$,
(e) *if* $G = B_n(2^m)$, $n \geqslant 3$, *then* $J(U) = \langle X(r) \mid r \geqslant r_n\rangle$,
(f) *if* $G = B_3(q)$, q *odd, then* $J(U) = \langle X(r) \mid r \geqslant r_1\rangle$,
(g) *if* $G = B_4(q)$, q *odd, then* $J(U) = \langle X(r) \mid r \geqslant r_1, r_3$ *or* $r_4\rangle$,
(h) *if* $G = B_n(q)$, q *odd and* $n \geqslant 5$, *then* $J(U) = \langle X(r) \mid r \geqslant r_{n-1}$ *or* $r \geqslant r_n\rangle$,
(i) *if* $G = C_n(q)$, q *odd and* $n \geqslant 3$, *then* $J(U) = \langle X(r) \mid r \geqslant r_n\rangle$,
(j) *if* $G = D_4(q)$, *then* $J(U) = \langle X(r) \mid r \geqslant r_1, r_3$ *or* $r_4\rangle$,
(k) *if* $G = D_n(q)$, $n \geqslant 5$, *then* $J(U) = \langle X(r) \mid r \geqslant r_{n-1}$ *or* $r \geqslant r_n\rangle$.

PROOF. By inspection of the results in Sections 2–5.

COROLLARY. 6.2. *If G is of type A_n, B_n, C_n or D_n then* $J(U) = \langle A \mid A \leqslant U, A$ *abelian of maximal rank*$\rangle$.

PROOF. $\mathrm{Ab}(U)$ in all cases contains elementary abelian subgroups of U. Thus if $A \leqslant U$, A abelian of maximal rank, then $A \in \mathrm{Ab}(U)$. Inspection of the results of Sections 2–5 then completes the proof.

Some have defined $J(U)$ as $\langle A \mid A \leqslant U, A$ abelian of maximal rank$\rangle$ and so Corollary 6.2 assures us that these two definitions yield the same subgroup for the Chevalley groups under consideration here.

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7. Conclusion

We have not considered the case of the twisted Chevalley groups ${}^2A_n(q^2)$ and ${}^2D_n(q^2)$ in this paper. The methods of Section 2 can be used to determine $\mathrm{Ab}(U)$ if $G = {}^2A_{n+1}(q^2)$. In this case $|\mathrm{Ab}(U)| = 1$. If $G = {}^2D_n(2^{2m})$ the methods of Section 4 work even though $|\mathrm{Ab}(U)| \neq 1$. However the solutions for ${}^2A_{2n}(q^2)$, any q, and ${}^2D_n(q^2)$, q odd, demand the introduction of more geometrical methods. Because of this and the fact that the inclusion of the twisted Chevalley groups would mean even more notation we felt that these cases should be presented in a separate paper.

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Department of Mathematics
Carysfort College
Blackrock, Co. Dublin
Ireland.

Author's current address:
21 Vernon Grove
Rathgar, Dublin
Ireland