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Exact Morphism Category and Gorenstein-projective Representations

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Abstract. Let Q be a finite acyclic quiver, let J be an ideal of kQ generated by all arrows in Q, and let A be a finite-dimensional k-algebra. The category of all finite-dimensional representations of (Q, J^2) over A is denoted by $\operatorname{rep}(Q, J^2, A)$. In this paper, we introduce the category $\operatorname{exa}(Q, J^2, A)$, which is a subcategory of $\operatorname{rep}(Q, J^2, A)$ of all exact representations. The main result of this paper explicitly describes the Gorenstein-projective representations in $\operatorname{rep}(Q, J^2, A)$, via the exact representations plus an extra condition. As a corollary, A is a self-injective algebra if and only if the Gorenstein-projective representations are exactly the exact representations of (Q, J^2) over A.

1 Introduction

In algebra representation theory, the representations of a quiver with relations (Q, I) over a field k (if Q has no relations, we take $I = 0$) is one of the fundamental methods for constructing representations. This is equivalent to constructing modules of the path algebra kQ/I. This idea can be extended to the representations of a quiver with relations (Q, I) over an algebra A. This is equivalent to constructing modules of the tensor algebra $A \otimes_k kQ/I$.

On the other hand, Gorenstein-projective modules enjoy more stable properties than the usual projective modules (see [AB]). They are a main ingredient in the relative homological algebra (see [EJ1, EJ2]) and in the representation theory of algebras (see [AR1, AR2, B, GZ, IKM]) and play a central role in the Tate cohomology of algebras (see [AM]). Let $\mathcal{P}(A)$ be the full subcategory of A-mod consisting of projective modules, and let $\mathcal{GP}(A)$ be the full subcategory of A-mod consisting of Gorenstein-projective modules. An important feature is that $\mathcal{GP}(A)$ is a Frobenius category with relative projective-injective objects being projective A-modules, and hence the stable category $\underline{\mathcal{GP}(A)}$ of $\mathcal{GP}(A)$ modulo $\mathcal{P}(A)$ is a triangulated category. By [Hap], the singularity category of Gorenstein algebra A is triangle equivalent to $\underline{\mathcal{GP}(A)}$. Thus, explicitly constructing all the Gorenstein-projective modules is a fundamental problem and will be useful in all of these applications.

For a finite acyclic quiver Q without relations, a field k and a finite-dimensional k-algebra A, the construction of Gorenstein-projective modules over $A \otimes_k kQ$ was described explicitly in [LZ2]. In this paper, we will construct all the Gorenstein-projective modules over $A \otimes_k kQ/J^2$, where (Q, J^2) is a quiver with relations J^2 and

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without multiple arrows, and J is generated by all arrows in Q_1. Let $\Lambda = A \otimes_k kQ/J^2$, where kQ/J^2 is the path algebra of (Q, J^2) over k. We call Λ the path algebra of (Q, J^2) over A. As in the case of $A = k$, Λ-mod is equivalent to the category $\mathrm{rep}(Q, J^2, A)$ of representations of (Q, J^2) over A. This interpretation permits us to introduce the so-called exact representations of (Q, J^2) over A (see Definition 2.1). Let $\mathrm{exa}(Q, J^2, A)$ be the full subcategory of $\mathrm{rep}(Q, J^2, A)$ of exact representations of (Q, J^2) over A. The main result of this paper, Theorem 3.2, explicitly describes all the Gorenstein-projective Λ-modules, via the exact representations of (Q, J^2) over A plus an extra condition. As a corollary, we see that A is self-injective if and only if $\mathcal{GP}(\Lambda) = \mathrm{exa}(Q, J^2, A)$. As another corollary, if A is a self-injective Gorenstein algebra, then $D^b_{sg}(\Lambda) \cong \underline{\mathrm{exa}(Q, J^2, A)}$ (see Corollary 4.2).

2 Exact Representations of a Quiver With Relations (Q, J^2) Over an Algebra A

Throughout this section, k is a field, Q is a finite acyclic quiver with relations J^2 and without multiple arrows, and A is a finite-dimensional k-algebra, where J is an ideal of kQ generated by all arrows in Q_1.

By definition, a representation X of (Q, J^2) over A is a datum

$$X = (X_i, X_{ji}, i, j \in Q_0),$$

where X_i is an A-module for each $i \in Q_0$, $X_{ji}\colon X_j \to X_i$ is an A-map if there is an arrow from j to i; otherwise, X_{ji} vanishes, and $X_{ik}X_{ji} = 0$ whenever there are arrows from j to i and i to k. It is *a finite-dimensional representation* if each X_i is finite-dimensional. We call X_i *the i-th branch* of X. A morphism f from representation X to representation Y is a datum $(f_i,\ i \in Q_0)$, where $f_i\colon X_i \to Y_i$ is an A-map for each $i \in Q_0$, such that for each arrow $\alpha\colon j \to i$, the diagram

$$\begin{array}{ccc} X_j & \xrightarrow{f_j} & Y_j \\ \downarrow{\scriptstyle X_{ji}} & & \downarrow{\scriptstyle Y_{ji}} \\ X_i & \xrightarrow{f_i} & Y_i \end{array} \tag{2.1}$$

commutes. We call f_i *the i-th branch* of f. Denote by $\mathrm{rep}(Q, J^2, A)$ the category of finite-dimensional representations of (Q, J^2) over A. Let $\Lambda = A \otimes_k kQ/J^2$. By the results in [LZ2], we know that Λ-mod $\cong \mathrm{rep}(Q, J^2, A)$.

In the following, if Q_0 is labeled as $1, \dots, n$, then we also write a representation X of (Q, J^2) over A as

$$\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}_{(X_{ji},\ j, i \in Q_0)},$$

and a morphism in $\mathrm{rep}(Q, J^2, A)$ as

$$\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}.$$

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The following is a central notion of this paper.

Definition 2.1 A representation $X = (X_i, X_{ji}, i, j \in Q_0)$ of (Q, J^2) over A is *an exact representation*, or an exact Λ-module, if the following two conditions are satisfied:

(m1) For each $i \in Q_0$, $\sum_{j\in Q_0} \operatorname{Im} X_{ji} = \bigoplus_{j\in Q_0} \operatorname{Im} X_{ji}$.
(m2) For each $j \in Q_0$, if j is a source, then X_{ji} is an injective A-map whenever there is an arrow from j to i. If j is not a source, then $\operatorname{Ker} X_{ji} = \bigoplus_{k\in Q_0} \operatorname{Im} X_{kj}$.

Denote by $\operatorname{exa}(Q, J^2, A)$ the full subcategory of $\operatorname{rep}(Q, J^2, A)$ of exact representations of (Q, J^2) over A. We call $\operatorname{exa}(Q, J^2, A)$ *the exact morphism category of* (Q, J^2) *over* A.

If (Q, J^2) is a quiver in which for any vertex i there is at most one arrow ending at i, then the condition (m1) vanishes. If

$$(Q, J^2) = \underset{n}{\bullet} \longrightarrow \cdots \longrightarrow \underset{1}{\bullet},$$

then an object in $\operatorname{exa}(Q, J^2, A)$ is just an exact sequence with an injective A-map from X_n to X_{n-1}.

Let (Q, J^2) be a finite acyclic quiver with relations J^2 and without multiple arrows, let A be a finite-dimensional algebra, and let $\Lambda = A \otimes_k kQ/J^2$. In the following, we label the vertices of (Q, J^2) as $1, 2, \dots, n$, such that if there is an arrow from j to i, then $j > i$. Denote by $P(i)$ the indecomposable projective kQ/J^2-module at $i \in Q_0$. It is clear that $P(i) \in \operatorname{exa}(Q, J^2, k)$; it follows that $M \otimes_k P(i) \in \operatorname{exa}(Q, J^2, A)$ for $M \in A$-mod. Thus, we have the following functors ($-i$: by taking the i-th branch)

$$- \otimes_k P(i) \colon A\text{-}\operatorname{mod} \longrightarrow \operatorname{exa}(Q, J^2, A), \quad -_i \colon \operatorname{rep}(Q, J^2, A) \longrightarrow A\text{-}\operatorname{mod}.$$

We need the adjoint pair $(- \otimes_k P(i), -_i)$.

Lemma 2.2 *For each object $X = (X_i, X_{ji}, i, j \in Q_0) \in \Lambda$-mod and each A-module M, we have the following isomorphisms of abelian groups, which are natural in both positions*

$$\operatorname{Hom}_\Lambda\big(M \otimes_k P(i), X\big) \cong \operatorname{Hom}_A(M, X_i), \quad i \in Q_0. \tag{2.2}$$

Proof For $f = (f_k, k \in Q_0) \in \operatorname{Hom}_\Lambda(M \otimes_k P(i), X)$, we have $f_i \in \operatorname{Hom}_A(M, X_i)$. Since $M \otimes_k P(i) = (M \otimes_k e_k(kQ/J^2)e_i, \operatorname{id}_M \otimes \alpha, k \in Q_0, \alpha \in Q_1)$, it follows from the commutative diagram (2.1) that

$$f_k = \begin{cases} 0, & \text{if there is no arrow from } i \text{ to } k \\ m \otimes_k \alpha \mapsto X_{ik} f_i(m), & \text{if there is an arrow } \alpha \text{ from } i \text{ to } k. \end{cases} \tag{2.3}$$

By (2.3) we see that $f \mapsto f_i$ gives an injective map

$$\operatorname{Hom}_\Lambda(M \otimes_k P(i), X) \longrightarrow \operatorname{Hom}_A(M, X_i).$$

This map is also surjective, since for a given $f_i \in \operatorname{Hom}_A(M, X_i)$, $f = (f_k, k \in Q_0)$ given by (2.3) is indeed a morphism in $\operatorname{rep}(Q, J^2, A)$ from $M \otimes_k P(i)$ to X. ∎

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Proposition 2.3 *The indecomposable projective Λ-modules have the form $P \otimes_k P(i)$, where P is an indecomposable projective A-module and $P(i)$ is the indecomposable projective kQ/J^2-module at $i \in Q_0$.*

Proof As a direct summand of the regular Λ-module ${}_\Lambda\Lambda$, we see that $P \otimes_k P(i)$ is a projective Λ-module, and each projective Λ-module has this form. By (2.2) we have

$$\operatorname{End}_\Lambda\big(P \otimes_k P(i)\big) \cong \operatorname{Hom}_A\big(P, (P \otimes_k P(i))_i\big) = \operatorname{End}_A(P),$$

from which we see that $P \otimes_k P(i)$ is indecomposable. ∎

3 Gorenstein-projective Modules in $\operatorname{rep}(Q, J^2, A)$

Let A and B be rings, let M be an A-B-bimodule, and let $\Lambda = \left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$ be the upper triangular matrix ring, where the addition and the multiplication are given by those of matrices. We assume that Λ is an Artin algebra ([ARS, p. 72]) and consider finitely generated Λ-modules. A Λ-module can be identified with a triple $\binom{X}{Y}_\phi$, or simply $\binom{X}{Y}$ if ϕ is clear, where $X \in A$-mod, $Y \in B$-mod, and $\phi: M \otimes_B Y \to X$ is an A-map. A Λ-map $\binom{X}{Y}_\phi \to \binom{X'}{Y'}_{\phi'}$ can be identified with a pair $\binom{f}{g}$, where $f \in \operatorname{Hom}_A(X, X')$, $g \in \operatorname{Hom}_B(Y, Y')$, such that the diagram

$$\begin{array}{ccc} M \otimes_B Y & \xrightarrow{\phi} & X \\ {\scriptstyle \mathrm{id} \otimes g}\downarrow & & \downarrow{\scriptstyle f} \\ M \otimes_B Y' & \xrightarrow{\phi'} & X' \end{array}$$

commutes. A sequence of Λ-maps

$$0 \longrightarrow \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}_{\phi_1} \xrightarrow{\binom{f_1}{g_1}} \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}_{\phi_2} \xrightarrow{\binom{f_2}{g_2}} \begin{pmatrix} X_3 \\ Y_3 \end{pmatrix}_{\phi_3} \longrightarrow 0$$

is exact if and only if $0 \longrightarrow X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \longrightarrow 0$ is an exact sequence of A-maps and $0 \longrightarrow Y_1 \xrightarrow{g_1} Y_2 \xrightarrow{g_2} Y_3 \longrightarrow 0$ is an exact sequence of B-maps. Indecomposable projective Λ-modules are exactly $\binom{P}{0}$ and $\binom{M \otimes_B Q}{Q}_{\mathrm{id}}$, where P runs over indecomposable projective A-modules, and Q runs over indecomposable projective B-modules.

Note that an algebra Λ is of the form above if and only if there is an idempotent decomposition $1 = e + f$ such that $f\Lambda e = 0$; and in this case $\Lambda = \left(\begin{smallmatrix} e\Lambda e & e\Lambda f \\ 0 & f\Lambda f \end{smallmatrix}\right)$.

The Gorenstein-projective Λ-modules have been studied in many papers, including [LZ1, LZ2, XZ, Z1, Z2]. In [Z2], Zhang researched $\mathcal{GP}(\Lambda)$ in a more general setup. He described the Gorenstein-projective Λ-modules when ${}_AM_B$ is a compatible A-B-bimodule. For an A-B-bimodule M with $\operatorname{proj.dim} M_B < \infty$, if $\operatorname{proj.dim}_A M < \infty$, then M is compatible.

Theorem 3.1 ([Z2]) *Let M be a compatible A-B-bimodule, and $\Lambda = \left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$. Then $\binom{X}{Y}_\phi \in \mathcal{GP}(\Lambda)$ if and only if $\phi: M \otimes_B Y \to X$ is an injective A-map,* Coker $\phi \in \mathcal{GP}(A)$, *and $Y \in \mathcal{GP}(B)$.*

The aim of this section is to prove the following characterization of Gorenstein-projective Λ-modules, where Λ is the path algebra of a finite acyclic quiver with relations (Q, J^2) over a finite-dimensional algebra A. That is to say, $\Lambda = A \otimes_k kQ/J^2$ and it is not assumed to be Gorenstein.

Theorem 3.2 *Let (Q, J^2) be a finite acyclic quiver with relations J^2 and without multiple arrows, and let A be a finite-dimensional algebra over a field k. Let $\Lambda = A \otimes_k kQ/J^2$, and let $X = (X_i, X_{ji}, i, j \in Q_0)$ be a Λ-module. Then $X \in \mathcal{GP}(\Lambda)$ if and only if $X \in \mathrm{exa}(Q, J^2, A)$ and X satisfies the following condition (Gp), where*

(Gp) *For each $i \in Q_0$, $X_i \in \mathcal{GP}(A)$ and the quotient $X_i/(\bigoplus_{j \in Q_0} \operatorname{Im} X_{ji}) \in \mathcal{GP}(A)$.*

Theorem 3.2 will be proved by using Theorem 3.1 and induction on the number of vertices $|Q_0|$.

Remember that we label Q_0 as $1, \dots, n$, such that $j > i$ if $\alpha\colon j \to i$ is an arrow in Q_1. Thus n is a source of Q. Denote by Q' the quiver obtained from Q by deleting the vertex n, and $\Lambda' = A \otimes_k kQ'/J^2$. Let $P(n)$ be the indecomposable projective (left) kQ/J^2-module at vertex n. Put $P = A \otimes_k \operatorname{rad} P(n)$. Clearly, P is a Λ'-A-bimodule and $\Lambda = \begin{pmatrix} \Lambda' & P \\ 0 & A \end{pmatrix}$.

Since the global dimension of $\Lambda' = kQ/J^2$ is finite, $\operatorname{rad} P(n)$ has finite projective dimension as a Λ'-module, and hence $P = A \otimes_k \operatorname{rad} P(n)$ has finite projective dimension as a left Λ'-module. Since as a right A-module, P is a direct sum of copies of A_A, then P is a right projective A-module. That is to say, M is a compatible A-B-bimodule. So we can apply Theorem 3.1. For this, we write a Λ-module $X = (X_i, X_{ji}, i, j \in Q_0)$ as $X = \binom{X'}{X_n}_\phi$, where $X' = (X_i, X_{ji}, i, j \in Q_0')$ is a Λ'-module, and $\phi\colon P \otimes_A X_n \to X'$ is a Λ'-map. The explicit expression of ϕ will be given in the proof of Lemma 3.4 below. We keep all these notations of Q', Λ', $P(n)$, P, X' and ϕ, throughout this section.

By a direct translation from Theorem 3.1 in this special case, we have the following lemma.

Lemma 3.3 *Let $X = \binom{X'}{X_n}_\phi$ be a Λ-module. Then $X \in \mathcal{GP}(\Lambda)$ if and only if X satisfies the following conditions:*

(i) *$X_n \in \mathcal{GP}(A)$;*
(ii) *$\phi\colon P \otimes_A X_n \to X'$ is an injective Λ'-map;*
(iii) *$\operatorname{Coker} \phi \in \mathcal{GP}(\Lambda')$.*

Lemma 3.4 *Let $X = (X_i, X_{ji}, i, j \in Q_0)$ be a Λ-module. Then X_{ni} is an injective A-map whenever there is an arrow from n to i if and only if $\phi\colon P \otimes_A X_n \to X'$ is an injective Λ'-map.*

Proof For $i \in Q_0'$, let

$$m_i = \begin{cases} 0, & \text{if there is no arrow from } n \text{ to } i, \\ 1, & \text{if there is an arrow from } n \text{ to } i. \end{cases}$$

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As a kQ'/J^2-module, $\operatorname{rad} P(n)$ can be written as $\begin{pmatrix} k^{m_1} \\ \vdots \\ k^{m_{n-1}} \end{pmatrix}$ (when $m_i = 0$, we regard as $k^{m_i} = 0$), hence we have isomorphisms of Λ'-modules

$$P \otimes_A X_n \cong (\operatorname{rad} P(n) \otimes_k A) \otimes_A X_n \cong \operatorname{rad} P(n) \otimes_k X_n \cong \begin{pmatrix} X_n^{m_1} \\ \vdots \\ X_n^{m_{n-1}} \end{pmatrix}.$$

Then ϕ is of the form

$$\begin{pmatrix} \phi_1 \\ \vdots \\ \phi_{n-1} \end{pmatrix} : P \otimes_A X_n \cong \begin{pmatrix} X_n^{m_1} \\ \vdots \\ X_n^{m_{n-1}} \end{pmatrix} \longrightarrow \begin{pmatrix} X_1 \\ \vdots \\ X_{n-1} \end{pmatrix},$$

where $\phi_i = X_{ni}: X_n \to X_i$ if there is an arrow from n to i, and otherwise, $X_n^{mi} = 0$ implies $\phi_i = 0$. So ϕ is injective if and only if X_{ni} is injective whenever there is an arrow from n to i. ■

Remark From the proof of the lemma above, we know that

$$\operatorname{Coker} \phi = (X_i/\operatorname{Im} X_{ni}^{m_i}, \widetilde{X_{ji}}, i, j \in Q_0').$$

where $\widetilde{X_{ji}}: X_j/\operatorname{Im} X_{nj}^{m_j} \to X_i/\operatorname{Im} X_{ni}^{m_i}$ is induced by the A-map $X_{ji}: X_j \to X_i$.

Lemma 3.5 *Let $X = \binom{X'}{X_n}_\phi$ be an exact Λ-module. Then we have*

(i) *ϕ is an injective Λ'-map;*

(ii) *$\operatorname{Coker} \phi$ is an exact Λ'-module.*

Proof By the definition of exact Λ-modules, (i) follows directly from Lemma 3.4.

For (ii), we need to prove the following:

(a) For each $i \in Q_0'$, $\sum_{j \in Q_0'} \operatorname{Im} \widetilde{X_{ji}} = \bigoplus_{j \in Q_0'} \operatorname{Im} \widetilde{X_{ji}}$.

(b) For each source i in Q_0', $\widetilde{X_{ik}}$ is an injective A-map whenever there is an arrow from i to k.

(c) For each $i \in Q_0'$ that is not a source, $\operatorname{Ker} \widetilde{X_{ik}} = \bigoplus_{j \in Q_0'} \operatorname{Im} \widetilde{X_{ji}}$.

For (a), we assume that

$$\sum_{j \in Q_0'} \overline{X_{ji}(x_j)} = 0,$$

where $\overline{X_{ji}(x_j)}$ is the image of $x_j \in X_j$ in $X_i/\operatorname{Im} X_{ni}^{m_i}$. Then

$$\sum_{j \in Q_0'} X_{ji}(x_j) \in \operatorname{Im} X_{ni}^{m_i}.$$

If $m_i = 0$, then $X_{ni}^{m_i} = 0$ and $\sum_{j \in Q_0'} X_{ji}(x_j) = \sum_{j \in Q_0} X_{ji}(x_j)$. Since X is an exact Λ-module, then $X_{ji}(x_j) = 0$ for all possible $j \in Q_0'$. If $m_i = 1$, then $\sum_{j \in Q_0'} X_{ji}(x_j) = X_{ni}(x_n')$ for some $x_n' \in X_n$. Then $\sum_{j \in Q_0} X_{ji}(x_j) = 0$, where $x_n = -x_n'$. Since X is an exact Λ-module, then $X_{ji}(x_j) = 0$ for all possible $j \in Q_0'$. The assertion (a) is proved.

Since i is a source in Q', then there is either an arrow from n to i or i is a source in Q. We prove (b) in the following four cases:

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(1) If there is an arrow from n to i and one from n to k, then

$$\widetilde{X_{ik}}: X_i/\operatorname{Im} X_{ni} \to X_k/\operatorname{Im} X_{nk},$$

which is induced by $X_{ik}: X_i \to X_k$ in X. Since X is an exact Λ-module, then $\operatorname{Im} X_{ik} \cap \operatorname{Im} X_{nk} = 0$. Hence $\operatorname{Ker} \widetilde{X_{ik}} = \{x \in X_i | X_{ik}(x) = 0\}/\operatorname{Im} X_{ni} = \operatorname{Ker} X_{ik}/\operatorname{Im} X_{ni}$. Since X is an exact representation and there is only one arrow in Q ending at i, then $\operatorname{Ker} X_{ik} = \operatorname{Im} X_{ni}$. Hence $\operatorname{Ker} \widetilde{X_{ik}} = 0$. That is to say, $\widetilde{X_{ik}}$ is an injective A-map.

(2) If there is an arrow from n to i and no arrow from n to k, then

$$\widetilde{X_{ik}}: X_i/\operatorname{Im} X_{ni} \to X_k,$$

which is induced by $X_{ik}: X_i \to X_k$ in X. So $\operatorname{Ker} \widetilde{X_{ik}} = \operatorname{Ker} X_{ik}/\operatorname{Im} X_{ni}$. Since there is only one arrow ending at i and X is an exact Λ-module, then $\operatorname{Ker} X_{ik} = \operatorname{Im} X_{ni}$. So $\widetilde{X_{ik}}$ is an injective A-map.

(3) If i is a source in Q and there is an arrow from n to k, then

$$\widetilde{X_{ik}}: X_i \to X_k/\operatorname{Im} X_{nk},$$

which is induced by $X_{ik}: X_i \to X_k$ in X. So $\operatorname{Ker} \widetilde{X_{ik}} = \{x \in X_i | X_{ik}(x) \in \operatorname{Im} X_{nk}\}$. Since $\operatorname{Im} X_{ik} \cap \operatorname{Im} X_{nk} = 0$, then $\operatorname{Ker} \widetilde{X_{ik}} = \operatorname{Ker} X_{ik}$. Since i is source in Q, then $\operatorname{Ker} X_{ik} = 0$. So $\widetilde{X_{ik}}$ is an injective A-map.

(4) If i is a source in Q and there is no arrow from n to k, then $\widetilde{X_{ik}} = X_{ik}$. Since X is an exact Λ-module, then $\widetilde{X_{ik}}$ is an injective A-map.

For (c), since X is a Λ-module, then $\bigoplus_{j\in Q_0'} \operatorname{Im} \widetilde{X_{ji}} \subseteq \operatorname{Ker} \widetilde{X_{ik}}$. Let $x_i \in X_i$ and $\widetilde{X_{ik}}(\overline{x_i}) = 0$, *i.e.*, $X_{ik}(x_i) \in \operatorname{Im} X_{nk}^{m_k}$. If $m_k = 0$, then $X_{ik}(x_i) = 0$, namely, $x_i \in \operatorname{Ker} X_{ik}$. If $m_k = 1$, then $X_{ik}(x_i) = X_{nk}(x_n)$ for some $x_n \in X_n$. So $X_{ik}(x_i) = 0$ which follows from $\operatorname{Im} X_{ik} \cap \operatorname{Im} X_{nk} = 0$. Since X is an exact Λ-module, then $x_i \in \bigoplus_{j\in Q_0} \operatorname{Im} X_{ji}$. Hence $\overline{x_i} \in \bigoplus_{j\in Q_0'} \operatorname{Im} \widetilde{X_{ji}}$. This completes the proof. ∎

Lemma 3.6 *Let* $X = \binom{X'}{X_n}_\phi$ *be an exact* Λ*-module satisfying* (Gp)*. Then* $\operatorname{Coker}\phi$ *satisfies* (Gp), i.e., *for each* $i \in Q_0'$, $X_i/\operatorname{Im} X_{ni}^{m_i}$ *and* $(X_i/\operatorname{Im} X_{ni}^{m_i})/(\bigoplus_{j\in Q_0'} \operatorname{Im} \widetilde{X_{ji}})$ *are Gorenstein-projective modules.*

Proof Following from the short exact sequence

$$0 \longrightarrow \bigoplus_{j\in Q_0} \operatorname{Im} X_{ji} \longrightarrow X_i \longrightarrow X_i \Big/ \Big(\bigoplus_{j\in Q_0} \operatorname{Im} X_{ji}\Big) \longrightarrow 0$$

and that X satisfies (Gp), we know that $\bigoplus_{j\in Q_0} \operatorname{Im} X_{ji}$ is Gorenstein-projective. So $X_i/\operatorname{Im} X_{ni}^{m_i}$ is Gorenstein-projective following from the short exact sequence

$$0 \longrightarrow \Big(\bigoplus_{j\in Q_0} \operatorname{Im} X_{ji}\Big) \Big/ \operatorname{Im} X_{ni}^{mi} \longrightarrow X_i \Big/ \operatorname{Im} X_{ni}^{m_i} \longrightarrow X_i \Big/ \Big(\bigoplus_{j\in Q_0} \operatorname{Im} X_{ji}\Big) \longrightarrow 0.$$

Since for each $i \in Q_0'$,

$$\bigoplus_{j\in Q_0'} \operatorname{Im} \widetilde{X_{ji}} = \Big(\sum_{j\in Q_0'} \operatorname{Im} X_{ji} + \operatorname{Im} X_{ni}^{m_i}\Big) \Big/ \operatorname{Im} X_{ni}^{m_i} = \sum_{j\in Q_0} \operatorname{Im} X_{ji} \Big/ \operatorname{Im} X_{ni}^{m_i},$$

then $(X_i/\operatorname{Im} X_{ni}^{m_i})/(\bigoplus_{j\in Q_0'} \operatorname{Im} \widetilde{X_{ji}}) \cong X_i/(\bigoplus_{j\in Q_0} \operatorname{Im} X_{ji})$ is a Gorenstein-projective module, because X satisfies (Gp). So $\operatorname{Coker}\phi$ satisfies (Gp). ∎

Lemma 3.7 *The sufficiency in Theorem 3.2 holds. That is, if $X = (X_i,\ X_{ji},\ i, j \in Q_0)$ is an exact Λ-module satisfying (Gp), then X is Gorenstein-projective.*

Proof Using induction on $n = |Q_0|$. The assertion clearly holds for $n = 1$. Suppose that the assertion holds for $n-1$ with $n \geq 2$. It suffices to prove that X satisfies conditions (i), (ii), and (iii) of Lemma 3.3.

Condition (i) is contained in (Gp), and condition (ii) follows from Lemma 3.5(i). By Lemma 3.5(ii), Coker ϕ is an exact Λ'-module, and by Lemmas 3.6, we know that Coker ϕ satisfies (Gp). It follows from the inductive hypothesis that condition (iii) in Lemma 3.3 is satisfied. ∎

Proof of Theorem 3.2 By Lemma 3.7, it remains to prove the necessity, *i.e.*, if X is a Gorenstein-projective Λ-module, then X is an exact Λ-module satisfying (Gp). We use induction on $n = |Q_0|$. The assertion is clear for $n = 1$. Suppose that the assertion holds for $n-1$ with $n \geq 2$. We write as $X = \binom{X'}{X_n}_{\phi}$. Then X satisfies conditions (i), (ii), and (iii) in Lemma 3.3.

By condition (ii) and Lemma 3.4 we know that:

(1) X_{ni} is an injective A-map whenever there is an arrow from n to i.

Since Coker $\phi = (X_i/\operatorname{Im} X_{ni}^{m_i}, \widetilde{X_{ji}}, i, j \in Q_0')$ is a Gorenstein-projective Λ'-module, it follows from the inductive hypothesis that the following properties hold:

(2) For each source $i \in Q_0'$, $\widetilde{X_{ik}}$ is injective whenever there is an arrow from i to k in Q'.

(3) For each $i \in Q_0'$ which is not a source, $\sum_{j\in Q_0'} \operatorname{Im} \widetilde{X_{ji}} = \bigoplus_{j\in Q_0'} \operatorname{Im} \widetilde{X_{ji}}$.

(4) For each $i \in Q_0'$ which is not a source, if there is an arrow from i to k, then we have $\operatorname{Ker} \widetilde{X_{ik}} = \bigoplus_{j\in Q_0'} \operatorname{Im} \widetilde{X_{ji}}$.

(5) For each $i \in Q_0'$, $X_i/\operatorname{Im} X_{ni}^{m_i}$ and $(X_i/\operatorname{Im} X_{ni}^{m_i})/\bigoplus_{j\in Q_0'} \operatorname{Im} \widetilde{X_{ji}}$ are Gorenstein-projective A-modules.

Claim 1: For each $i \in Q_0$ which is not a source, $\sum_{j\in Q_0} \operatorname{Im} X_{ji} = \bigoplus_{j\in Q_0} \operatorname{Im} X_{ji}$.

If there is no arrow from n to i, then

$$\widetilde{X_{ji}}\colon X_j/\operatorname{Im} X_{nj}^{m_j} \to X_i$$

with $\operatorname{Im} \widetilde{X_{ji}} = \operatorname{Im} X_{ji}$. So by (3),

$$\sum_{j\in Q_0} \operatorname{Im} X_{ji} = \bigoplus_{j\in Q_0} \operatorname{Im} X_{ji}.$$

If there is an arrow from n to i, then $\widetilde{X_{ji}}\colon X_j/\operatorname{Im} X_{nj}^{m_j} \to X_i/\operatorname{Im} X_{ni}$ with $\operatorname{Im} \widetilde{X_{ji}} = (\operatorname{Im} X_{ji} + \operatorname{Im} X_{ni})/\operatorname{Im} X_{ni}$. Let $\sum_{j\in Q_0} X_{ji}(x_j) = 0$ with $x_j \in X_j$, then $\sum_{j\in Q_0'} X_{ji}(x_j) = -X_{ni}(x_n)$. So $\sum_{j\in Q_0'} \overline{X_{ji}(x_j)} = 0$, where $\overline{X_{ji}(x_j)}$ is the image of $x_j \in X_j$ in $X_i/\operatorname{Im} X_{ni}$. By (3), we have $\overline{X_{ji}(x_j)} = 0$, *i.e.*, $x_j \in \operatorname{Ker} \widetilde{X_{ji}}$. By (4), we have $x_j \in \sum_{k\in Q_0'} \operatorname{Im} X_{kj} + \operatorname{Im} X_{nj}^{m_j}$, namely, $x_j \in \sum_{k\in Q_0} \operatorname{Im} X_{kj}$. Hence there is some $x_k' \in X_k$ such that $x_j \in \sum_{k\in Q_0} X_{kj}(x_k')$. Since X is a Λ-module, $X_{ji}(x_j) = \sum_{k\in Q_0} X_{ji}X_{kj}(x_k') = 0$ for $j \in Q_0'$, moreover, $X_{ni}(x_n) = 0$. This proves Claim 1.

Claim 2: For each source $i \in Q_0$, X_{ik} is an injective A-map whenever there is an arrow from i to k in Q_1.

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If $i \in Q_0$ is a source in Q and $i \neq n$, then by (2)

$$\widetilde{X_{ik}}: X_i \to X_k / \operatorname{Im} X_{nk}^{m_k}$$

is an injective A-map induced by $X_{ik}: X_i \to X_k$. So X_{ik} is injective. Together with (1), we know that Claim 2 is true.

Claim 3: For each $i \in Q_0$ which is not a source, $\operatorname{Ker} X_{ik} = \bigoplus_{j \in Q_0} \operatorname{Im} X_{ji}$.

Since

$$\widetilde{X_{ik}}: X_i \big/ \operatorname{Im} X_{ni}^{mi} \to X_k / \operatorname{Im} X_{nk}^{mk}$$

is induced by $X_{ik}: X_i \to X_k$ in X and by Claim 2, $\operatorname{Im} X_{ik} \bigcap \operatorname{Im} X_{nk}^{mk} = 0$, then $\operatorname{Ker} \widetilde{X_{ik}} = \operatorname{Ker} X_{ik} / \operatorname{Im} X_{ni}^{m_i}$. By (4), we have

$$\operatorname{Ker} \widetilde{X_{ik}} = \Big(\bigoplus_{j \in Q_0'} \operatorname{Im} X_{ji} + \operatorname{Im} X_{ni}^{m_i} \Big) \Big/ \operatorname{Im} X_{ni}^{m_i}.$$

Hence $\operatorname{Ker} X_{ik} = \bigoplus_{j \in Q_0} \operatorname{Im} X_{ji}$.

Claim 4: X satisfies (Gp), namely, for each $i \in Q_0$, X_i and $X_i / \bigoplus_{j \in Q_0} \operatorname{Im} X_{ji}$ are Gorenstein-projective A-modules. Since

$$\begin{aligned}(X_i / \operatorname{Im} X_{ni}^{m_i}) \Big/ \Big(\bigoplus_{j \in Q_0'} \operatorname{Im} \widetilde{X_{ji}} \Big) &\cong (X_i / \operatorname{Im} X_{ni}^{m_i}) \Big/ \Big(\bigoplus_{j \in Q_0} \operatorname{Im} X_{ji} / \operatorname{Im} X_{ni}^{m_i} \Big) \\ &\cong X_i \Big/ \Big(\bigoplus_{j \in Q_0} \operatorname{Im} X_{ji} \Big),\end{aligned}$$

$X_i / (\bigoplus_{j \in Q_0} \operatorname{Im} X_{ji})$ is a Gorenstein-projective A-module by (5). If there is no arrow from n to i, then $X_i / \operatorname{Im} X_{ni}^{m_i} = X_i$ is a Gorenstein-projective A-module by (5). If there is an arrow from n to i, then $X_i / \operatorname{Im} X_{ni}^{m_i} = X_i / \operatorname{Im} X_{ni}$. Since X_n is a Gorenstein-projective A-module and X_{ni} is an injective A-map, then

$$0 \to X_n \to X_i \to X_i / \operatorname{Im} X_{ni} \to 0$$

is a short exact sequence. Note that $\mathcal{G}p(A)$ is closed under extension. So X_i is a Gorenstein-projective A-module for each $i \in Q_0$. Hence, Claim 4 holds.

Summarizing the above claims, we have that X is an exact Λ-module satisfying (Gp). ∎

4 Corollaries

As a consequence of Theorem 3.2 and Proposition 2.3, we have the following characterization of self-injectivity.

Corollary 4.1 *Let A be a finite-dimensional algebra and Q a finite acyclic quiver. Then the following are equivalent:*

(i) *A is self-injective;*
(ii) $\mathcal{GP}(A \otimes_k kQ/J^2) = \operatorname{exa}(Q, J^2, A)$.

Proof (i) ⇒ (ii): If A is self-injective, then every A-module is Gorenstein-projective, and hence (ii) follows from Theorem 3.2.

(ii) ⇒ (i): Take a sink of Q, say vertex 1, and consider $D(A_A) \otimes_k P(1)$. By the definition of exact representations, we know that $D(A_A) \otimes_k P(1) \in \operatorname{exa}(Q, J^2, A)$. By

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(ii), $D(A_A) \otimes_k P(1)$ can be embedded into a projective Λ-module P. So $D(A_A)$ can be embedded into the first branch P_1 of P. Since $D(A_A)$ is an injective A-module, then it is a direct summand of P_1. By Lemma 2.3, we know that P_1 is a projective A-module. This implies that $D(A_A)$ is a projective A-module, namely, A is self-injective. ∎

Let $D^b(\Lambda)$ be the bounded derived category of Λ, and let $K^b(\mathcal{P}(\Lambda))$ be the bounded homotopy category of $\mathcal{P}(\Lambda)$. By definition the singularity category $D^b_{sg}(\Lambda)$ of Λ is the Verdier quotient $D^b(\Lambda)/K^b(\mathcal{P}(\Lambda))$. In [Hap], Happel has proved that if Λ is Gorenstein, then there is a triangle-equivalence $D^b_{sg}(\Lambda) \cong \underline{\mathcal{GP}(\Lambda)}$, where $\underline{\mathcal{GP}(\Lambda)}$ is the stable category of $\mathcal{GP}(\Lambda)$ modulo $\mathcal{P}(\Lambda)$ (see also [Hap, Theorem 4.6]). Note that if A is Gorenstein, then $\Lambda = A \otimes_k kQ/J^2$ is Gorenstein (see [AR2]). So we have the following corollary.

Corollary 4.2 *Let A be a finite-dimensional Gorenstein algebra, and let (Q, J^2) be a finite acyclic quiver with relations J^2 and without multiple arrows. Let $\Lambda = A \otimes_k kQ/J^2$. Then there is a triangle-equivalence $D^b_{sg}(\Lambda) \cong \underline{\mathcal{GP}(\Lambda)}$. In particular, if A is self-injective, then there is a triangle-equivalence $D^b_{sg}(\Lambda) \cong \underline{\operatorname{exa}(Q, J^2, A)}$.*

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