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ORIGINAL RESEARCH PAPER

A new approach in two-dimensional heavy-tailed distributions

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Abstract

We consider a new approach in the definition of two-dimensional heavy-tailed distributions. Specifically, we introduce the classes of two-dimensional long-tailed, of two-dimensional dominatedly varying, and of two-dimensional consistently varying distributions. Next, we define the closure property with respect to two-dimensional convolution and to joint max-sum equivalence in order to study whether they are satisfied by these classes. Further, we examine the joint-tail behavior of two random sums, under generalized tail asymptotic independence. Afterward, we study the closure property under scalar product and two-dimensional product convolution, and by these results, we extended our main result in the case of jointly randomly weighted sums. Our results contained some applications where we establish the asymptotic expression of the ruin probability in a two-dimensional discrete-time risk model.

Keywords: Two-dimensional heavy-tailed distributions; closedness with respect to convolution; joint max-sum equivalence; generalized tail asymptotic independence; ruin probability

1. Introduction

1.1 Preliminaries

The heavy-tailed distributions accurately describe complicated situations. One of the most important applications is related to the risk theory in actuarial science. Although several one-dimensional problems remain still open, the multidimensional case has gained popularity from both theoretical and practical aspects. Especially, with respect to a practical point of view, the modern insurance industry does not operate with a single portfolio.

On this line, there are some recent papers, as, for example, Hu and Jiang (2013), Konstantinides and Li (2016), and Yang and Su (2023). In this direction, we introduce some two-dimensional distribution classes, with heavy tails, that are convenient for calculations and permit direct and consistent generalization of the one-dimensional concepts.

In Subsection 1.2, we remind some basic definitions for one-dimensional heavy-tailed distributions, for easy comparison with the two-dimensional ones. In Section 2, we introduce the closure property with respect to the two-dimensional convolution and the two-dimensional max-sum equivalence. Next, we present some results on these classes of distributions. In Section 3, we estimate the joint-tail asymptotic behavior of two random sums, under a dependence structure that generalizes the tail asymptotic independence, and we establish an asymptotic expression for the ruin probabilities, in a discrete-time two-dimensional risk model without stochastic discount factors. Furthermore in Section 5, we study the closure property of some of new classes with respect to scalar product, and in Section 6, we extended some of our results in Section 4, in the case which we have a common discount factor for the two portfolios. Last but not least, we limited ourselves to the non-negative case, and we study the closure property of new classes with respect to product convolution in two dimensions, and some previous results are extended.

Before passing to the next subsection, we give some notations that we need for the rest of the paper. We denote by $\overline{F} := 1 - F$ the distribution tail, hence $\overline{F}(x) = \mathbf{P}[X > x]$ and holds $\overline{F}(x) > 0$ for any $x \geq 0$, except it is referred to differently. For two positive functions $f(x)$ and $g(x)$, the asymptotic relation $f(x) = o[g(x)]$, as $x \to \infty$ means

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0,$$

the asymptotic relation $f(x) = O[g(x)]$, as $x \to \infty$ holds if

$$\limsup_{x\to\infty} \frac{f(x)}{g(x)} < \infty.$$

and the asymptotic relation $f(x) \asymp g(x)$, as $x \to \infty$ if both $f(x) = O[g(x)]$ and $g(x) = O[f(x)]$. Similarly, for the bivariate functions $f(x,\ y)$, $g(x,\ y)$, the corresponding asymptotic relations hold with $\min\{x,\ y\} \to \infty$, as, for example, $f(x,\ y) = o[g(x,\ y)]$, if it holds

$$\lim_{x\wedge y\to\infty} \frac{f(x,\ y)}{g(x,\ y)} = 0.$$

For a real number x, y, we denote $x^+ := \max\{x, 0\}$, $x \wedge y := \min\{x, y\}$, $x \vee y := \max\{x, y\}$. With bold letters, we denote vectors, and further for the unit and zero vectors, we write $\mathbf{1}$ and $\mathbf{0}$, respectively.

1.2 One-dimensional heavy-tailed distributions

The following properties are to be extended in two dimensions:

(1) For two random variables X_1, X_2 with distributions F_1, F_2, respectively, the distribution of the sum is defined by $F_{X_1+X_2}(x) = \mathbf{P}[X_1 + X_2 \leq x]$ with tail $\overline{F_{X_1+X_2}}(x) = \mathbf{P}[X_1 + X_2 > x]$. If X_1, X_2 are independent, we write $F_1 * F_2$ instead of $F_{X_1+X_2}$.

(2) We say that the random variables X_1, X_2 or their distributions F_1, F_2 are max-sum equivalent if $\overline{F_1 * F_2}(x) \sim \overline{F}_1(x) + \overline{F}_2(x)$, as $x \to \infty$. (In some cases, the max-sum equivalence is extended also to $\overline{F}_{X_1+X_2}(x) \sim \overline{F}_1(x) + \overline{F}_2(x)$, for weakly dependent random variables X_1, X_2).

Now we consider some classes of heavy-tailed distributions. We say that a distribution F is heavy-tailed, and we write $F \in \mathcal{K}$, if it holds

$$\int_{-\infty}^{\infty} e^{\varepsilon\, x}\, F(dx) = \infty,$$

for any $\varepsilon > 0$. A large enough class of heavy-tailed distributions is the class of long tails, denoted by $\mathcal{L}$. We have $F \in \mathcal{L}$ if it holds

$$\lim_{x\to\infty} \frac{\overline{F}(x-a)}{\overline{F}(x)} = 1,$$

for any (or, equivalently, for some) $a > 0$. It is well-known that if $F \in \mathcal{L}$, then there exists a function $a : [0,\ \infty) \longrightarrow [0,\ \infty)$, such that $a(x) \to \infty$, $\overline{F}(x \pm a(x)) \sim \overline{F}(x)$, as $x \to \infty$. This kind of function $a(x)$ is called an insensitivity function for F; see further in Cline and Samorodnitsky (1994), Foss et al. (2013), or Konstantinides (2018).

A little smaller class than $\mathcal{L}$ is the class of subexponential distributions, introduced in Chistyakov (1964). We say that a distribution F with support on the interval $[0,\ \infty)$ belongs to

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the class of subexponential distributions, symbolically $F \in \mathcal{S}$ if it holds

$$\lim_{x\to\infty} \frac{\overline{F^{n*}}(x)}{\overline{F}(x)} = n,$$

for any $n \in \mathbb{N}$, where F^{n*} represents the n-th order convolution power for F. The class $\mathcal{S}$ has found several applications in the risk models, as, for example, in Li et al. (2010), Geng et al. (2023), and Ji et al. (2023).

We say that the distribution F belongs to the class of the dominatedly varying distributions, symbolically $F \in \mathcal{D}$, if it holds

$$\limsup_{x\to\infty} \frac{\overline{F}(b\,x)}{\overline{F}(x)} < \infty,$$

for some (or equivalently, for all) $b \in (0,\ 1)$. It is well known that $\mathcal{D} \cap \mathcal{L} = \mathcal{D} \cap \mathcal{S} \subset \mathcal{K}$; see Goldie (1978,Th. 1).

Further, a smaller class of heavy-tailed distributions represents the class of consistently varying distributions, symbolically $F \in \mathcal{C}$. We say that $F \in \mathcal{C}$, if it holds

$$\lim_{y\uparrow 1} \limsup_{x\to\infty} \frac{\overline{F}(y\,x)}{\overline{F}(x)} = 1,$$

or equivalently

$$\lim_{y\downarrow 1} \liminf_{x\to\infty} \frac{\overline{F}(y\,x)}{\overline{F}(x)} = 1.$$

Finally, we say that a distribution F belongs to the class of regularly varying distributions, with index $\alpha > 0$, symbolically $F \in \mathcal{R}_{-\alpha}$ if it holds

$$\lim_{x\to\infty} \frac{\overline{F}(t\,x)}{\overline{F}(x)} = t^{-\alpha},$$

for any $t > 0$.

For these classes, we obtain the following inclusions (see Bingham et al., 1987; Leipus et al., 2023):

$$\mathcal{R} := \bigcup_{\alpha\geq 0} \mathcal{R}_{-\alpha} \subsetneq \mathcal{C} \subsetneq \mathcal{D} \cap \mathcal{L} \subsetneq \mathcal{S} \subsetneq \mathcal{L} \subsetneq \mathcal{K},$$

where $\mathcal{R}_0$ is the class of slowly varying distributions. We can find numerous classes of heavy-tailed distributions; however, we mentioned the most popular in the literature. In this paper, we extend into two dimensions the classes $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{L}$.

In Cai and Tang (2004), we find the following results.

Proposition 1.1. *If $F_1 \in \mathcal{D}$ and $F_2 \in \mathcal{D}$ are distributions with support on the interval $[0,\ \infty)$, then $F_{X_1+X_2} \in \mathcal{D}$.*

In Proposition 1.1 we find that for non-negative random variables, the class $\mathcal{D}$ satisfies the closure property with respect to sum. As was mentioned in Cai and Tang (2004), the class $\mathcal{D}$ does NOT satisfy the max-sum equivalence, as it follows from the fact that $\mathcal{D} \not\subset \mathcal{S}$ and $\mathcal{S} \not\subset \mathcal{D}$; therefore, the relation $\overline{F^{2*}}(x) \sim 2\,\overline{F}(x)$, as $x \to \infty$, does NOT hold for $F \in \mathcal{D} \setminus \mathcal{S}$. In opposite to the dominated variation, the class of the consistently varying distributions satisfies both these properties.

Proposition 1.2. *If $F_1 \in \mathcal{C}$ and $F_2 \in \mathcal{C}$ are distributions with support on the interval $[0,\ \infty)$, then it holds $F_1 * F_2 \in \mathcal{C}$ and $\overline{F_1 * F_2}(x) \sim \overline{F}_1(x) + \overline{F}_2(x)$, as $x \to \infty$.*

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2. Two-dimensional heavy tails

The reason why the multivariate distributions have been so popular is their ability to describe better multidimensional phenomena. This happens because of the interdependence among the components of the random vectors, which affect significantly the final outcome.

The first heavy-tailed distributions class that was extended to a multidimensional frame is the regular variation. We say that the random vector $\mathbf{X} = (X_1, \ldots, X_d)$ represents a multivariate regularly varying vector with index α and non-degenerate, Radon measure ν, symbolically $\mathbf{X} \in MRV(\alpha, F, \nu)$ if it holds

$$\lim_{x\to\infty} \frac{1}{\overline{F}(x)} \mathbf{P}\left[\frac{\mathbf{X}}{x} \in \mathbb{B}\right] = \nu(\mathbb{B}),$$

for any ν-continuous Borel set $\mathbb{B} \subset [0, \infty]^d \setminus \{\mathbf{0}\}$, with $F \in \mathcal{R}_{-\alpha}$. The measure ν is homogeneous; namely, it holds $\nu(\lambda\,\mathbb{B}) = \lambda^{-\alpha}\,\nu(\mathbb{B})$, for any $\lambda > 0$.

The frame of multivariate regular variation was introduced in De Haan and Resnick (1981). Under this definition, the multivariate regular variation was used in the study of several issues in multivariate risk models and in risk management, as, for example, in Li (2016), Tang and Yang (2019), and Yang and Su (2023).

Although this kind of extension to multidimensional setup is well-established, it does not happen to other multidimensional distribution classes. Most of the extensions cover the multivariate subexponential distribution class and the multivariate long-tailed distribution class.

Initially, these two distribution classes were introduced in Cline and Resnick (1992) as essential extension of the multivariate regular variation, namely, using vague convergence and point processes. Later, in Omey (2006), three different formulations appear for the multivariate subexponentiality and the multivariate long-tailedness. The formulations, which are close to our definitions, are given in classes $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{L}(\mathbb{R}^d)$. We say that the multivariate distribution F belongs to class $\mathcal{S}(\mathbb{R}^d)$, if it holds

$$\lim_{x\to\infty} \frac{\overline{F^{2*}}(\mathbf{t}\,x)}{\overline{F}(\mathbf{t}\,x)} = 2,$$

for any $\mathbf{t} > \mathbf{0}$, with $\min_{1\leq i\leq d}\{t_i\} < \infty$, and that the multivariate distribution F belongs to class $\mathcal{L}(\mathbb{R}^d)$, if it holds

$$\lim_{x\to\infty} \frac{\overline{F}(\mathbf{t}\,x - \mathbf{a})}{\overline{F}(\mathbf{t}\,x)} = 1,$$

for any $\mathbf{a} \geq \mathbf{0}$ and for any $\mathbf{t} > \mathbf{0}$, with $\min_{1\leq i\leq d}\{t_i\} < \infty$.

This approach was used to study the asymptotic behavior of the tail of a randomly stopped sum of random vectors, namely, $S_N = \sum_{i=1}^N \mathbf{X}_i$, where N is a discrete random variable with support $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and the $\mathbf{X}_i$ are independent, identically distributed random vectors with multivariate distribution F. For applications of this class, see Omey et al. (2006).

Finally, another formulation of multivariate subexponential distributions was provided in Samorodnitsky and Sun (2016), which represents the only approach with results for the ruin probability in a multivariate continuous-time risk model. Although the approach by Samorodnitsky and Sun (2016) is clearly stronger than the previous two, it describes in some sense the linear multivariate single big jump, but it cannot cover the distributions through their joint tail; see Konstantinides and Passalidis (2024, Sec. 5) for comments about this approach, indicating the complementary function to Samorodnitsky and Sun (2016) of our approach, found below. In the present paper, we confine ourselves to the two dimensions, and we stay close to the formulation in Omey (2006); however, we keep two important differences.

First, we follow a direct approach to the one-dimensional distribution classes' definitions.

Second, in the case of $d=2$, the formulation in Omey (2006), and in the definition of multivariate regular variation, the convention $\mathbf{F}(x,\ y)=\mathbf{P}[X\leq x,\ Y\leq y]$ is adopted, and the distribution tail $1-\mathbf{F}(x,\ y)$, denoted $\overline{\mathbf{F}}(x,\ y)$, is applied on the event $\{X>x\}\cup\{Y>y\}$. We consider only the case in which there exist excesses of both random variables $\{X>x\}\cap\{Y>y\}$; namely, we define by $\overline{\mathbf{F}}_1(x,\ y):=\mathbf{P}[X>x,\ Y>y]$, as the distribution tail of $\mathbf{F}$, with notation $\overline{\mathbf{F}}_b(x,\ y):=\mathbf{P}[X>b_1\,x,\ Y>b_2\,y]$, for all $\mathbf{b}=(b_1,\ b_2)\in(0,\ \infty)^2$. The choice of such a definition is due to both the consistency with the univariate case and the ease in asymptotic calculation of the joint tail of random sums as well. We intend that our approach becomes more consistent with the ruin of all portfolios, which represents the worst event that can happen for an insurance company with multiple businesses. In some sense, this is the reason why our classes lead to a nonlinear approach of the single big jump in multidimensional setup.

Next, we introduce the first bivariate heavy-tailed distribution class. From now on and further by the notation $\mathbf{a}=(a_1,\ a_2)>(0,\ 0)$, we mean that $(a_1,\ a_2)\in[0,\ \infty)^2\setminus\{\mathbf{0}\}$, except it is referred to differently.

Definition 2.1. *We say that the random pair $(X,\ Y)$ with marginal distributions F, G belongs to the bivariate long-tailed distributions, symbolically $(X,\ Y)\in\mathcal{L}^{(2)}$, if the following conditions hold*

(1) *$F\in\mathcal{L}$ and $G\in\mathcal{L}$.*

(2) *It holds*

$$\lim_{x\wedge y\to\infty}\frac{\overline{\mathbf{F}}_1(x-a_1,\ y-a_2)}{\overline{\mathbf{F}}_1(x,\ y)}=\lim_{x\wedge y\to\infty}\frac{\mathbf{P}[X>x-a_1,\ Y>y-a_2]}{\mathbf{P}[X>x,\ Y>y]}=1,$$

for some, or equivalently for any, $\boldsymbol{a}=(a_1,\ a_2)>(0,0)$, with a_1 not necessarily equal to a_2.

Remark 2.1. *From the previous definition we wonder if by the two-dimensional property of class $\mathcal{L}^{(2)}$ follows directly the inclusion $F,\ G\in\mathcal{L}$. The answer to this question is no because it holds for any, or equivalently for some, $(a_1,\ a_2)>(0,\ 0)$, as follows from Definition 2.1.*

Let $F\in\mathcal{L}$ be a distribution and G be another distribution, not necessarily from class $\mathcal{L}$. We assume that the two distributions stem from the independent random variables X and Y; thus, if $a_1>0$ and $a_2=0$, we find that

$$\lim_{x\wedge y\to\infty}\frac{\mathbf{P}[X>x-a_1,\ Y>y]}{\mathbf{P}[X>x,\ Y>y]}=\lim_{x\wedge y\to\infty}\frac{\overline{F}(x-a_1)\,\overline{G}(y)}{\overline{F}(x)\,\overline{G}(y)}=1,$$

however, if it holds $G\notin\mathcal{L}$, then we have not this pair in the class $\mathcal{L}^{(2)}$.

The reason why we require that the marginals belong to class $\mathcal{L}$ is to secure some two-dimensional closure properties that could fail if the $\mathcal{L}$ condition is missing.

From Definition 2.1 we obtain that if $(F,\ G)\in\mathcal{L}^{(2)}$, then for any $(A_1,\ A_2)>(0,\ 0)$, it holds

$$\sup_{|a_1|<A_1,\,|a_2|<A_2}\left|\mathbf{P}[X>x-a_1,\ Y>y-a_2]-\mathbf{P}[X>x,\ Y>y]\right|=o\left(\mathbf{P}[X>x,\ Y>y]\right),\quad(2.1)$$

as $x\wedge y\to\infty$, which follows from the uniformity of the convergence

$$\lim_{x\wedge y\to\infty}\frac{\mathbf{P}[X>x-a_1,\ Y>y-a_2]}{\mathbf{P}[X>x,\ Y>y]}=1,$$

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over the parallelogram $[-A_1, A_1] \times [-A_2, A_2]$. Indeed, for $-A_1 \leq a_1 \leq A_1$ and $-A_2 \leq a_2 \leq A_2$, we obtain $x - A_1 \leq x + a_1 \leq x + A_1$ and $y - A_2 \leq y + a_2 \leq y + A_2$. Hence,

$$\frac{\mathbf{P}[X > x - A_1,\ Y > y - A_2]}{\mathbf{P}[X > x,\ Y > y]} \geq \frac{\mathbf{P}[X > x + a_1,\ Y > y + a_2]}{\mathbf{P}[X > x,\ Y > y]} \geq \frac{\mathbf{P}[X > x + A_1,\ Y > y + A_2]}{\mathbf{P}[X > x,\ Y > y]},$$

where the first fraction tends to unity, as $x \wedge y \to \infty$, by Definition 2.1, and the last fraction also tends to unity, as $x \wedge y \to \infty$, after the change of variables $x' = x + A_1$ and $y' = y + A_2$ and by Definition 2.1.

Definition 2.2 provides the insensitivity property in joint distributions; see the univariate analogue, for example, in Foss et al. (2013) or in Konstantinides (2018).

Definition 2.2. *Let $a_F(x),\ a_G(y) > 0$ for any $x > 0,\ y > 0$ be two non-decreasing function. We say that the joint distribution $\boldsymbol{F} = (F,\ G)$ of $(X,\ Y)$, with right endpoint $r_{\boldsymbol{F}} := (r_F,\ r_G) = (\infty,\ \infty)$, satisfies $(a_F,\ a_G)$-joint insensitivity, if*

$$\sup_{|a_1| \leq a_F(x),\, |a_2| \leq a_G(y)} \left|\mathbf{P}[X > x - a_1, Y > y - a_2] - \mathbf{P}[X > x, Y > y]\right| = o\left(\mathbf{P}[X > x, Y > y]\right),$$

as $x \wedge y \to \infty$.

Now we show that class $\mathcal{L}^{(2)}$ satisfies the $(a_F,\ a_G)$-joint insensitive property.

Lemma 2.1. *Let assume that $(X,\ Y) \in \mathcal{L}^{(2)}$. Then there exist some functions $a_F(x),\ a_G(y)$ such that $a_F(x) \to \infty$ and $a_G(y) \to \infty$, as $x \wedge y \to \infty$, and $(F,\ G)$ satisfies the $(a_F,\ a_G)$-joint insensitive property.*

Proof. For any integer $n \in \mathbb{N}$, from relation (2.1), we can choose an increasing to infinity sequence $\{u_n\}$, such that the inequality

$$\sup_{|a_1| \leq n,\, |a_2| \leq n} \left|\mathbf{P}[X > x - a_1,\ Y > y - a_2] - \mathbf{P}[X > x,\ Y > y]\right| \leq \frac{\mathbf{P}[X > x,\ Y > y]}{n},$$

holds for any $x \geq u_n$ and any $y \geq u_n$. Without loss of generality, we consider that the sequence $\{u_n\}$ increases to infinity. We put $a_F(x) = a_G(y) = n$, for any $(x,\ y) \in (u_n,\ u_{n+1}]^2$. From the fact that $u_n \to \infty$, as $n \to \infty$, we obtain that $a_F(x) \to \infty$, as $x \to \infty$, and $a_G(y) \to \infty$, as $y \to \infty$.

So, from the construction of $a(\cdot)$, we conclude that

$$\sup_{|a_1| \leq a_F(x),\, |a_2| \leq a_G(y)} \left|\mathbf{P}[X > x - a_1,\ Y > y - a_2] - \mathbf{P}[X > x,\ Y > y]\right| \leq \frac{\mathbf{P}[X > x,\ Y > y]}{n},$$

for any $x > u_n$ and any $y > u_n$, which is the required result. □

Remark 2.2. *From the $(a_F,\ a_G)$-joint insensitivity, it does not follow necessarily that a_F and a_G are insensitivity functions for the marginal distributions $F,\ G$, respectively. Furthermore, Lemma 2.1 asserts that*

$$\lim_{x \wedge y \to \infty} \frac{\mathbf{P}[X > x \pm a_F(x),\ Y > y \pm a_G(y)]}{\mathbf{P}[X > x,\ Y > y]} = 1.$$

Let us see now two examples that help either to understanding or to constructing of such bivariate distributions. The first case is the simplest, as we construct $(X,\ Y) \in \mathcal{L}^{(2)}$ through the independence between X and Y.

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Example 2.1. *Let X and Y be random variables with distributions $F \in \mathcal{L}$ and $G \in \mathcal{L}$, respectively. We assume that X and Y are independent, to obtain*

$$\lim_{x\wedge y\to\infty} \frac{\overline{\mathbf{F}}_1(x-a_1,\ y-a_2)}{\overline{\mathbf{F}}_1(x,\ y)} = \lim_{x\wedge y\to\infty} \frac{\mathbf{P}[X > x-a_1,\ Y > y-a_2]}{\mathbf{P}[X > x,\ Y > y]}$$
$$= \lim_{x\wedge y\to\infty} \frac{\mathbf{P}[X > x-a_1]}{\mathbf{P}[X > x]}\,\frac{\mathbf{P}[Y > y-a_2]}{\mathbf{P}[Y > y]} = 1.$$

Therefore $(X,\ Y) \in \mathcal{L}^{(2)}$.

The next example makes sense, as it cannot be reduced into univariate distributions. The following dependence structure can be found in Li (2018). We say that the random variables X and Y are strongly asymptotic independent (SAI) if $\mathbf{P}[X^- > x,\ Y > y] = O[F(-x)\,\overline{G}(y)]$, $\mathbf{P}[X > x,\ Y^- > y] = O[\overline{F}(x)\,G(-y)]$ hold as $x \wedge y \to \infty$, and there exists a constant $C > 0$ such that if it holds

$$\mathbf{P}[X > x,\ Y > y] \sim C\,\overline{F}(x)\,\overline{G}(y), \tag{2.2}$$

as $x \wedge y \to \infty$.

If the X and Y are bounded from below, then (2.2) is enough to be SAI.

Example 2.2. *Let X and Y be random variables with strongly asymptotic independence, with some constant $C > 0$ and distributions $F \in \mathcal{L}$ and $G \in \mathcal{L}$, respectively. Then*

$$\lim_{x\wedge y\to\infty} \frac{\overline{\mathbf{F}}_1(x-a_1,\ y-a_2)}{\overline{\mathbf{F}}_1(x,\ y)} = \lim_{x\wedge y\to\infty} \frac{\mathbf{P}[X > x-a_1,\ Y > y-a_2]}{\mathbf{P}[X > x,\ Y > y]}$$
$$= \lim_{x\wedge y\to\infty} \frac{C\,\overline{F}(x-a_1)\,\overline{G}(y-a_2)}{C\,\overline{F}(x)\,\overline{G}(y)} = 1.$$

Therefore $(X,\ Y) \in \mathcal{L}^{(2)}$.

The first two examples restrict themselves either in the independent case or in some kind of asymptotic independence. Notice that in the next example as class $\mathcal{L}^{(2)}$, we understand the class from Definition 2.1, but with a restriction with respect to convergence, instead of $x \wedge y$ to $x = y$ only. In Li and Yang (2015), the dependence structure from relation (2.3) was used, through the survival copula $\widehat{C}$, to depict the dependence relation among claims in a bivariate, continuous-time risk model. We assume that for two random variables $X,\ Y$ following a survival copula $\widehat{C}$, there exists some constant $\gamma \geq 1$ and a positive measurable function $h(\cdot,\ \cdot)$, such that the asymptotic relation holds

$$\widehat{C}(t_1\,x,\ t_2\,x) \sim x^{\gamma}\,h(t_1,\ t_2), \tag{2.3}$$

as $x \downarrow 0$ holds, for any $(t_1,\ t_2) \in (0,\ \infty)$.

Example 2.3. *Let the random variables $X,\ Y$ follow a survival copula from relation (2.3) and $F,\ G$ be their marginal distributions. Furthermore, we assume that it holds*

$$\lim_{x\to\infty} \frac{\overline{G}(x)}{\overline{F}(x)} = c, \tag{2.4}$$

for some positive constant $c > 0$ and either $F \in \mathcal{L}$ or $G \in \mathcal{L}$ is true. Finally, we suppose that relation (2.3) holds with $\gamma = 1$. Then we obtain $F,\ G \in \mathcal{L}$, which follows from the closure property of class

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$\mathcal{L}$ with respect to strong equivalence of (2.4); see Leipus et al. (2023). From Li and Yang (2015, Prop. 3.1), we have the random variables X, Y to be asymptotic dependent, and further, they satisfy

$$\lim_{x\to\infty} \frac{\mathbf{P}[X > x,\ Y > x]}{\mathbf{P}[X > x]} = h(1,\ c) > 0,$$

hence by the last formulas, for any $(a_1,\ a_2) > (0,\ 0)$, it holds

$$\lim_{x\to\infty} \frac{\mathbf{P}[X > x - a_1,\ Y > x - a_2]}{\mathbf{P}[X > x,\ Y > x]} = \lim_{x\to\infty} \frac{h(1,\ c)\,\mathbf{P}[X > x - a_1]}{h(1,\ c)\,\mathbf{P}[X > x]} = 1,$$

so we find $(X,\ Y) \in \mathcal{L}^{(2)}$, in the sense that in Definition 2.1, the convergence is valid with $x = y$.

We can find several dependence structures that satisfy the $\mathcal{L}^{(2)}$ condition. However, we choose to pursue theoretical results.

Now we pass to the bivariate subexponential distribution class $\mathcal{S}^{(2)}$.

Definition 2.3. *We say that the random pair $(X,\ Y)$, with marginal distributions F and G, respectively, belongs to the class of bivariate subexponential distributions, symbolically $(X,\ Y) \in \mathcal{S}^{(2)}$, if*

(1) $F \in \mathcal{S}$ and $G \in \mathcal{S}$.
(2) $(X,\ Y) \in \mathcal{L}^{(2)}$.
(3) It holds

$$\lim_{x\wedge y\to\infty} \frac{\mathbf{P}[X_1 + X_2 > x,\ Y_1 + Y_2 > y]}{\mathbf{P}[X > x,\ Y > y]} = 2^2, \tag{2.5}$$

where $(X_1,\ Y_1)$ and $(X_2,\ Y_2)$ are independent and identically distributed copies of $(X,\ Y)$.

Remark 2.3. *In case of d-variate distribution, relation (2.5) becomes*

$$\lim_{x_1\wedge\ldots\wedge x_n \text{ to } \infty} \frac{\mathbf{P}[X_{1,1} + X_{1,2} > x_1,\ \ldots,\ X_{d,1} + X_{d,2} > x_d]}{\mathbf{P}[X_{1,1} > x_1,\ \ldots,\ X_{d,1} > x_d]} = 2^d.$$

Conjecture 2.1. *In Definition 2.3, we suppose that the* (1), (3) *do NOT imply directly the property* (2) *and the membership in $\mathcal{L}^{(2)}$. Although it is not proved, we consider that this conjecture could be established through a special counterexample, in which the* (1), (3) *are satisfied, and the $(X,\ Y)$ satisfy properties of some special kind of copulas that belong to SAI in (2.2), but now with $C = 0$; see Li (2018b), Ji et al. (2023), and Li (2024) for examples of such dependence through copulas.*

Now we come to the bivariate dominatedly varying distribution class $\mathcal{D}^{(2)}$.

Definition 2.4. *We say that the random pair $(X,\ Y)$, with marginal distributions F and G, respectively, belongs to the class of bivariate dominatedly varying distributions, symbolically $(X,\ Y) \in \mathcal{D}^{(2)}$, if*

(1) $F \in \mathcal{D}$ and $G \in \mathcal{D}$.
(2) It holds

$$\limsup_{x\wedge y\to\infty} \frac{\overline{\mathbf{F}}_b(x,\ y)}{\overline{\mathbf{F}}_1(x,\ y)} = \limsup_{x\wedge y\to\infty} \frac{\mathbf{P}[X > b_1\,x,\ Y > b_2\,y]}{\mathbf{P}[X > x,\ Y > y]} < \infty, \tag{2.6}$$

for some, or equivalently for all $\boldsymbol{b} = (b_1,\ b_2) \in (0,\ 1)^2$, with b_1 not necessarily equal to b_2.

It is obvious that 2.6 is equivalently with:

$$\liminf_{x\wedge y\to\infty} \frac{\overline{\mathbf{F}}_b(x,\ y)}{\overline{\mathbf{F}}_1(x, y)} > 0$$

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for some, or equivalently for all $\mathbf{b} = (b_1,\ b_2) \in (1,\ \infty)^2$, with b_1 not necessarily equal to b_2.

Remark 2.4. *In Konstantinides and Passalidis (2024b), the class $\mathcal{D}_n$ (for some $n \in \mathbb{N}$) of multivariate dominatedly varying random vectors was introduced. It is obvious that in case $n = 2$, our approach includesthis definition. Specifically*

$$\mathcal{D}_2 \subset \mathcal{D}^{(2)}$$

Definition 2.5. *We say that the random pair $(X,\ Y)$, with marginal distributions F and G, respectively, belongs to the class of bivariate consistently varying distributions, symbolically $(X,\ Y) \in \mathcal{C}^{(2)}$, if*

(1) $F \in \mathcal{C}$ and $G \in \mathcal{C}$.
(2) It holds

$$\lim_{\mathbf{z} \uparrow \mathbf{1}} \limsup_{x \wedge y \to \infty} \frac{\overline{\mathbf{F}}_{\mathbf{z}}(x,\ y)}{\overline{\mathbf{F}}_{\mathbf{1}}(x,\ y)} = 1,$$

or equivalently

$$\lim_{\mathbf{z} \downarrow \mathbf{1}} \liminf_{x \wedge y \to \infty} \frac{\overline{\mathbf{F}}_{\mathbf{z}}(x,\ y)}{\overline{\mathbf{F}}_{\mathbf{1}}(x,\ y)} = 1,$$

where $\mathbf{z} = (z_1,\ z_2)$, and $\mathbf{1} = (1,\ 1)$.

Examples 2.1 and 2.2 remain intact in classes $\mathcal{D}^{(2)}$ and $\mathcal{C}^{(2)}$; hence, they keep functioning in class $(\mathcal{D} \cap \mathcal{L})^{(2)} := \mathcal{D}^{(2)} \cap \mathcal{L}^{(2)}$.

Theorem 2.1. *It holds $\mathcal{C}^{(2)} \subsetneq \mathcal{L}^{(2)}$.*

Proof. Let consider that $(F,\ G) \in \mathcal{C}^{(2)}$. Then, for $\mathbf{a} = (a_1,\ a_2) > (0,\ 0)$ for any distributions $F,\ G$, we obtain

$$1 \le \liminf_{x \wedge y \to \infty} \frac{\mathbf{P}[X > x - a_1,\ Y > y - a_2]}{\mathbf{P}[X > x,\ Y > y]}. \tag{2.7}$$

Hence, we have to show that the upper bound of the last fraction is equal to unity. We observe that for any small enough $\delta_1,\ \delta_2 > 0$, there exist some $x_0 > 0$, such that $x\,(1 - \delta_1) \le x - a_1$ and $y\,(1 - \delta_2) \le y - a_2$, for any $x \wedge y \ge x_0$. Therefore, we find

$$\limsup_{x \wedge y \to \infty} \frac{\mathbf{P}[X > x - a_1, Y > y - a_2]}{\mathbf{P}[X > x, Y > y]} \le \limsup_{x \wedge y \to \infty} \frac{\mathbf{P}[X > x(1 - \delta_1),\ Y > y(1 - \delta_2)]}{\mathbf{P}[X > x,\ Y > y]} \to 1, \tag{2.8}$$

as $(\delta_1,\ \delta_2) \to (0,\ 0)$, where in the last step, we use the properties of class $\mathcal{C}^{(2)}$ for the pair of distributions $(F,\ G)$. So, by relations (2.7) and (2.8), we conclude that $(X,\ Y) \in \mathcal{L}^{(2)}$. □

3. Max-sum equivalence and closure properties with respect to convolution

Now, we present two definitions. In the first one, we define the closure property with respect to convolution in bivariate distributions. In this case, we formulate the main result, showing that the class $\mathcal{D}^{(2)}$ is closed. The second definition, given at the end of the section, under concrete dependence structures, also presented later, is fulfilled with respect to classes $(\mathcal{D} \cap \mathcal{L})^{(2)}$ and $\mathcal{C}^{(2)}$.

Definition 3.1. *Let $X_1,\ X_2,\ Y_1,\ Y_2$ be random variables, with distributions $F_1,\ F_2,\ G_1$ and G_2, respectively. If the following conditions are true*

(1) $F_1 \in \mathcal{B}$, $F_2 \in \mathcal{B}$, $G_1 \in \mathcal{B}$, $G_2 \in \mathcal{B}$ and for any $k,\ l \in \{1,\ 2\}$, holds $(X_k,\ Y_l) \in \mathcal{B}^{(2)}$,
(2) Holds $(X_1 + X_2,\ Y_1 + Y_2) \in \mathcal{B}^{(2)}$,

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where $\mathcal{B}^{(2)}$ is some bivariate class, defined in Section 2, then we say that the class $\mathcal{B}^{(2)}$ is closed with respect to sum. If X_1, X_2 are independent random variables, and Y_1, Y_2 are also independent, then we say that $\mathcal{B}^{(2)}$ is closed with respect to convolution, symbolically $(F_1 * F_1,\ G_1 * G_2) \in \mathcal{B}^{(2)}$.

In the last definition, although the check of $F_{X_1+X_2} \in \mathcal{B}$, $G_{Y_1+Y_2} \in \mathcal{B}$ is implied directly by the univariate closure properties, the check of $(F_k,\ G_l) \in \mathcal{B}^{(2)}$, for any $k,\ l \in \{1,\ 2\}$, is still NOT implied. Also, it is NOT implied that the joint tail of $(X_1 + X_2,\ Y_1 + Y_2)$ has the desired property of $\mathcal{B}^{(2)}$. Hence, we find out that the dependence structures among the components play a crucial role in the closure properties of bivariate vectors.

Next we see that the class $\mathcal{D}^{(2)}$ is closed with respect to sum (of arbitrarily dependent random vectors with arbitrarily non-negative dependent components), under the condition that the point (1) in Definition 3.1 is satisfied.

Theorem 3.1. *Let non-negative random variables X_1, X_2, Y_1, Y_2 with distributions F_1, F_2, G_1, and G_2, from class $\mathcal{D}$, respectively. We assume that $(X_k,\ Y_l) \in \mathcal{D}^{(2)}$ for any $k,\ l \in \{1,\ 2\}$, then $(X_1 + X_2,\ Y_1 + Y_2) \in \mathcal{D}^{(2)}$.*

Proof. At first, for the first condition of $\mathcal{D}^{(2)}$, we obtain $F_{X_1+X_2} \in \mathcal{D}$ and $G_{Y_1+Y_2} \in \mathcal{D}$ because of Proposition 1.1.

Taking into consideration that all the distributions have support on the interval $[0,\ \infty)$, by the elementary inequalities

$$\mathbf{P}[X_1 + X_2 > x] \leq \mathbf{P}\left[X_1 > \frac{x}{2}\right] + \mathbf{P}\left[X_2 > \frac{x}{2}\right],$$

$$\mathbf{P}[X_1 + X_2 > x] \geq \frac{1}{2}\left(\mathbf{P}[X_1 > x] + \mathbf{P}[X_2 > x]\right),$$

we find

$$\begin{aligned}\mathbf{P}[X_1 + X_2 > x,\ Y_1 + Y_2 > y] &\leq \mathbf{P}[X_1 > x/2,\ Y_1 + Y_2 > y] + \mathbf{P}[X_2 > x/2,\ Y_1 + Y_2 > y] \\ &\leq \mathbf{P}\left[X_1 > \frac{x}{2},\ Y_1 > \frac{y}{2}\right] + \mathbf{P}\left[X_1 > \frac{x}{2},\ Y_2 > \frac{y}{2}\right] \\ &\quad + \mathbf{P}\left[X_2 > \frac{x}{2},\ Y_1 > \frac{y}{2}\right] + \mathbf{P}\left[X_2 > \frac{x}{2},\ Y_2 > \frac{y}{2}\right],\end{aligned}$$

hence

$$\mathbf{P}[X_1 + X_2 > x,\ Y_1 + Y_2 > y] \leq \sum_{k=1}^{2}\sum_{l=1}^{2} \mathbf{P}\left[X_k > \frac{x}{2},\ Y_l > \frac{y}{2}\right]. \tag{3.1}$$

From the other side

$$\begin{aligned}&\mathbf{P}[X_1 + X_2 > x,\ Y_1 + Y_2 > y] \geq \frac{\mathbf{P}[X_1 > x,\ Y_1 + Y_2 > y] + \mathbf{P}[X_2 > x,\ Y_1 + Y_2 > y]}{2} \\ &\geq \frac{\mathbf{P}[X_1 > x,\ Y_1 > y] + \mathbf{P}[X_1 > x,\ Y_2 > y] + \mathbf{P}[X_2 > x,\ Y_1 > y] + \mathbf{P}[X_2 > x,\ Y_2 > y]}{4},\end{aligned}$$

from where we obtain

$$\mathbf{P}[X_1 + X_2 > x,\ Y_1 + Y_2 > y] \geq \frac{1}{4}\sum_{k=1}^{2}\sum_{l=1}^{2} \mathbf{P}[X_k > x,\ Y_l > y]. \tag{3.2}$$

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Therefore by relations (3.1) and (3.2), due to $(X_k,\ Y_l) \in \mathcal{D}^{(2)}$ for any $k,\ l \in \{1,\ 2\}$, and $\mathbf{b} = (b_1,\ b_2) \in (0,\ 1)^2$, we find

$$\limsup_{x\wedge y\to\infty} \frac{\mathbf{P}[X_1+X_2>b_1\,x,\ Y_1+Y_2>b_2\,y]}{\mathbf{P}[X_1+X_2>x,\ Y_1+Y_2>y]}$$

$$\leq 4\ \limsup_{x\wedge y\to\infty} \frac{\sum_{k=1}^{2}\sum_{l=1}^{2}\mathbf{P}\left[X_k>\frac{b_1}{2}\,x,\ Y_l>\frac{b_2}{2}\,y\right]}{\sum_{k=1}^{2}\sum_{l=1}^{2}\mathbf{P}[X_k>x,\ Y_l>y]}$$

$$\leq 4\ \max_{k,\,l\in\{1,\,2\}}\left\{\limsup_{x\wedge y\to\infty}\frac{\mathbf{P}\left[X_k>\frac{b_1}{2}\,x,\ Y_l>\frac{b_2}{2}\,y\right]}{\mathbf{P}[X_k>x,\ Y_l>y]}\right\}<\infty.$$

So we conclude $(X_1+X_2,\ Y_1+Y_2) \in \mathcal{D}^{(2)}$ □.

Remark 3.1. *Let us notice here that the vectors $(X_k,\ Y_l)$ for $k,\ l=1,\ 2$ are NOT necessarily under the same dependence structure; for example, we can have $(X_1,\ Y_1)$ with independent components and $(X_1,\ Y_2)$ to be SAI, with $C>0$. A case where we see that $(X_k,\ Y_l) \in \mathcal{D}^{(2)}$ for any $k,\ l=1,\ 2$ is the following. Let $X_1,\ X_2,\ Y_1,\ Y_2$ with distributions from class $\mathcal{D}$ and also the $X_1,\ X_2$ and the $Y_1,\ Y_2$ are arbitrarily dependent. If $(X_k,\ Y_l)$ are SAI with $C_{k,l}>0$, not necessarily the same for each pair, then $(X_k,\ Y_l) \in \mathcal{D}^{(2)}$ for any $k,\ l=1,\ 2$.*

Now we are ready to define the max-sum equivalence in two dimensions.

Definition 3.2. *Let $X_1,\ X_2,\ Y_1,\ Y_2$ be random variables. Then we say that they are joint max-sum equivalent if*

$$\mathbf{P}[X_1+X_2>x,\ Y_1+Y_2>y] \sim \sum_{k=1}^{2}\sum_{l=1}^{2}\mathbf{P}[X_k>x,\ Y_l>y],$$

as $x\wedge y\to\infty$.

This kind of asymptotic relation will be established for classes $(\mathcal{D}\cap\mathcal{L})^{(2)}$ and $\mathcal{C}^{(2)}$, under the assumption of some specific dependence structure.

4. Joint behavior of random sums

In one dimension, the following asymptotic relation attracted attention:

$$\mathbf{P}\left[\sum_{i=1}^{n} X_i > x\right] \sim \sum_{i=1}^{n}\mathbf{P}[X_i>x], \tag{4.1}$$

as $x\to\infty$. Therefore, we study the behavior of both the maximum $\bigvee_{i=1}^{n} X_i$ and the maximum of sums

$$\bigvee_{i=1}^{n} S_i := \max_{1\leq k\leq n}\sum_{i=1}^{k} X_i,$$

for some distributions and correspondingly with some dependence structures to examine if it holds

$$\mathbf{P}\left[\sum_{i=1}^{n} X_i>x\right] \sim \mathbf{P}\left[\bigvee_{i=1}^{n} X_i>x\right] \sim \mathbf{P}\left[\bigvee_{i=1}^{n} S_i>x\right] \sim \sum_{i=1}^{n}\mathbf{P}\left[X_i>x\right], \tag{4.2}$$

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as $x \to \infty$. Relations (4.1) and (4.2) have been studied extensively; see, for example, in Geluk and Ng (2006), Geluk and Tang (2009), Ng et al. (2002), and Jiang et al. (2014). A similar interest has been appeared for weighted sums of the form

$$S_n^{\Theta} := \sum_{i=1}^{n} \Theta_i X_i, \qquad \bigvee_{i=1}^{n} S_i^{\Theta} := \max_{1 \le k \le n} \sum_{i=1}^{k} \Theta_i X_i,$$

and for the circumstances when they satisfy relations (4.1) and (4.2), see, for example, Tang and Yuan (2014), Tang and Tsitsiashvili (2003), Yang et al. (2012), and Zhang et al. (2009).

In this section, we study relation (4.2) in two dimensions. This can be achieved for the class $(\mathcal{D} \cap \mathcal{L})^{(2)}$ under generalized tail asymptotic independence (GTAI). Although the univariate randomly weighted sums are well studied, this is not true for the multivariate case.

Let us mention some papers involved in the asymptotic behavior of the joint-tail probability

$$\mathbf{P}\left[\sum_{i=1}^{n} \Theta_i X_i > x, \sum_{j=1}^{n} \Delta_i Y_i > y\right],$$

as, for example, Chen and Yang (2019), Li (2018), Shen and Du (2023), Shen et al. (2020), and Yang et al. (2024).

We restrict ourselves at the moment in the study of non-weighted random sums of the following form:

$$\mathbf{P}\left[\sum_{i=1}^{n} X_i > x, \sum_{j=1}^{n} Y_j > y\right].$$

Let us remind that, as before, the X_i, Y_j follow distributions with infinite right endpoint.

We note that, in almost all existing papers, the dependence structure for the main variables X_i, Y_j is either of the form: $\{(X_i, Y_i), i \in \mathbb{N}\}$ independent random vectors, and there exists some dependence structure in each random pair, or there exists dependence among $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$, but the X_i and Y_j are independent for any i, j. Using *GTAI*, introduced in Konstantinides and Passalidis (2024), both dependence structures are simultaneously permitted. *GTAI* is defined as follows. Let us consider two sequences of random variables $\{X_n, n \in \mathbb{N}\}$, $\{Y_m, m \in \mathbb{N}\}$. We say that the random variables $X_1, \ldots, X_n, Y_1, \ldots, Y_m$ follow the *GTAI*, if

(1) It holds

$$\lim_{\min\{x_i, x_k, y_j\} \to \infty} \mathbf{P}[|X_i| > x_i \mid X_k > x_k, \; Y_j > y_j] = 0,$$

for any $1 \le i \ne k \le n, j = 1, \ldots, m$.

(2) It holds

$$\lim_{\min\{x_i, y_k, y_j\} \to \infty} \mathbf{P}[|Y_j| > y_j \mid X_i > x_i, \; Y_k > y_k] = 0,$$

for any $1 \le j \ne k \le m, i = 1, \ldots, n$.

The aim of this dependence structure is to model the dependence both within each sequence of random variables and the interdependence between the sequences. We have to notice that if the X_i and Y_j are independent for any i, j, then each sequence of random variables follows tail asymptotic dependence (TAI) (see definition below); however, in any other case, the *GTAI* does not restrict each sequence to *TAI*, but in a more general form of dependence.

It is easy to find that *GTAI* contains the case when $X_1, \ldots, X_n$ are independent or when $Y_1, \ldots, Y_n$ are independent or both. Even more, this dependence structure indicates that the

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probability to happen three extreme events is negligible with respect to the probability to happen two extreme events, one in each sequence, and in some sense, *GTAI* belongs to the dependencies of second-order asymptotic independence.

In most of our results, we use the *TAI* dependence structure as an extra assumption, which characterizes the dependence of the terms of each sequence. This dependence structure was introduced by Geluk and Tang (2009). We say that $X_1, \ldots, X_n$ are tail asymptotic independent, symbolically *TAI* (and sometimes named strong quasi-asymptotically independent), if for any pair $i, j = 1, \ldots, n$, with $i \neq j$, it holds the limit

$$\lim_{x_i \wedge x_j \to \infty} \mathbf{P}[|X_i| > x_i \mid X_j > x_j] = 0.$$

The next result provides an asymptotic relation for the maximum of two sequences of random variables under the *GTAI*, *WITHOUT* imposing any assumption on the distributions of $X_1, \ldots, X_n, Y_1, \ldots, Y_m$ (except the infinite right point).

Theorem 4.1. *If $X_1, \ldots, X_n$ are random variables with distributions $F_1, \ldots, F_n$, respectively, and $Y_1, \ldots, Y_m$ are random variables with distributions $G_1, \ldots, G_m$ and $X_1, \ldots, X_n, Y_1, \ldots, Y_m$ are GTAI, it holds*

$$\mathbf{P}\left[\bigvee_{i=1}^{n} X_i > x,\ \bigvee_{j=1}^{m} Y_j > y\right] \sim \sum_{i=1}^{n}\sum_{j=1}^{m} \mathbf{P}\left[X_i > x,\ Y_j > y\right],$$

as $x \wedge y \to \infty$.

Proof. For $x > 0,\ y > 0$ holds

$$\mathbf{P}\left[\bigvee_{i=1}^{n} X_i > x,\ \bigvee_{j=1}^{m} Y_j > y\right] \leq \sum_{i=1}^{n}\sum_{j=1}^{m} \mathbf{P}\left[X_i > x,\ Y_j > y\right]. \tag{4.3}$$

Further for the lower bound, we use Bonferroni's inequality

$$\begin{aligned}
&\mathbf{P}\left[\bigvee_{i=1}^{n} X_i > x,\ \bigvee_{j=1}^{m} Y_j > y\right] \\
&\geq \sum_{i=1}^{n} \mathbf{P}\left[X_i > x,\ \bigvee_{j=1}^{m} Y_j > y\right] - \sum\sum_{i<l=1}^{n} \mathbf{P}\left[X_i > x,\ X_l > x,\ \bigvee_{j=1}^{m} Y_j > y\right] \\
&\geq \sum_{i=1}^{n}\sum_{j=1}^{m} \mathbf{P}\left[X_i > x,\ Y_j > y\right] - \sum_{i=1}^{n}\sum\sum_{j<k=1}^{m} \mathbf{P}\left[X_i > x,\ Y_j > y,\ Y_k > y\right] \\
&-\sum\sum_{l<i=1}^{n}\sum_{j=1}^{m} \mathbf{P}\left[X_i > x,\ X_l > x,\ Y_j > y\right] =: I_1(x, y) - I_2(x, y) - I_3(x, y).
\end{aligned}$$

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For $I_2(x, y)$, we obtain

$$I_2(x, y) = \sum_{i=1}^{n} \sum \sum_{j<k=1}^{m} \mathbf{P}\left[X_i > x,\ Y_j > y,\ Y_k > y\right]$$

$$= \sum_{i=1}^{n} \sum \sum_{j<k=1}^{m} \mathbf{P}\left[Y_k > y \mid X_i > x,\ Y_j > y\right] \mathbf{P}\left[X_i > x,\ Y_j > y\right]$$

$$= o\left(\sum_{i=1}^{n} \sum_{j=1}^{m} \mathbf{P}\left[X_i > x,\ Y_j > y\right]\right) = o[I_1(x, y)],$$

as $x \wedge y \to \infty$, where in the last step, we use the *GTAI* property. In a similar way, we can find

$$I_3(x, y) = o[I_1(x, y)],$$

as $x \wedge y \to \infty$. Hence we conclude

$$\mathbf{P}\left[\bigvee_{i=1}^{n} X_i > x,\ \bigvee_{j=1}^{m} Y_j > y\right] \gtrsim \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbf{P}\left[X_i > x,\ Y_j > y\right], \tag{4.4}$$

as $x \wedge y \to \infty$. Now, from relations (4.3) and (4.4), we have the result. □

Before next theorem, we need some preliminary lemmas. The next lemma provides an important property of the *GTAI* structure, presenting itself as closure property with respect to sum.

Lemma 4.1. *If $X_1, \ldots, X_n, Y_1, \ldots, Y_m$ follow the GTAI, then holds*

$$\lim_{\min\{x_I, x_k, y_j\} \to \infty} \mathbf{P}\left[\left|\sum_{i \in I} X_i\right| > x_I \,\middle|\, X_k > x_k,\ Y_j > y_j\right] = 0, \tag{4.5}$$

for $I \subsetneq \{1, \ldots, n\}$ and $k \in \{1, \ldots, n\} \setminus I$, $j = 1, \ldots, m$. Similarly holds

$$\lim_{\min\{x_i, y_k, y_J\} \to \infty} \mathbf{P}\left[\left|\sum_{j \in J} Y_j\right| > y_J \,\middle|\, Y_k > y_k,\ X_i > x_i\right] = 0, \tag{4.6}$$

for $J \subsetneq \{1, \ldots, m\}$ and $k \in \{1, \ldots, m\} \setminus J$, $i = 1, \ldots, n$.

Proof. It is enough to show relation (4.5) as relation (4.6) follows by similar way. Indeed, we observe that

$$\lim_{\min\{x_I, x_k, y_j\} \to \infty} \mathbf{P}\left[\left|\sum_{i \in I} X_i\right| > x_I \,\middle|\, X_k > x_k,\ Y_j > y_j\right]$$

$$\leq \lim_{\min\{x_I, x_k, y_j\} \to \infty} \sum_{i \in I} \mathbf{P}\left[|X_i| > \frac{x_I}{n} \,\middle|\, X_k > x_k,\ Y_j > y_j\right] = 0,$$

where the last step follows from *GTAI* property. □

In most of the following results, we assume that the random variables $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ are *GTAI* and follow distributions from some class $\mathcal{B} \in \{\mathcal{C}, \mathcal{D} \cap \mathcal{L}, \mathcal{L}\}$, and at the same time, it holds $(X_k, Y_l) \in \mathcal{B}^{(2)}$, for any $k, l \in \{1, \ldots, n\}$. Following a referee's advice, we provide some examples to show that *GTAI* and $(X_k, Y_l) \in \mathcal{B}^{(2)}$ are possible simultaneously. For the sake of simplicity, we consider only the case $n = 2$ with non-negative random variables.

Example 4.1. *Let X_1, X_2, Y_1, Y_2 be non-negative random variables with distributions from class $\mathcal{B} \in \{\mathcal{C}, \mathcal{D} \cap \mathcal{L}, \mathcal{L}\}$. Further, we suppose that the X_1, X_2 are TAI, the Y_1, Y_2 are also TAI, while the (X_1, X_2) and (Y_1, Y_2) are independent random pairs. Then, we directly find that $(X_k, Y_l) \in \mathcal{B}^{(2)}$, and additionally by TAI, we obtain that for any $\varepsilon > 0$, there exists some $x_0 > 0$, such that for any $1 \leq i \neq k \leq 2$, it holds $\mathbf{P}[X_i > x_i \mid X_k > x_k] < \varepsilon$, for any $x_i \wedge x_k \geq x_0$. Hence for any $1 \leq i \neq k \leq 2$, $j = 1, 2$, we conclude*

$$\mathbf{P}\left[X_i > x_i,\, X_k > x_k,\, Y_j > y_j\right] = \mathbf{P}\left[X_i > x_i,\, X_k > x_k\right]\, \mathbf{P}\left[Y_j > y_j\right]$$

$$< \varepsilon\, \mathbf{P}\left[X_k > x_k\right]\, \mathbf{P}\left[Y_j > y_j\right] = \varepsilon\, \mathbf{P}\left[X_k > x_k,\, Y_j > y_j\right],$$

for any $x_i \wedge x_k \geq x_0$. From the last relation, because of the arbitrary choice of ε, we get $\mathbf{P}\left[X_i > x_i \mid X_k > x_k,\, Y_j > y_j\right] \to 0$, as $x_i \wedge x_k \wedge y_j \to \infty$.

Similarly, by symmetry, we obtain for any $1 \leq j \neq k \leq 2$, $i = 1, 2$ the convergence $\mathbf{P}\left[Y_j > y_j \mid X_i > x_i,\, Y_k > y_k\right] \to 0$, as $x_i \wedge y_k \wedge y_j \to \infty$. Hence, the X_1, X_2, Y_1, Y_2 satisfy the GTAI.

Example 4.2. *Let X_1, X_2, Y_1, Y_2 be non-negative random variables with distributions from class $\mathcal{B} \in \{\mathcal{C}, \mathcal{D} \cap \mathcal{L}, \mathcal{L}\}$. We suppose that $Z_i, Z_j, Z_k \in \{X_1, X_2, Y_1, Y_2\}$ with $Z_i \neq Z_j \neq Z_k$ and $z_i, z_j, z_k \in \{x_1, x_2, y_1, y_2\}$, where the z_i, z_j, z_k are allowed to be equal. Let us assume the SAI property for any duo or trio of them, namely*

$$\mathbf{P}[Z_i > z_i,\, Z_j > z_j] \sim C_{ij}\, \mathbf{P}[Z_i > z_i]\, \mathbf{P}[Z_j > z_j],$$

as $z_i \wedge z_j \to \infty$, with $C_{ij} > 0$ and

$$\mathbf{P}[Z_i > z_i,\, Z_j > z_j,\, Z_k > z_k] \sim C_{ijk}\, \mathbf{P}[Z_i > z_i]\, \mathbf{P}[Z_j > z_j]\, \mathbf{P}[Z_k > z_k],$$

as $z_i \wedge z_j \wedge z_k \to \infty$, with $C_{ijk} > 0$. Then, by SAI in duo mode, we obtain $(X_k, Y_l) \in \mathcal{B}^{(2)}$, for $k, l \in \{1, 2\}$ (see Example 2.2, for the case of class $\mathcal{L}$). Next, by SAI in trio mode, we obtain directly the GTAI.

From here on, we study only the case $n = m$. In the next lemma, we find the lower asymptotic bound of the joint tail of the random sums

$$S_n := \sum_{k=1}^{n} X_k, \qquad T_n := \sum_{l=1}^{n} Y_l,$$

when the summands follow distributions with long tails and the $\mathcal{L}^{(2)}$ property is true for any pair of the summands distribution. A similar result, for the unidimensional case, can be found in Geluk and Tang (2009), where the dependence structure is *TAI*. In the next result, we find generalization to two dimensions and furthermore the *GTAI* assumption. Next, we introduce the notations

$$S_{n,k} := S_n - X_k, \qquad T_{n,l} := T_n - Y_l,$$

for some $k \in \{1, \ldots, n\}$ and some $l \in \{1, \ldots, n\}$. In what follows, we can choose

$$a = (a_F, a_G) := \left(\min_{1 \leq k \leq n} a_{F_k},\, \min_{1 \leq l \leq n} a_{G_l} \right), \tag{4.7}$$

namely, the minimum of all the joint insensitivity functions, which means that the function $a(\cdot)$ is insensitive for all the distribution pairs (F_k, G_l), for $k, l \in \{1, \ldots, n\}$. In what follows, for the sake of simplicity, the function $a(\cdot)$ is understood either as a_F for the X_k or as a_G for the Y_l.

Lemma 4.2. *Let $X_1, \ldots, X_n$, $Y_1, \ldots, Y_n$ be random variables with distributions $F_1, \ldots, F_n$, $G_1, \ldots, G_n$ from class $\mathcal{L}$, respectively. We also assume that $X_1, \ldots, X_n$, $Y_1, \ldots, Y_n$ satisfy the GTAI property and it holds*

$$(X_k, Y_l) \in \mathcal{L}^{(2)},$$

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for any $k,\ l \in \{1, \ldots, n\}$. *Then it holds*

$$\mathbf{P}\left[S_n > x,\ T_n > y\right] \gtrsim \sum_{k=1}^{n}\sum_{l=1}^{n} \mathbf{P}\left[X_k > x,\ Y_l > y\right],$$

as $x \wedge y \to \infty$.

Proof. We choose as $a(\cdot)$ a function with joint insensitivity property for any random pair $(X_k,\ Y_l)$ for any $k,\ l \in \{1, \ldots, n\}$. A possible choice of this function is by (4.7). Next, we apply twice Bonferroni's inequality to obtain

$$\begin{aligned}
&\mathbf{P}\left[S_n > x,\ T_n > y\right] \geq \mathbf{P}\left[S_n > x,\ T_n > y,\ \bigvee_{k=1}^{n} X_k > x + a(x),\ \bigvee_{l=1}^{n} Y_l > y + a(y)\right] \\
&\geq \sum_{k=1}^{n}\sum_{l=1}^{n} \mathbf{P}\left[S_n > x,\ T_n > y,\ X_k > x + a(x),\ Y_l > y + a(y)\right] \\
&- \sum \sum_{1 \leq k < i \leq n} \sum_{l=1}^{n} \mathbf{P}\left[X_i > x + a(x),\ X_k > x + a(x),\ Y_l > y + a(y)\right] \\
&- \sum_{k=1}^{n} \sum \sum_{1 \leq l < i \leq n} \mathbf{P}\left[X_k > x + a(x),\ Y_l > y + a(y),\ Y_i > y + a(y)\right] \\
&=: \sum_{i=1}^{3} J_i(x,\ y). \qquad (4.8)
\end{aligned}$$

Now for each term of $J_2(x,\ y)$, we find

$$\begin{aligned}
&\mathbf{P}\left[X_i > x + a(x),\ X_k > x + a(x),\ Y_l > y + a(y)\right] \\
&= \mathbf{P}\left[X_i > x + a(x) \mid X_k > x + a(x),\ Y_l > y + a(y)\right]\ \mathbf{P}\left[X_k > x + a(x),\ Y_l > y + a(y)\right] \\
&= o(\mathbf{P}\left[X_k > x,\ Y_l > y\right]),
\end{aligned}$$

as $x \wedge y \to \infty$, which follows from *GTAI* property, $\mathcal{L}^{(2)}$ membership and the definition of function $a(\cdot)$. So

$$J_2(x,\ y) = o(\mathbf{P}\left[X_k > x,\ Y_l > y\right]), \qquad (4.9)$$

as $x \wedge y \to \infty$. Similarly, due to symmetry, we have

$$J_3(x,\ y) = o(\mathbf{P}\left[X_k > x,\ Y_l > y\right]), \qquad (4.10)$$

as $x \wedge y \to \infty$.

Finally, for the first term, we obtain

$$\begin{aligned}
J_1(x,\ y) \geq &\sum_{k=1}^{n}\sum_{l=1}^{n} \mathbf{P}[X_k > x + a(x),\ Y_l > y + a(y)] \\
&- \sum_{1 \leq k < i \leq n}^{n} \sum \sum_{l=1}^{n} \mathbf{P}\left[X_k > x + a(x),\ Y_l > y + a(y),\ X_i < -\frac{a(x)}{n}\right] \\
&- \sum_{k=1}^{n} \sum_{1 \leq l < i \leq n}^{n} \sum \mathbf{P}\left[X_k > x + a(x),\ Y_l > y + a(y),\ Y_i < -\frac{a(y)}{n}\right], \qquad (4.11)
\end{aligned}$$

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hence, the last two terms in (4.11), from the *GTAI* structure and the definition of the function a, in combination with properties of class $\mathcal{L}^{(2)}$, become negligible with respect to the first term in (4.11). Therefore, it holds

$$J_1(x,\, y) \gtrsim \sum_{k=1}^{n} \sum_{l=1}^{n} \mathbf{P}[X_k > x + a(x),\; Y_l > y + a(y)], \tag{4.12}$$

as $x \wedge y \to \infty$. Thus, relations (4.9), (4.10), and (4.12), together with relation (4.8), render the desired lower bound. $\square$

Lemma 4.3. *Let $X_1,\ X_2,\ Y_1,\ Y_2$ be non-negative random variables, with GTAI property, such that the pair $(X_k,\ Y_l) \in (\mathcal{D} \cap \mathcal{L})^{(2)}$, for any $k,\ l \in \{1,\ 2\}$. If by a we denote the insensitivity function from (4.7), then it holds*

$$\begin{aligned}\mathbf{P}\left[X_1 > x - a(x),\; Y_1 > a(y),\; Y_2 \leq \frac{y}{2}\right] &\sim \mathbf{P}\left[X_1 > x,\; Y_1 > a(y),\; Y_2 \leq \frac{y}{2}\right] \\ &\sim \mathbf{P}[X_1 > x,\; Y_1 > a(y)],\end{aligned} \tag{4.13}$$

as $x \wedge y \to \infty$, and further, it holds

$$\begin{aligned}\mathbf{P}\left[Y_1 > y - a(y),\; X_1 > a(x),\; X_2 \leq \frac{x}{2}\right] &\sim \mathbf{P}\left[Y_1 > y,\; X_1 > a(x),\; X_2 \leq \frac{x}{2}\right] \\ &\sim \mathbf{P}[Y_1 > y,\; X_1 > a(x)],\end{aligned}$$

as $x \wedge y \to \infty$.

Proof. We show only the first relation (4.13), since the second follows along similar steps, due to symmetry. At first, by definition of insensitivity function $a(\cdot)$ from (4.7), and if the pair $(X,\ Y) \in (\mathcal{D} \cap \mathcal{L})^{(2)}$, we obtain

$$1 \leq \lim_{x \wedge y \to \infty} \frac{\mathbf{P}\left[X > x - a(x),\; Y > y\right]}{\mathbf{P}\left[X > x,\; Y > y\right]} \leq \lim_{x \wedge y \to \infty} \frac{\mathbf{P}\left[X > x - a(x),\; Y > y - a(y)\right]}{\mathbf{P}\left[X > x,\; Y > y\right]} = 1, \tag{4.14}$$

Hence, because of $(X_1,\ Y_1) \in (\mathcal{D} \cap \mathcal{L})^{(2)} \subsetneq \mathcal{L}^{(2)}$, we have through (4.14) that it holds

$$\mathbf{P}\left[X_1 > x - a(x),\; Y_1 > a(y)\right] \sim \mathbf{P}\left[X_1 > x,\; Y_1 > a(y)\right],$$

or equivalently

$$\begin{aligned}&\mathbf{P}\left[X_1 > x - a(x),\; Y_1 > a(y),\; Y_2 \leq \frac{y}{2}\right] + \mathbf{P}\left[X_1 > x - a(x),\; Y_1 > a(y),\; Y_2 > \frac{y}{2}\right] \\ &\sim \mathbf{P}\left[X_1 > x,\; Y_1 > a(y),\; Y_2 \leq \frac{y}{2}\right] + \mathbf{P}\left[X_1 > x,\; Y_1 > a(y),\; Y_2 > \frac{y}{2}\right],\end{aligned} \tag{4.15}$$

as $x \wedge y \to \infty$. We compare the first terms of each side of (4.15), to find

$$\begin{aligned}&\mathbf{P}\left[X_1 > x - a(x),\; Y_1 > a(y),\; Y_2 \leq \frac{y}{2}\right] = \mathbf{P}\left[X_1 > x - a(x),\; Y_1 > a(y)\right] \\ &-\mathbf{P}\left[X_1 > x - a(x),\; Y_1 > a(y),\; Y_2 > \frac{y}{2}\right] = \mathbf{P}\left[X_1 > x - a(x),\; Y_1 > a(y)\right] \\ &-\mathbf{P}\left[Y_2 > \frac{y}{2} \,\middle|\, X_1 > x - a(x),\; Y_1 > a(y)\right] \mathbf{P}\left[X_1 > x - a(x),\; Y_1 > a(y)\right] \\ &\sim \mathbf{P}\left[X_1 > x,\; Y_1 > a(y)\right] - o\left(\mathbf{P}\left[X_1 > x,\; Y_1 > a(y)\right]\right),\end{aligned}$$

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as $x \wedge y \to \infty$, where in the pre-last step, we used the class $(\mathcal{D} \cap \mathcal{L})^{(2)}$ property, relation (4.14), and the *GTAI* property. Similarly, we get

$$\begin{aligned}
&\mathbf{P}\left[X_1 > x,\ Y_1 > a(y),\ Y_2 \leq \frac{y}{2}\right] \\
&= \mathbf{P}\left[X_1 > x,\ Y_1 > a(y)\right] - \mathbf{P}\left[X_1 > x,\ Y_1 > a(y),\ Y_2 > \frac{y}{2}\right] \\
&= \mathbf{P}\left[X_1 > x,\ Y_1 > a(y)\right] - \mathbf{P}\left[Y_2 > \frac{y}{2}\ \middle|\ X_1 > x,\ Y_1 > a(y)\right] \mathbf{P}\left[X_1 > x,\ Y_1 > a(y)\right] \\
&\sim \mathbf{P}\left[X_1 > x,\ Y_1 > a(y)\right] - o\left(\mathbf{P}\left[X_1 > x,\ Y_1 > a(y)\right]\right),
\end{aligned} \tag{4.16}$$

as $x \wedge y \to \infty$. Therefore, considering all together relations (4.15)–(4.16), we conclude relation (4.13) $\square$.

Remark 4.1. *Taking into account relations (4.15)–(4.16), together with the fact that the $X_1,\ X_2,\ Y_1,\ Y_2$ are non-negative random variables, which are GTAI, it follows that*

$$\mathbf{P}\left[X_1 > x - a(x),\ Y_1 > a(y),\ Y_2 > \frac{y}{2}\right] = o(\mathbf{P}\left[X_1 > x,\ Y_1 > a(y)\right]),$$
$$\mathbf{P}\left[Y_1 > y - a(y),\ X_1 > a(x),\ X_2 > \frac{x}{2}\right] = o(\mathbf{P}\left[X_1 > a(x),\ Y_1 > y\right]),$$

as $x \wedge y \to \infty$.

The next result shows that in the non-negative part of class $(\mathcal{D} \cap \mathcal{L})^{(2)}$, the property of joint max-sum equivalence as also under an extra assumption the closure property with respect to convolution are satisfied, as soon as the *GTAI* holds.

Lemma 4.4. *Let $X_1,\ X_2,\ Y_1,\ Y_2$ be non-negative random variables, with the following distributions $F_1,\ F_2,\ G_1,\ G_2$ from class $\mathcal{D} \cap \mathcal{L}$, respectively. Further, we assume that the random variables $X_1,\ X_2,\ Y_1,\ Y_2$ satisfy the GTAI and*

$$(X_k,\ Y_l) \in (\mathcal{D} \cap \mathcal{L})^{(2)},$$

for any $k,\ l \in \{1,\ 2\}$ properties. Then it holds

$$\mathbf{P}\left[X_1 + X_2 > x,\ Y_1 + Y_2 > y\right] \sim \sum_{k=1}^{2} \sum_{l=1}^{2} \mathbf{P}\left[X_k > x,\ Y_l > y\right], \tag{4.17}$$

as $x \wedge y \to \infty$. If further X_1, X_2 are TAI and Y_1, Y_2 are TAI, then

$$(X_1 + X_2,\ Y_1 + Y_2) \in (\mathcal{D} \cap \mathcal{L})^{(2)},$$

Proof. From Lemma 4.2 and the fact that $(\mathcal{D} \cap \mathcal{L})^{(2)} \subsetneq \mathcal{L}^{(2)}$, we find

$$\mathbf{P}[X_1 + X_2 > x,\ Y_1 + Y_2 > y] \gtrsim \sum_{k=1}^{2} \sum_{l=1}^{2} \mathbf{P}\left[X_k > x,\ Y_l > y\right], \tag{4.18}$$

as $x \wedge y \to \infty$, which provides the lower asymptotic bound.

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Let us examine now the upper asymptotic bound

$$\begin{aligned}
&\mathbf{P}[X_1+X_2>x,\ Y_1+Y_2>y] \leq \mathbf{P}\left[X_1>x-a(x),\ Y_1+Y_2>y\right]\\
&+\mathbf{P}\left[X_2>x-a(x),\ Y_1+Y_2>y\right]+\mathbf{P}\left[X_1>a(x),\ X_2>\frac{x}{2},\ Y_1+Y_2>y\right]\\
&+\mathbf{P}\left[X_1>\frac{x}{2},\ X_2>a(x),\ Y_1+Y_2>y\right] \leq \mathbf{P}\left[X_1>x-a(x),\ Y_1>y-a(y)\right]\\
&+\mathbf{P}\left[X_1>x-a(x),\ Y_2>y-a(y)\right]+\mathbf{P}\left[X_1>x-a(x),\ Y_1>a(y),\ Y_2>\frac{y}{2}\right]\\
&+\mathbf{P}\left[X_1>x-a(x),\ Y_1>\frac{y}{2},\ Y_2>a(y)\right]+\mathbf{P}\left[X_2>x-a(x),\ Y_1>y-a(y)\right]\\
&+\mathbf{P}\left[X_2>x-a(x),\ Y_2>y-a(y)\right]+\mathbf{P}\left[X_2>x-a(x),\ Y_1>a(y),\ Y_2>\frac{y}{2}\right]\\
&+\mathbf{P}\left[X_2>x-a(x),\ Y_1>\frac{y}{2},\ Y_2>a(y)\right]+\mathbf{P}\left[X_1>a(x),\ X_2>\frac{x}{2},\ Y_1>y-a(y)\right]\\
&+\mathbf{P}\left[X_1>a(x),\ X_2>\frac{x}{2},\ Y_2>y-a(y)\right]\\
&+\mathbf{P}\left[X_1>a(x),\ X_2>\frac{x}{2},\ Y_1>a(y),\ Y_2>\frac{y}{2}\right]\\
&+\mathbf{P}\left[X_1>a(x),\ X_2>\frac{x}{2},\ Y_1>\frac{y}{2},\ Y_2>a(y)\right]\\
&+\mathbf{P}\left[X_1>\frac{x}{2},\ X_2>a(x),\ Y_1>y-a(y)\right]+\mathbf{P}\left[X_1>\frac{x}{2},\ X_2>a(x),\ Y_2>y-a(y)\right]\\
&+\mathbf{P}\left[X_1>\frac{x}{2},\ X_2>a(x),\ Y_1>a(y),\ Y_2>\frac{y}{2}\right]\\
&+\mathbf{P}\left[X_1>\frac{x}{2},\ X_2>a(x),\ Y_1>\frac{y}{2},\ Y_2>a(y)\right] =: \sum_{i=1}^{16} I_i(x,\ y).
\end{aligned} \tag{4.19}$$

Taking into account the property $\mathcal{L}^{(2)}$ and the definition of function $a(x)$, we find the asymptotic expressions for $I_1(x,\ y)\sim\mathbf{P}\left[X_1>x, Y_1>y\right]$, $I_2(x,\ y)\sim\mathbf{P}\left[X_1>x, Y_2>y\right]$, $I_5(x,\ y)\sim\mathbf{P}\left[X_2>x, Y_1>y\right]$, $I_6(x,\ y)\sim\mathbf{P}\left[X_2>x, Y_2>y\right]$, as $x\wedge y\to\infty$. Hence

$$I_1(x,\ y)+I_2(x,\ y)+I_5(x,\ y)+I_6(x,\ y)\sim\sum_{k=1}^{2}\sum_{l=1}^{2}\mathbf{P}\left[X_k>x,\ Y_l>y\right], \tag{4.20}$$

as $x\wedge y\to\infty$.

Next, we follow a similar approach for $I_3(x,\ y)$, $I_4(x,\ y)$, $I_7(x,\ y)$, $I_8(x,\ y)$, $I_9(x,\ y)$, $I_{10}(x,\ y)$, $I_{13}(x,\ y)$ and $I_{14}(x,\ y)$. Now, we obtain by Lemma 4.3

$$\begin{aligned}
&I_3(x,\ y)\sim\mathbf{P}\left[X_1>x,\ Y_1>a(y),\ Y_2>\frac{y}{2}\right]\\
&\quad=\mathbf{P}\left[Y_1>a(y)\,\middle|\,X_1>x,\ Y_2>\frac{y}{2}\right]\mathbf{P}\left[X_1>x,\ Y_2>\frac{y}{2}\right]=o\left(\mathbf{P}\left[X_2>x, Y_2>y\right]\right),
\end{aligned}$$

as $x\wedge y\to\infty$, which follows because of properties $(\mathcal{D}\cap\mathcal{L})^{(2)}$ and *GTAI*.

In similar way, we find $I_4(x,\ y)=o(\mathbf{P}\left[X_1>x, Y_2>y\right])$, $I_7(x,\ y)=o(\mathbf{P}\left[X_2>x, Y_2>y\right])$, $I_8(x,\ y)=o(\mathbf{P}\left[X_2>x, Y_2>y\right])$, $I_9(x,\ y)=o(\mathbf{P}\left[X_2>x, Y_1>y\right])$ and finally $I_{10}(x,\ y)=o(\mathbf{P}\left[X_1>x, Y_2>y\right])$, as $x\wedge y\to\infty$. Hence

$$\sum_{j=3}^{4} I_j(x,\ y)+\sum_{i=7}^{10} I_i(x,\ y)+I_{13}(x,\ y)+I_{14}(x,\ y)=o\left(\sum_{k=1}^{2}\sum_{l=1}^{2}\mathbf{P}\left[X_k>x,\ Y_l>y\right]\right), \tag{4.21}$$

as $x\wedge y\to\infty$.

The $I_{11}(x,\ y)$, $I_{12}(x,\ y)$, $I_{15}(x,\ y)$, $I_{16}(x,\ y)$ can be handled also similarly

$$I_{11}(x,\ y)\leq \mathbf{P}\left[X_2>\frac{x}{2},\ Y_1>a(y),\ Y_2>\frac{y}{2}\right]$$
$$=\mathbf{P}\left[Y_1>a(y)\,\middle|\,X_2>\frac{x}{2},\ Y_2>\frac{y}{2}\right]\mathbf{P}\left[X_2>\frac{x}{2},Y_2>\frac{y}{2}\right],$$

or equivalently $I_{11}(x,\ y)=o(\mathbf{P}\left[X_2>x,Y_2>y\right])$, as $x\wedge y\to\infty$, which follows because of properties $(\mathcal{D}\cap\mathcal{L})^{(2)}$ and *GTAI*. Similarly, we find $I_{1j}(x,\ y)=o(\mathbf{P}\left[X_k>x,Y_l>y\right])$, for some $k,\ l\in\{1,\ 2\}$ and for any $j\in\{1,\ 2,\ 5,\ 6\}$. Therefore, we obtain

$$I_{1j}(x,\ y)=o\left(\sum_{k=1}^{2}\sum_{l=1}^{2}\mathbf{P}\left[X_k>x,\ Y_l>y\right]\right), \tag{4.22}$$

as $x\wedge y\to\infty$, for any $j\in\{1,\ 2,\ 5,\ 6\}$.

From (4.20), (4.21), and (4.22), in combination with (4.19), we find that

$$\mathbf{P}[X_1+X_2>x,\ Y_1+Y_2>y]\lesssim\sum_{k=1}^{2}\sum_{l=1}^{2}\mathbf{P}\left[X_k>x,\ Y_l>y\right]$$

as $x\wedge y\to\infty$, which in combination with (4.18) leads to (4.17).

Now we check the validity of relation $(X_1+X_2,\ Y_1+Y_2)\in(\mathcal{D}\cap\mathcal{L})^{(2)}$. At first, by (4.17), we obtain

$$\limsup_{x\wedge y\to\infty}\frac{\mathbf{P}\left[X_1+X_2>b_1\,x,\ Y_1+Y_2>b_2\,y\right]}{\mathbf{P}\left[X_1+X_2>x,\ Y_1+Y_2>y\right]}=\limsup_{x\wedge y\to\infty}$$
$$\frac{\sum_{k=1}^{2}\sum_{l=1}^{2}\mathbf{P}\left[X_k>b_1\,x,\ Y_l>b_2\,y\right]}{\sum_{k=1}^{2}\sum_{l=1}^{2}\mathbf{P}\left[X_k>x,\ Y_l>y\right]}\leq\max_{k,\,l\in\{1,2\}}\left\{\limsup_{x\wedge y\to\infty}\frac{\mathbf{P}\left[X_k>b_1x,\ Y_l>b_2y\right]}{\mathbf{P}\left[X_k>x,\ Y_l>y\right]}\right\}<\infty,$$

for any $\mathbf{b}=(b_1,\ b_2)\in(0,\ 1)^2$, this means that we have one of two conditions of the closure property with respect to $\mathcal{D}^{(2)}$.

Next, we check the closure property with respect to $\mathcal{L}^{(2)}$. From (4.17), we obtain

$$\limsup_{x\wedge y\to\infty}\frac{\mathbf{P}\left[X_1+X_2>x-a_1,\ Y_1+Y_2>y-a_2\right]}{\mathbf{P}\left[X_1+X_2>x,\ Y_1+Y_2>y\right]}$$
$$=\limsup_{x\wedge y\to\infty}\frac{\sum_{k=1}^{2}\sum_{l=1}^{2}\mathbf{P}\left[X_k>x-a_1,\ Y_l>y-a_2\right]}{\sum_{k=1}^{2}\sum_{l=1}^{2}\mathbf{P}\left[X_k>x,\ Y_l>y\right]},$$

for any $\mathbf{a}=(a_1,\ a_2)>(0,\ 0)$, and therefore

$$\limsup_{x\wedge y\to\infty}\frac{\mathbf{P}\left[X_1+X_2>x-a_1,\ Y_1+Y_2>y-a_2\right]}{\mathbf{P}\left[X_1+X_2>x,\ Y_1+Y_2>y\right]}$$
$$\leq\max_{k,\,l\in\{1,\,2\}}\left\{\limsup_{x\wedge y\to\infty}\frac{\mathbf{P}\left[X_k>x-a_1,\ Y_l>y-a_2\right]}{\mathbf{P}\left[X_k>x,\ Y_l>y\right]}\right\}=1,$$

and always

$$\liminf_{x\wedge y\to\infty}\frac{\mathbf{P}\left[X_1+X_2>x-a_1,\ Y_1+Y_2>y-a_2\right]}{\mathbf{P}\left[X_1+X_2>x,\ Y_1+Y_2>y\right]}\geq 1,$$

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which means that we have one of two conditions of the closure property with respect to $\mathcal{L}^{(2)}$ true. So by the extra assumption of *TAI* between X_1, X_2 and Y_1, Y_2 by Lemma 4.1 of Geluk and Tang (2009), we have that $X_1 + X_2 \in \mathcal{D} \cap \mathcal{L}$ and $Y_1 + Y_2 \in \mathcal{D} \cap \mathcal{L}$; as a result, we conclude $(X_1 + X_2,\ Y_1 + Y_2) \in (\mathcal{D} \cap \mathcal{L})^{(2)}$. □

An easy example, where we combine the *GTAI* property for the $X_1, \ldots, X_n, Y_1, \ldots, Y_m$ with the *TAI* property for each sequence, is found in case of each sequence to be *TAI* but the two sequences to be independent.

Next, we provide a corollary, following from Lemma 4.4, where we establish the closure property with respect to $\mathcal{C}^{(2)}$ and the joint max-sum equivalence, under condition *GTAI*.

Corollary 4.1. *Let X_1, X_2, Y_1, Y_2 be non-negative random variables, with the distributions F_1, F_2, G_1, G_2 from class $\mathcal{C}$, respectively, and they satisfy the GTAI condition. If it holds $(X_k,\ Y_l) \in \mathcal{C}^{(2)}$, for any $k,\ l \in \{1,\ 2\}$, then*

$$\mathbf{P}\left[X_1 + X_2 > x,\ Y_1 + Y_2 > y\right] \sim \sum_{k=1}^{2}\sum_{l=1}^{2} \mathbf{P}\left[X_k > x,\ Y_l > y\right], \tag{4.23}$$

as $x \wedge y \to \infty$. If further X_1, X_2 are TAI and Y_1, Y_2 are TAI, then it holds $(X_1 + X_2,\ Y_1 + Y_2) \in \mathcal{C}^{(2)}$.

Proof. Relation (4.23) follows from the fact that $\mathcal{C}^{(2)} \subsetneq (\mathcal{D} \cap \mathcal{L})^{(2)}$ (see Theorem 2.1) and by application of Lemma 4.4.

Next, we check the closure property with respect to convolution. From (4.23), we obtain

$$\mathbf{P}[X_1 + X_2 > d_1\, x,\ Y_1 + Y_2 > d_2\, y] \sim \sum_{k=1}^{2}\sum_{l=1}^{2} \mathbf{P}\left[X_k > d_1\, x,\ Y_l > d_2\, y\right],$$

as $x \wedge y \to \infty$, for any $\mathbf{d} = (d_1,\ d_2) \in (0,\ 1)^2$. Hence

$$\begin{aligned}
&\limsup_{x\wedge y\to\infty} \frac{\mathbf{P}\left[X_1 + X_2 > d_1\, x,\ Y_1 + Y_2 > d_2\, y\right]}{\mathbf{P}\left[X_1 + X_2 > x,\ Y_1 + Y_2 > y\right]} \\
&\qquad = \limsup_{x\wedge y\to\infty} \frac{\sum_{k=1}^{2}\sum_{l=1}^{2} \mathbf{P}\left[X_k > d_1\, x,\ Y_l > d_2\, y\right]}{\sum_{k=1}^{2}\sum_{l=1}^{2} \mathbf{P}\left[X_k > x,\ Y_l > y\right]} \\
&\qquad \leq \limsup_{x\wedge y\to\infty} \max_{k,\, l\in\{1,\,2\}} \left\{ \frac{\mathbf{P}\left[X_k > d_1\, x,\ Y_l > d_2\, y\right]}{\mathbf{P}\left[X_k > x,\ Y_l > y\right]} \right\},
\end{aligned}$$

Thus, because of the definition of $\mathcal{C}^{(2)}$, we get

$$\begin{aligned}
1 &\leq \lim_{\mathbf{d}\uparrow \mathbf{1}} \limsup_{x\wedge y\to\infty} \frac{\mathbf{P}\left[X_1 + X_2 > d_1\, x,\ Y_1 + Y_2 > d_2\, y\right]}{\mathbf{P}\left[X_1 + X_2 > x,\ Y_1 + Y_2 > y\right]} \\
&\leq \lim_{\mathbf{d}\uparrow \mathbf{1}} \limsup_{x\wedge y\to\infty} \max_{k,\, l\in\{1,\,2\}} \left(\frac{\mathbf{P}\left[X_k > d_1\, x,\ Y_l > d_2\, y\right]}{\mathbf{P}\left[X_k > x,\ Y_l > y\right]} \right) \\
&\leq \max_{k,\, l\in\{1,\,2\}} \left(\lim_{\mathbf{d}\uparrow \mathbf{1}} \limsup_{x\wedge y\to\infty} \frac{\mathbf{P}\left[X_k > d_1\, x,\ Y_l > d_2\, y\right]}{\mathbf{P}\left[X_k > x,\ Y_l > y\right]} \right) = 1,
\end{aligned}$$

this means that one of two conditions of closedness under convolution holds. By the assumptions of *TAI* in each sequence and by $\mathcal{C} \subsetneq \mathcal{D} \cap \mathcal{L}$, we use Lemma 4.1 of Geluk and Tang (2009), and we

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take

$$1 \leq \lim_{d_1 \uparrow 1} \limsup_{x\to\infty} \frac{\mathbf{P}\left[X_1 + X_2 > d_1\, x\right]}{\mathbf{P}\left[X_1 + X_2 > x\right]} = \lim_{d_1 \uparrow 1} \limsup_{x\to\infty} \frac{\mathbf{P}\left[X_1 > d_1\, x\right] + \mathbf{P}\left[X_1 > d_1\, x\right]}{\mathbf{P}\left[X_1 > x\right] + \mathbf{P}\left[X_1 > x\right]}$$
$$\leq \lim_{d_1 \uparrow 1} \limsup_{x\to\infty} \max_{k\in\{1,2\}} \left(\frac{\mathbf{P}\left[X_k > d_1\, x\right]}{\mathbf{P}\left[X_k > x\right]}\right) \leq \max_{k\in\{1,2\}} \left(\lim_{d_1 \uparrow 1} \limsup_{x\to\infty} \frac{\mathbf{P}\left[X_k > d_1\, x\right]}{\mathbf{P}\left[X_k > x\right]}\right) = 1,$$

which gives that $(X_1 + X_2) \in \mathcal{C}$. With the same argument, we have $(Y_1 + Y_2) \in \mathcal{C}$. That means $(X_1 + X_2,\ Y_1 + Y_2) \in \mathcal{C}^{(2)}$. □

Now we can give the main result, where we find an analogue to relation (4.2) in two dimensions.

Theorem 4.2. *Let $X_1, \ldots, X_n$, $Y_1, \ldots, Y_n$ be random variables with the following distributions $F_1, \ldots, F_n$, $G_1, \ldots, G_n$ from class $\mathcal{D} \cap \mathcal{L}$, respectively, and they satisfy the GTAI condition, with $(X_k,\ Y_l) \in (\mathcal{D} \cap \mathcal{L})^{(2)}$, for any $k,\ l \in \{1, \ldots, n\}$. If further $X_1, \ldots, X_n$ are TAI and $Y_1, \ldots, Y_n$ are TAI, then*

$$\mathbf{P}\left[\sum_{k=1}^{n} X_k > x,\ \sum_{l=1}^{n} Y_l > y\right] \sim \mathbf{P}\left[\bigvee_{i=1}^{n} S_i > x,\ \bigvee_{j=1}^{n} T_j > y\right]$$
$$\sim \mathbf{P}\left[\bigvee_{k=1}^{n} X_k > x,\ \bigvee_{l=1}^{n} Y_l > y\right] \sim \sum_{k=1}^{n}\sum_{l=1}^{n} \mathbf{P}\left[X_k > x,\ Y_l > y\right],$$

as $x \wedge y \to \infty$.

Proof. By Lemma 4.2, we find

$$\mathbf{P}\left[\sum_{k=1}^{n} X_k > x,\ \sum_{l=1}^{n} Y_l > y\right] \gtrsim \sum_{k=1}^{n}\sum_{l=1}^{n} \mathbf{P}\left[X_k > x,\ Y_l > y\right],$$

as $x \wedge y \to \infty$. Because of closure property of $(\mathcal{D} \cap \mathcal{L})^{(2)}$ with respect to convolution in the positive part, under *GTAI* condition, we can apply Lemmas 4.4 and 4.1, and employing induction, we find

$$\mathbf{P}\left[\sum_{k=1}^{n} X_k > x, \sum_{l=1}^{n} Y_l > y\right] \leq \mathbf{P}\left[\sum_{k=1}^{n} X_k^{+} > x, \sum_{l=1}^{n} Y_l^{+} > y\right] \sim \sum_{k=1}^{n}\sum_{l=1}^{n} \mathbf{P}\left[X_k > x, Y_l > y\right]$$

as $x \wedge y \to \infty$. Now, taking into consideration Theorem 4.1, we find

$$\mathbf{P}\left[\sum_{k=1}^{n} X_k > x,\ \sum_{l=1}^{n} Y_l > y\right] \sim \sum_{k=1}^{n}\sum_{l=1}^{n} \mathbf{P}\left[X_k > x,\ Y_l > y\right] \sim \mathbf{P}\left[\bigvee_{k=1}^{n} X_k > x,\ \bigvee_{l=1}^{n} Y_l > y\right],$$

as $x \wedge y \to \infty$. Finally, due to the inequality

$$\mathbf{P}\left[\sum_{k=1}^{n} X_k > x, \sum_{l=1}^{n} Y_l > y\right] \leq \mathbf{P}\left[\bigvee_{i=1}^{n} S_i > x, \bigvee_{j=1}^{n} T_j > y\right] \leq \mathbf{P}\left[\sum_{k=1}^{n} X_k^{+} > x, \sum_{l=1}^{n} Y_l^{+} > y\right]$$

we get the asymptotic relation

$$\mathbf{P}\left[\bigvee_{i=1}^{n} S_i > x,\ \bigvee_{j=1}^{n} T_j > y\right] \sim \sum_{k=1}^{n}\sum_{l=1}^{n} \mathbf{P}\left[X_k > x,\ Y_l > y\right],$$

as $x \wedge y \to \infty$. □

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Recently more and more researchers study two-dimensional risk models; we refer to the reader Hu and Jiang (2013), Cheng and Yu (2019), and Cheng (2021), among others. For

$$U_1(k,\ x) := x - \sum_{i=1}^{k} X_i, \qquad U_2(k,\ y) := y - \sum_{j=1}^{k} Y_j,$$

for $1 \leq k \leq n$, we define now two ruin times,

$$T_{max} := \inf\left\{1 \leq k \leq n : U_1(k,\ x) \wedge U_2(k,\ y) < 0\right\},$$

which denote the first moment when both portfolios are found with negative surplus, and for each portfolio, we define

$$T_1(x) := \inf\left\{1 \leq k \leq n : U_1(k,\ x) < 0 | U_1(0,\ x) = x\right\},$$

$$T_2(y) := \inf\left\{1 \leq k \leq n : U_2(k,\ y) < 0 | U_2(0,\ y) = y\right\},$$

as a result, the second type of ruin type is

$$T_{and} := \max\left\{T_1(x),\ T_2(y)\right\},$$

which corresponds to the first moment, when both portfolios have been with negative surplus, but not necessarily simultaneously. Hence we define the ruin probabilities as

$$\psi_{max}(x,\ y,\ n) = \mathbf{P}[T_{max} \leq n], \qquad \psi_{and}(x,\ y,\ n) = \mathbf{P}[T_{and} \leq n] \tag{4.24}$$

for any $n \in \mathbb{N}$ and $x,\ y > 0$. From (4.24), we easily find out that

$$\psi_{and}(x,\ y,\ n) = \mathbf{P}\left[\bigvee_{i=1}^{n} S_i > x,\ \bigvee_{i=1}^{n} T_i > y\right].$$

Therefore, by Theorem 4.2, we obtain the following result.

Corollary 4.2. *Under conditions of Theorem 4.2, we obtain*

$$\psi_{and}(x,\ y,\ n) \sim \sum_{k=1}^{n}\sum_{l=1}^{n} \mathbf{P}\left[X_k > x,\ Y_l > y\right], \tag{4.25}$$

as $x \wedge y \to \infty$.

Remark 4.2. *From relation (4.25) and the definitions for* T_{max} *and* T_{and}*, we can easily observe that* $\psi_{max}(x,\ y,\ n) \leq \psi_{and}(x,\ y,\ n)$*, for any* $x,\ y > 0$ *and any* $n \in \mathbb{N}$*. Thus, for* $\psi_{max}(x,\ y,\ n)$*, we find the asymptotic upper bound*

$$\psi_{max}(x,\ y,\ n) \lesssim \sum_{k=1}^{n}\sum_{l=1}^{n} \mathbf{P}\left[X_k > x,\ Y_l > y\right],$$

as $x \wedge y \to \infty$*, for any* $n \in \mathbb{N}$.

5. Scalar product

Now we examine the closure property of scalar product in $\mathcal{L}^{(2)}$, $\mathcal{D}^{(2)}$, and in their intersection. Later, we check the same for random sums in two dimensions.

The scalar product has the following tail:

$$\overline{\mathbf{H}}(x,\ y) := \mathbf{P}[\Theta\, X > x,\ \Theta\, Y > y]. \tag{5.1}$$

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Here, we set Θ to be a non-negative random variable with distribution B, such that $B(0-)=0$ and $B(0)<1$. We assume also that Θ is independent of $(X,\ Y)$. These products in relation (5.1) have many applications in actuarial mathematics, in risk management, and in stochastic fields. Next, we use an assumption from Konstantinides and Passalidis (2024b).

Assumption 5.1. *Let us suppose that it holds* $\overline{B}[c\,(x\wedge y)] = o\left(\mathbf{P}\left[\Theta\, X > x,\ \Theta\, Y > y\right]\right) =: o\left[\overline{H}(x,\ y)\right]$, *as* $x\wedge y\to\infty$, *for any* $c>0$.

Remark 5.1. *From Assumption 5.1, it is implied that*

$$\overline{B}(c\,x) = o\left(\mathbf{P}\left[\Theta\, X > x\right]\right), \tag{5.2}$$

as $x\to\infty$, *for any* $c>0$, *and similarly*

$$\overline{B}(c\,y) = o\left(\mathbf{P}\left[\Theta\, Y > y\right]\right), \tag{5.3}$$

as $y\to\infty$ *for any* $c>0$. *This condition is well-known; see, for example,* Tang (2006). *Further, we can see that Assumption 5.1 holds immediately when the distribution B has support bounded from above (because of the unbounded support of* $F,\ G$*).*

The next lemma helps our argumentation and presents a multivariate extension of Tang (2006, Lem. 3.2), providing the existence of an auxiliary function. For a similar paper on auxiliary functions, we refer to Zhou et al. (2012).

Lemma 5.1. *For two distributions B and* $\mathbf{H}$, *with* $\overline{B}(x)>0$, $\overline{H}(x,\ y)>0$ *for any* $x,\ y>0$, *then, Assumption 5.1 holds if and only if there exists a function* $b:[0,\ \infty)\to(0,\ \infty)$, *such that*

$b(x)\to\infty$, *as* $x\to\infty$,
$b(x)=o(x)$, *as* $x\to\infty$,
$\overline{B}[b(x\wedge y)] = o[\overline{H}(x,\ y)]$, *as* $x\wedge y\to\infty$.

Proof.
($\Leftarrow$). The existence of such an auxiliary function easily implies Assumption 5.1; for example, we consider the function $x\wedge y/n$.
($\Rightarrow$). Let suppose that Assumption 5.1 is satisfied. Then we obtain

$$\lim_{x\wedge y\to\infty} \frac{\overline{B}((x\wedge y)/n)}{\overline{H}(x,\ y)} = 0.$$

Let an increasing sequence of positive numbers $\{\lambda_n,\ n\in\mathbb{N}\}$ with $\lambda_{n+1} > (n+1)\,\lambda_n$, for any $n\in\mathbb{N}$, such that for any $x\wedge y\geq \lambda_n$, we have

$$\frac{\overline{B}(x\wedge y/n)}{\overline{H}(x,\ y)} \leq \frac{1}{n}.$$

Therefore, the points (1), (2), and (3) are satisfied with

$$b(x\wedge y) := \sup_{0\leq k\leq x\wedge y} z(k), \qquad z(x\wedge y) = \sum_{n=1}^{\infty} \frac{x\wedge y}{n}\, \mathbf{1}_{\{\lambda_n\leq x\wedge y\leq \lambda_{n+1}\}},$$

which completes the proof. □

Remark 5.2. *We have to mention that in case of distribution B with support bounded from above, the existence of function* $b(\cdot)$ *follows immediately.*

Now we study the closure property of class $\mathcal{D}^{(2)}$ with respect to scalar product.

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Theorem 5.1. *Let* $(X,\ Y)$ *be random vector and* Θ *be random variable, with tail distribution* $\overline{\mathbf{F}}_1(x,\ y) = \mathbf{P}[X > x,\ Y > y]$ *and B, respectively, and assume* $B(0-) = 0, B(0) < 1$. *If* Θ *and* $(X,\ Y)$ *are independent, Assumption 5.1 holds and* $(X,\ Y) \in \mathcal{D}^{(2)}$, *then* $\mathbf{H}(x,\ y) \in \mathcal{D}^{(2)}$.

Proof. Initially, we get from $F,\ G \in \mathcal{D}$ and by Cline and Samorodnitsky (1994, Th. 3.3 (i)) or Leipus, Šiaulys, and Konstantinides (2023, Prop. 5.4 (i)) that the products $\Theta\, X$ and $\Theta\, Y$ follow distributions from $\mathcal{D}$. From Assumption 5.1, we obtain that for any $\mathbf{b} \in (0,\ 1)^n$

$$\limsup_{x\wedge y\to\infty} \frac{\overline{H}_b(x,\ y)}{\overline{H}(x,\ y)} = \limsup_{x\wedge y\to\infty} \frac{\mathbf{P}[\Theta\, X > b_1\, x,\ \ \Theta\, Y > b_2\, y]}{\mathbf{P}[\Theta\, X > x,\ \Theta\, Y > y]} = \limsup_{x\wedge y\to\infty}$$
$$\frac{\left(\int_0^{b(x\wedge y)} + \int_{b(x\wedge y)}^{\infty}\right) \mathbf{P}\left[X > \frac{b_1\, x}{s},\ Y > \frac{b_2\, y}{s}\right]\, B(ds)}{\mathbf{P}[\Theta\, X > x,\ \Theta\, Y > y]} =: \limsup_{x\wedge y\to\infty} \frac{I_1 + I_2}{\mathbf{P}[\Theta\, X > x, \Theta\, Y > y]}. \tag{5.4}$$

Further, we calculate

$$I_2 \leq \int_{b(x\wedge y)}^{\infty} B(ds) = \overline{B}[b(x\wedge y)] = o\left[\overline{H}(x,\ y)\right],$$

as $x \wedge y \to \infty$, due to Assumption 5.1. Hence, taking into account also relation (5.4), we find

$$\limsup_{x\wedge y\to\infty} \frac{\overline{H}_b(x,\ y)}{\overline{H}(x,\ y)} \leq \limsup_{x\wedge y\to\infty} \frac{\int_0^{b(x\wedge y)} \mathbf{P}\left[X > \frac{b_1\, x}{s},\ Y > \frac{b_2\, y}{s}\right]\, B(ds)}{\int_0^{b(x\wedge y)} \mathbf{P}\left[X > \frac{x}{s},\ Y > \frac{y}{s}\right]\, B(ds)}$$
$$\leq \limsup_{x\wedge y\to\infty} \sup_{0<s\leq b(x\wedge y)} \frac{\mathbf{P}\left[X > b_1\, \frac{x}{s},\ Y > b_2\, \frac{y}{s}\right]}{\mathbf{P}\left[X_1 > \frac{x}{s},\ Y > \frac{y}{s}\right]}$$
$$\leq \limsup_{x\wedge y\to\infty} \frac{\mathbf{P}\left[X > b_1\, x,\ Y > b_2\, y\right]}{\mathbf{P}\left[X > x,\ Y > y\right]} < \infty,$$

where in the last step, we used the condition $(X,\ Y) \in \mathcal{D}^{(2)}$. So we get $\mathbf{H}(x,\ y) \in \mathcal{D}^{(2)}$. □

Let us observe, that if Θ has upper bounded support, the proof of Theorem 5.1 (as also of Theorem 5.2) is implied by similar manipulations, replacing $b\,(x \wedge y)$ by the right endpoint of the distribution B. Next, we provide an analogue for class $\mathcal{L}^{(2)}$.

Theorem 5.2. *Let* $(X,\ Y)$ *be a random vector and* Θ *be a random variable, with distributions* $\boldsymbol{F}, B$, *respectively, under condition* $B(0-) = 0,\ B(0) < 1$. *If* Θ *and* $(X,\ Y)$ *are independent, Assumption 5.1 holds, and* $(X,\ Y) \in \mathcal{L}^{(2)}$, *then* $\mathbf{H}(x,\ y) \in \mathcal{L}^{(2)}$.

Proof. From the fact that $(X,\ Y)$ is independent of Θ, hold $F,\ G \in \mathcal{L}$ and relations (5.2) and (5.3), using Cline and Samorodnitsky (1994, Th 2.2 (iii)), we find that distributions of $\Theta\, X$ and $\Theta\, Y$ belong to $\mathcal{L}$. Let $\mathbf{a} = (a_1,\ a_2) > (0,\ 0)$. Then we easily obtain

$$\liminf_{x\wedge y\to\infty} \frac{\overline{H}(x - a_1,\ y - a_2)}{\overline{H}(x,\ y)} = \lim_{x\wedge y\to\infty} \frac{\mathbf{P}[\Theta\, X > x - a_1,\ \Theta\, Y > y - a_2]}{\mathbf{P}[\Theta\, X > x,\ \Theta\, Y > y]} \geq 1. \tag{5.5}$$

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Next, we show the opposite asymptotic inequality. Using Assumption 5.1, we obtain

$$\begin{aligned}
&\limsup_{x\wedge y\to\infty} \frac{\overline{H}(x-a_1,\, y-a_2)}{\overline{H}(x,\, y)} \\
&= \lim_{x\wedge y\to\infty} \frac{1}{\overline{H}(x,\, y)} \left(\int_0^{b(x\wedge y)} + \int_{b(x\wedge y)}^{\infty} \right) \mathbf{P}\left[X > \frac{x-a_1}{s},\, Y > \frac{y-a_2}{s}\right] B(ds) \qquad (5.6)\\
&=: \lim_{x\wedge y\to\infty} \frac{I_1(x,\, y)+I_2(x,\, y)}{\overline{H}(x,\, y)}.
\end{aligned}$$

Thus, by Assumption 5.1, we find

$$I_2(x,\, y) = \int_{b(x\wedge y)}^{\infty} \mathbf{P}\left[X > \frac{x-a_1}{s},\, Y > \frac{y-a_2}{s}\right] B(ds) \le \overline{B}[b(x\wedge y)] = o\left[\overline{H}(x,\, y)\right],$$

hence,

$$\frac{I_2(x,\, y)}{\overline{H}(x,\, y)} = o(1),$$

as $x\wedge y\to\infty$. As a consequence, taking into account also (5.6), we get

$$\begin{aligned}
\limsup_{x\wedge y\to\infty} \frac{\overline{H}_1(x-a_1,\, y-a_2)}{\overline{H}(x,\, y)} &= \limsup_{x\wedge y\to\infty} \int_0^{b(x\wedge y)} \mathbf{P}\left[X > \frac{x-a_1}{s},\, Y > \frac{y-a_2}{s}\right] \frac{B(ds)}{\overline{H}(x,\, y)} \\
&\le \limsup_{x\wedge y\to\infty} \frac{\int_0^{b(x\wedge y)} \mathbf{P}\left[X > \frac{x-a_1}{s},\, Y > \frac{y-a_2}{s}\right] B(ds)}{\int_0^{b(x\wedge y)} \mathbf{P}\left[X > \frac{x}{s},\, Y > \frac{y}{s}\right] B(ds)} \\
&\le \limsup_{x\wedge y\to\infty} \sup_{0<s\le b(x\wedge y)} \frac{\mathbf{P}\left[X > \frac{x-a_1}{s},\, Y > \frac{y-a_2}{s}\right]}{\mathbf{P}\left[X > \frac{x}{s},\, Y > \frac{y}{s}\right]} \\
&= \limsup_{x\wedge y\to\infty} \frac{\mathbf{P}\left[X > x-a_1,\, Y > y-a_2\right]}{\mathbf{P}\left[X > x,\, Y > y\right]} = 1.
\end{aligned}$$

where in the last step, we consider the fact that $(X,\, Y)\in\mathcal{L}^{(2)}$. So we have

$$\limsup_{x\wedge y\to\infty} \frac{\overline{H}(x-a_1,\, y-a_2)}{\overline{H}(x,\, y)} \le 1. \qquad (5.7)$$

From relations (5.5) and (5.7), we conclude $\mathbf{H}(x,\, y)\in\mathcal{L}^{(2)}$. □

The next statement stems from a combination of previous results.

Corollary 5.1. *Let $(X,\, Y)$ be a random vector and Θ be a non-negative random variable with distributions (F, G), B, respectively, under condition $B(0)<1$. If $(X,\, Y)$ and Θ are independent, with $(X,\, Y)\in(\mathcal{D}\cap\mathcal{L})^{(2)}$ and satisfy the Assumption 5.1, then $\mathbf{H}(x,\, y)\in(\mathcal{D}\cap\mathcal{L})^{(2)}$.*

Proof. This follows directly from Theorems 5.1 and 5.2. □

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6. Randomly weighted sums

Finally, we extend Theorem 4.2 to weighted sums. The first kind of weighted sums takes the form

$$S_n(\Theta) = \sum_{k=1}^{n} \Theta\, X_k, \qquad T_n(\Theta) = \sum_{l=1}^{n} \Theta\, Y_l.$$

These quantities have the same discount factor Θ; hence, the $(X_k,\ Y_l)$, for $k,\ l = 1,\ \ldots,\ n$, are the losses or gains of the two lines of business during the k-th period. If the $(x,\ y)$ represents the two initial capitals, respectively, then the ruin probability in this model comes in the form

$$\psi_{and}(x,\ y,\ n) := \mathbf{P}\left[\bigvee_{i=1}^{n} S_i(\Theta) > x,\ \bigvee_{j=1}^{n} T_j(\Theta) > y\right]. \tag{6.1}$$

The ruin probability, in models with insurance and financial risks, plays a significant role in risk theory. For example, we refer to Li and Tang (2015), Yang and Konstantinides (2015), Cheng (2021), and Ji et al. (2023) for discrete-time or continuous-time models, respectively.

The next result is based on Theorem 4.2 and Corollary 5.1. We have to notice that there exists the asymptotic behavior of the ruin probability in (6.1) as well.

Corollary 6.1. *Let $X_1,\ \ldots,\ X_n,\ Y_1,\ \ldots,\ Y_n$ be random variables with the following distributions $F_1,\ \ldots,\ F_n,\ G_1,\ \ldots,\ G_n$, respectively, from class $\mathcal{D} \cap \mathcal{L}$, and they satisfy the GTAI dependence structure. We assume that Θ represents a non-negative random variable with upper bounded support, it is independent of $X_1,\ \ldots,\ X_n,\ Y_1,\ \ldots,\ Y_n$ and $(X_k,\ Y_l) \in (\mathcal{D} \cap \mathcal{L})^{(2)}$, for $k,\ l = 1,\ \ldots,\ n$. If further $X_1,\ \ldots,\ X_n$ are TAI and $Y_1,\ \ldots,\ Y_n$ are TAI, then the following asymptotic relation is true:*

$$\mathbf{P}\left[S_n(\Theta) > x,\ T_n(\Theta) > y\right] \sim \mathbf{P}\left[\bigvee_{i=1}^{n} S_i(\Theta) > x,\ \bigvee_{j=1}^{n} T_j(\Theta) > y\right]$$
$$\sim \mathbf{P}\left[\bigvee_{k=1}^{n} \Theta\, X_k > x,\ \bigvee_{l=1}^{n} \Theta\, Y_l > y\right] \sim \sum_{k=1}^{n}\sum_{l=1}^{n} \mathbf{P}\left[\Theta\, X_k > x,\ \Theta\, Y_l > y\right], \tag{6.2}$$

as $x \wedge y \to \infty$.

Proof. We start from Konstantinides and Passalidis (2024, Lem. 2.1), and because of the upper-bound of Θ, we obtain that the products $\Theta\, X_1,\ \ldots,\ \Theta\, X_n,\ \Theta\, Y_1,\ \ldots,\ \Theta\, Y_n$ are *GTAI*. Now we can apply Cline and Samorodnitsky (1994, Th. 2.2 (iii), Th.3.3 (ii)) to find $\Theta\, X_k \in \mathcal{D} \cap \mathcal{L}$, and $\Theta\, Y_l \in \mathcal{D} \cap \mathcal{L}$ for any $k = 1,\ \ldots,\ n$ and $l = 1,\ \ldots,\ n$. Because of the closedness of class $\mathcal{D}$ and using Theorem 2.2 of Li (2013), we conclude that $\Theta_1\, X_1,\ \ldots,\ \Theta_n\, X_n$ are *TAI* and $\Delta_1\, Y_1,\ \ldots,\ \Delta_n\, Y_n$ are *TAI*.

Next, since Θ is bounded from above, so Assumption 5.1 is fulfilled for Θ, $(X_k,\ Y_l)$, for any $k,\ l = 1,\ \ldots,\ n$, in order to obtain $(\Theta\, X_k,\ \Theta\, Y_l) \in \mathcal{D}^{(2)}$ for any $k = 1,\ \ldots,\ n$ and $l = 1,\ \ldots,\ n$, it is enough to apply Theorem 5.1, and similarly, by Theorem 5.2, we find $(\Theta\, X_k,\ \Theta\, Y_l) \in \mathcal{L}^{(2)}$ for any $k = 1,\ \ldots,\ n$ and $l = 1,\ \ldots,\ n$. Therefore, the $(\Theta\, X_k,\ \Theta\, Y_l) \in (\mathcal{D} \cap \mathcal{L})^{(2)}$ and the $\Theta\, X_1,\ \ldots,\ \Theta\, X_n,\ \Theta\, Y_1,\ \ldots,\ \Theta\, Y_n$ are *GTAI*. Finally, applying Theorem 4.2, we conclude (6.2). □

Now we need some preliminary results. Several times before proving that the convolution product satisfies $H \in \mathcal{B}$, with $\mathcal{B}$ some distribution class, we need to prove that $H_\varepsilon(x) := \mathbf{P}[(\Theta \vee \varepsilon)\, X \leq x]$ belongs to this class $\mathcal{B}$ for any $\varepsilon > 0$. Following the approach in Cline and Samorodnitsky (1994), we show that for some constant $\delta > 0$, if $H_\varepsilon \in \mathcal{L}^{(2)}$, for any $\varepsilon \in (0,\ \delta)$, then $H \in \mathcal{L}^{(2)}$. However, the next results deserve theoretical attention by its own merit.

From here until the end of paper, we assume that X, Y are non-negative random variables.

Lemma 6.1. *If for some constant vector* $\boldsymbol{\delta}=(\delta_1,\ \delta_2)>(0,\ 0)$*, for any* $\varepsilon_1 \in (0,\ \delta_1)$ *and for any* $\varepsilon_2 \in (0,\ \delta_2)$*, holds* $((\Theta \vee \varepsilon_1)\, X,\ (\Delta \vee \varepsilon_2)\, Y) \in \mathcal{L}^{(2)}$*, with* X, Y, Θ, Δ *non-negative random variables, then we conclude that* $(\Theta\, X,\ \Delta\, Y) \in \mathcal{L}^{(2)}$.

Proof. Keeping in mind that $((\Theta \vee \varepsilon_1)\, X,\ (\Delta \vee \varepsilon_2)\, Y) \in \mathcal{L}^{(2)}$, we start by Cline and Samorodnitsky (1994, th. 2.2 (i)) to establish that due to $(\Theta \vee \varepsilon_1)\, X \in \mathcal{L}$, $(\Delta \vee \varepsilon_2)\, Y \in \mathcal{L}$, we get $\Theta\, X \in \mathcal{L}$ and $\Delta\, Y \in \mathcal{L}$. Next, we check the second property of class $\mathcal{L}^{(2)}$. Let $(a_1,\ a_2)>(0,\ 0)$, then

$$\liminf_{x\wedge y\to\infty} \frac{\mathbf{P}\left[\Theta\, X > x-a_1,\ \Delta\, Y > y-a_2\right]}{\mathbf{P}\left[\Theta\, X > x,\ \Delta\, Y > y\right]} \geq 1, \tag{6.3}$$

Next, for any $(\varepsilon_1,\ \varepsilon_2)>(0,\ 0)$, we find

$$\mathbf{P}[(\Theta \vee \varepsilon_1)\, X > x,\ (\Delta \vee \varepsilon_2)\, Y > y] \geq \mathbf{P}[\Theta\, X > x,\ \Delta\, Y > y]$$

$$\geq \mathbf{P}[\Theta\, X > x,\ \Theta > \varepsilon_1,\ \Delta\, Y > y]$$

$$= \mathbf{P}[(\Theta \vee \varepsilon_1)\, X > x,\ \Delta\, Y > y] - \mathbf{P}[\Theta \leq \varepsilon_1]\, \mathbf{P}[X\, \varepsilon_1 > x,\ \Delta\, Y > y]$$

$$\geq \mathbf{P}[\Theta > \varepsilon_1]\, \mathbf{P}[(\Theta \vee \varepsilon_1)\, X > x,\ \Delta\, Y > y,\ \Delta > \varepsilon_2] =$$

$$\mathbf{P}[\Theta > \varepsilon_1]\, \big(\mathbf{P}[(\Theta \vee \varepsilon_1)X > x, (\Delta \vee \varepsilon_2)Y > y] - \mathbf{P}[(\Theta \vee \varepsilon_1)X > x,\ \varepsilon_2 Y > y]\mathbf{P}[\Delta \leq \varepsilon_2]\big)$$

$$\geq \mathbf{P}[\Theta > \varepsilon_1]\, \mathbf{P}[\Delta > \varepsilon_2]\, \mathbf{P}[(\Theta \vee \varepsilon_1)\, X > x,\ (\Delta \vee \varepsilon_2)\, Y > y],$$

hence we conclude

$$\begin{aligned} &\mathbf{P}[(\Theta \vee \varepsilon_1)\, X > x,\ (\Delta \vee \varepsilon_2)\, Y > y] \geq \mathbf{P}[\Theta\, X > x,\ \Delta\, Y > y] \\ &\geq \mathbf{P}[\Theta > \varepsilon_1]\, \mathbf{P}[\Delta > \varepsilon_2]\, \mathbf{P}[(\Theta \vee \varepsilon_1)\, X > x,\ (\Delta \vee \varepsilon_2)\, Y > y]. \end{aligned} \tag{6.4}$$

Therefore, using (6.4) and due to properties of $\mathcal{L}^{(2)}$, for $((\Theta \vee \varepsilon_1)\, X,\ (\Delta \vee \varepsilon_2)\, Y)$, we obtain

$$\limsup_{x\wedge y\to\infty} \frac{\mathbf{P}\left[\Theta\, X > x-a_1,\ \Delta\, Y > y-a_2\right]}{\mathbf{P}\left[\Theta\, X > x,\ \Delta\, Y > y\right]} \leq$$

$$\limsup_{x\wedge y\to\infty} \frac{\mathbf{P}\left[(\Theta \vee \varepsilon_1)\, X > x-a_1,\ (\Delta \vee \varepsilon_2)\, Y > y-a_2\right]}{\mathbf{P}[\Theta > \varepsilon_1]\mathbf{P}[\Delta > \varepsilon_2]\mathbf{P}\left[(\Theta \vee \varepsilon_1)X > x, (\Delta \vee \varepsilon_2)Y > y\right]} = \frac{1}{\mathbf{P}[\Theta > \varepsilon_1]\mathbf{P}[\Delta > \varepsilon_2]},$$

and leaving ε_1 and ε_2 to tend to zero, we get

$$\limsup_{x\wedge y\to\infty} \frac{\mathbf{P}\left[\Theta\, X > x-a_1,\ \Delta\, Y > y-a_2\right]}{\mathbf{P}\left[\Theta\, X > x,\ \Delta\, Y > y\right]} \leq 1,$$

hence, from (6.3) and from last inequality, we reach to $(\Theta\, X,\ \Delta\, Y) \in \mathcal{L}^{(2)}$. □

Lemma 6.2. *Let* X *and* Y *be non-negative random variables, with* $(X,\ Y) \in \mathcal{L}^{(2)}$ *and* Θ *and* Δ *be non-negative, non-degenerated to zero random variables, independent of* $(X,\ Y)$*. We assume that*

$$\mathbf{P}\,[\Theta > x] = o(\mathbf{P}\left[\Theta\, X > c_1\, x,\ \Delta\, Y > c_2\, y\right]) = \mathbf{P}\left[\Delta > y\right], \tag{6.5}$$

as $x \wedge y \to \infty$*, for any* $c_1,\ c_2 > 0$*. Then* $(\Theta\, X,\ \Delta\, Y) \in \mathcal{L}^{(2)}$.

Proof. From (6.5), we obtain

$$\frac{\mathbf{P}\,[\Theta > x]}{\mathbf{P}\,[\Theta\, X > c_1\, x]} \leq \frac{\mathbf{P}\,[\Theta > x]}{\mathbf{P}\left[\Theta\, X > c_1\, x,\ \Delta\, Y > c_2\, y\right]} \longrightarrow 0, \tag{6.6}$$

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as $x \wedge y \to \infty$, and similarly, we find $\mathbf{P}\left[\Delta > y\right] = o(\mathbf{P}\left[\Delta\, Y > c_2\, y\right])$, as $x \wedge y \to \infty$, for any $c_1,\ c_2 > 0$. Hence, by Cline and Samorodnitsky (1994,Th. 2.2), we find $\Theta\, X \in \mathcal{L}$ and $\Delta\, Y \in \mathcal{L}$. Next, we show the second property of $(\Theta\, X,\ \Delta\, Y) \in \mathcal{L}^{(2)}$. Indeed, from Lemma 6.1, we see that it is enough to show this for any $\Theta \geq \varepsilon_1$ and $\Delta \geq \varepsilon_2$ almost surely for any $\varepsilon_1,\ \varepsilon_2 > 0$. Let consider some $a_1,\ a_2 > 0$ and some $k_1,\ k_2,\ k > 0$, such that for a large enough $x_0 \geq 0$, it holds

$$\mathbf{P}\left[X > x - \frac{a_1}{\varepsilon_1},\ Y > y - \frac{a_2}{\varepsilon_2}\right] \leq (1+k)\,\mathbf{P}\left[X > x,\ Y > y\right], \tag{6.7}$$

for any $x \wedge y \geq x_0$ and

$$\mathbf{P}\left[X > x - \frac{a_1}{\varepsilon_1}\right] \leq (1+k_1)\,\mathbf{P}\left[X > x\right], \quad \mathbf{P}\left[Y > y - \frac{a_2}{\varepsilon_2}\right] \leq (1+k_2)\,\mathbf{P}\left[Y > y\right], \tag{6.8}$$

for any $x \geq x_0$ and $y \geq x_0$, respectively. Then, for all $x \wedge y \geq x_0$, we obtain

$$\mathbf{P}\left[\Theta\, X > x - a_1,\ \Delta\, Y > y - a_2\right] = \left(\int_{\varepsilon_1}^{x/x_0} + \int_{x/x_0}^{\infty}\right)$$
$$\left(\int_{\varepsilon_2}^{y/x_0} + \int_{y/x_0}^{\infty}\right)\mathbf{P}\left[X > \frac{x-a_1}{s},\ Y > \frac{y-a_2}{t}\right]\mathbf{P}[\Theta \in ds,\ \Delta \in dt] =: \sum_{m=1}^{4} I_m(x,\ y), \tag{6.9}$$

where we find

$$I_4(x,\ y) = \int_{x/x_0}^{\infty}\int_{y/x_0}^{\infty}\mathbf{P}\left[X > \frac{x-a_1}{s},\ Y > \frac{y-a_2}{t}\right]\mathbf{P}[\Theta \in ds,\ \Delta \in dt],$$

that gives

$$I_4(x,\ y) \leq \mathbf{P}\left[\Theta \geq \frac{x}{x_0},\ \Delta \geq \frac{y}{x_0}\right]. \tag{6.10}$$

Now we estimate $I_1(x,\ y)$

$$\begin{aligned} I_1(x,\ y) &= \int_{\varepsilon_1}^{x/x_0}\int_{\varepsilon_2}^{y/x_0}\mathbf{P}\left[X > \frac{x-a_1}{s},\ Y > \frac{y-a_2}{t}\right]\mathbf{P}[\Theta \in ds,\ \Delta \in dt] \qquad (6.11)\\ &\leq \int_{\varepsilon_1}^{x/x_0}\int_{\varepsilon_2}^{y/x_0}\mathbf{P}\left[X > \frac{x}{s} - \frac{a_1}{\varepsilon_1},\ Y > \frac{y}{t} - \frac{a_2}{\varepsilon_2}\right]\mathbf{P}[\Theta \in ds,\ \Delta \in dt]\\ &\leq (1+k)\int_{\varepsilon_1}^{x/x_0}\int_{\varepsilon_2}^{y/x_0}\mathbf{P}\left[X > \frac{x}{s},\ Y > \frac{y}{t}\right]\mathbf{P}[\Theta \in ds,\ \Delta \in dt]\\ &\leq (1+k)\,\mathbf{P}\left[\Theta\, X > x,\ \Delta\, Y > y\right], \end{aligned}$$

thus we get $I_1(x,\ y) \leq (1+k)\,\mathbf{P}\left[\Theta\, X > x,\ \Delta\, Y > y\right]$, which follows from (6.7).

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Next we consider $I_2(x,\ y)$

$$I_2(x,\ y) = \int_{\varepsilon_1}^{x/x_0} \int_{y/x_0}^{\infty} \mathbf{P}\left[X > \frac{x-a_1}{s},\ Y > \frac{y-a_2}{t}\right] \mathbf{P}[\Theta \in ds,\ \Delta \in dt] \tag{6.12}$$

$$\leq \int_{\varepsilon_1}^{x/x_0} \mathbf{P}\left[X > \frac{x}{s} - \frac{a_1}{\varepsilon_1}\right] \mathbf{P}\left[\Theta \in ds,\ \Delta > \frac{y}{x_0}\right]$$

$$\leq (1+k_1) \int_{\varepsilon_1}^{x/x_0} \mathbf{P}\left[X > \frac{x}{s}\right] \mathbf{P}\left[\Theta \in ds,\ \Delta > \frac{y}{x_0}\right]$$

$$\leq (1+k_1)\, \mathbf{P}\left[\Theta\, X > x,\ \Delta > \frac{y}{x_0}\right] \leq (1+k_1)\, \mathbf{P}\left[\Delta > \frac{y}{x_0}\right],$$

which means $I_2(x,\ y) \leq (1+k_1)\, \mathbf{P}\left[\Delta > y/x_0\right]$, where in the pre-last step, we use the first relation in (6.8).

For $I_3(x,\ y)$, we use the second relation in (6.8), and due to symmetry with respect to (6.13), we find

$$I_3(x,\ y) = \int_{x/x_0}^{\infty} \int_{\varepsilon_2}^{\infty} \mathbf{P}\left[X > \frac{x-a_1}{s},\ Y > \frac{y-a_2}{t}\right] \mathbf{P}[\Theta \in ds,\ \Delta \in dt]$$

$$\leq (1+k_2)\, \mathbf{P}\left[\Theta > \frac{x}{x_0}\right]. \tag{6.13}$$

Therefore, putting the estimation from (6.10) to (6.15) into (6.9), we conclude

$$\mathbf{P}\left[\Theta\, X > x - a_1,\ \Delta\, Y > y - a_2\right] \leq \mathbf{P}\left[\Theta > \frac{x}{x_0},\ \Delta > \frac{y}{x_0}\right]$$
$$+(1+k)\, \mathbf{P}\left[\Theta\, X > x,\ \Delta\, Y > y\right] + (1+k_2)\, \mathbf{P}\left[\Theta > \frac{x}{x_0}\right] + (1+k_1)\, \mathbf{P}\left[\Delta > \frac{y}{x_0}\right],$$

Now, because of (6.5) and the relation

$$\frac{\mathbf{P}\left[\Theta > x,\ \Delta > y\right]}{\mathbf{P}\left[\Theta\, X > c_1\, x,\ \Delta\, Y > c_2\, y\right]} \leq \frac{\mathbf{P}\left[\Theta > x\right]}{\mathbf{P}\left[\Theta\, X > c_1\, x,\ \Delta\, Y > c_2\, y\right]} \longrightarrow 0,$$

as $x \wedge y \to \infty$, for any $c_1,\ c_2 > 0$, we find

$$\lim_{x \wedge y \to \infty} \frac{\mathbf{P}\left[\Theta\, X > x - a_1,\ \Delta\, Y > y - a_2\right]}{\mathbf{P}\left[\Theta\, X > x,\ \Delta\, Y > y\right]} \leq 1 + k.$$

By these inequalities and relation (6.6), in combination of the arbitrary choice of k and relation (6.3), we have $(\Theta\, X,\ \Delta\, Y) \in \mathcal{L}^{(2)}$.

Next, we consider the asymptotic joint-tail behavior of discounted aggregate claims in a two-dimensional discrete-time risk model, where the vector $(X_k,\ Y_k)$ represents losses in two lines of business at the k-th period, while the $(\Theta_k,\ \Delta_k)$ represents the discount factors of these two lines of business, respectively. In this risk model, we study only the aggregate claims, and we accept that the $\Theta_1,\ \ldots,\ \Theta_n,\ \Delta_1,\ \ldots,\ \Delta_m$ are independent of claims $X_1,\ \ldots,\ X_n,\ Y_1,\ \ldots,\ Y_m$. For further reading on risk models with dependence among the discount factors and main claims, see Chen (2011, 2017) and Yang et al. (2016), but only in one dimension. Specifically, we have the sums:

$$S_n^{\Theta} := \sum_{k=1}^{n} \Theta_k X_k, \qquad T_n^{\Delta} := \sum_{l=1}^{n} \Delta_l Y_l.$$

Assumption 6.1. *There exist constants $0 < \xi_k \leq \delta_k$ such that hold $\xi_k \leq \Theta_k \leq \delta_k$ almost surely, for any $k = 1, \ldots, n$, and there exist constants $0 < \gamma_l \leq \zeta_l$ such that hold $\gamma_l \leq \Delta_l \leq \zeta_l$ almost surely, for any $l = 1, \ldots, n$.*

Theorem 6.1. *Let $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ be non-negative, random variables with the following distributions $F_1, \ldots, F_n, G_1, \ldots, G_n$, respectively, from class $\mathcal{D} \cap \mathcal{L}$, and they satisfy the GTAI dependence structure, with $(X_k, Y_l) \in (\mathcal{D} \cap \mathcal{L})^{(2)}$ for any $k = 1, \ldots, n$ and $l = 1, \ldots, n$. We suppose that the random discount factors $\Theta_1, \ldots, \Theta_n, \Delta_1, \ldots, \Delta_n$ satisfy Assumption 6.1 and are independent of $X_1, \ldots, X_n, Y_1, \ldots, Y_n$. Then the products $\Theta_1 X_1, \ldots, \Theta_n X_n, \Delta_1 Y_1, \ldots, \Delta_n Y_n$ are GTAI with $(\Theta_k X_k, \Delta_l Y_l) \in (\mathcal{D} \cap \mathcal{L})^{(2)}$. If further $X_1, \ldots, X_n$ are TAI and $Y_1, \ldots, Y_n$ are TAI, then the following asymptotic relations hold*

$$\mathbf{P}\left[S_n^{\Theta} > x,\ T_n^{\Delta} > y\right] \sim \mathbf{P}\left[\bigvee_{i=1}^{n} S_i^{\Theta} > x,\ \bigvee_{j=1}^{n} T_j^{\Delta} > y\right] \sim \sum_{k=1}^{n}\sum_{l=1}^{n} \mathbf{P}\left[\Theta\, X_k > x,\ \Delta\, Y_l > y\right], \tag{6.14}$$

as $x \wedge y \to \infty$.

Proof. Taking into account the upper bound for discount factors $\Theta_1, \ldots, \Theta_n, \Delta_1, \ldots, \Delta_n$ and their independence from $X_1, \ldots, X_n, Y_1, \ldots, Y_n$, we apply Konstantinides and Passalidis (2024, Lem. 2.1) to find that the products $\Theta_1 X_1, \ldots, \Theta_n X_n, \Delta_1 Y_1, \ldots, \Delta_n Y_n$ are *GTAI*. Now by Cline and Samorodnitsky (1994, Th. 3.3 (i)), we get $\Theta_k X_k \in \mathcal{D} \cap \mathcal{L}$ and $\Delta_l Y_l \in \mathcal{D} \cap \mathcal{L}$, for any $k = 1, \ldots, n$ and for any $l = 1, \ldots, n$. As a result, by class $\mathcal{D}$, using Li (2013, Th. 2.2), the $\Theta_1 X_1, \ldots, \Theta_n X_n$ are *TAI*, and $\Delta_1 Y_1, \ldots, \Delta_n Y_n$ are *TAI*.

Next, we check if $(\Theta_k X_k, \Delta_l Y_l) \in (\mathcal{D} \cap \mathcal{L})^{(2)}$ for any $k = 1, \ldots, n$ and $l = 1, \ldots, n$. Let $\mathbf{b} = (b_1, b_2) \in (0, 1)^2$, then

$$\limsup_{x \wedge y \to \infty} \frac{\mathbf{P}\left[\Theta_k X_k > b_1 x,\ \Delta_l Y_l > b_2 y\right]}{\mathbf{P}\left[\Theta_k X_k > x,\ \Delta_l Y_l > y\right]} \leq \limsup_{x \wedge y \to \infty} \frac{\mathbf{P}\left[X_k > b_1 \dfrac{x}{\delta_k},\ Y_l > b_2 \dfrac{y}{\zeta_l}\right]}{\mathbf{P}\left[X_k > \dfrac{x}{\xi_k},\ Y_l > \dfrac{y}{\gamma_l}\right]} < \infty,$$

which follows from the inequalities

$$\frac{b_1}{\delta_k} < \frac{1}{\xi_k}, \qquad \frac{b_2}{\zeta_l} < \frac{1}{\gamma_l},$$

and the membership $(X_k, Y_l) \in (\mathcal{D} \cap \mathcal{L})^{(2)}$ for any $k = 1, \ldots, n$ and $l = 1, \ldots, n$. Hence, we find the relation $(\Theta_k X_k, \Delta_l Y_l) \in \mathcal{D}^{(2)}$ for any $k = 1, \ldots, n$ and $l = 1, \ldots, n$.

Now, noticing that relation (6.5) is satisfied because of Assumption 6.1, we obtain directly from Lemma 6.2 the inclusion $(\Theta_k X_k, \Delta_l Y_l) \in \mathcal{L}^{(2)}$ for any $k = 1, \ldots, n$ and $l = 1, \ldots, n$. Hence $(\Theta_k X_k, \Delta_l Y_l) \in (\mathcal{D} \cap \mathcal{L})^{(2)}$ for any $k = 1, \ldots, n$ and $l = 1, \ldots, n$ and by application of Theorem 4.2 for the products, we conclude relation (6.16). □

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