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Compositio Math. **148** (2012), 640–668.

[doi:10.1112/S0010437X11005458](https://doi.org/10.1112/S0010437X11005458)



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ABSTRACT

We define an analytic index and prove a topological index theorem for a non-compact manifold M_0 with poly-cylindrical ends. Our topological index theorem depends only on the principal symbol, and establishes the equality of the topological and analytical index in the group $K_0(C^*(M))$, where $C^*(M)$ is a canonical C^* -algebra associated to the canonical compactification M of M_0 . Our topological index is thus, in general, not an integer, unlike the usual Fredholm index appearing in the Atiyah–Singer theorem, which is an integer. This will lead, as an application in a subsequent paper, to the determination of the K -theory groups $K_0(C^*(M))$ of the groupoid C^* -algebra of the manifolds with corners M . We also prove that an elliptic operator P on M_0 has an invertible perturbation $P + R$ by a lower-order operator if and only if its analytic index vanishes.

Introduction

Let M_0 be a smooth, *compact* manifold and D be an elliptic differential operator of order m acting between smooth sections of vector bundles on M_0 . Then D is continuous and Fredholm as a map between suitable Sobolev spaces (more precisely $H^s \rightarrow H^{s-m}$). In particular, the kernel and cokernel of D are finite dimensional. This allows one to define $\text{ind}(D)$, the *Fredholm index* of D , by

$$\text{ind}(D) := \dim \ker(D) - \dim \text{coker}(D) = \dim \ker(D) - \dim(H^{s-m}/DH^s).$$

The knowledge of the Fredholm index is relevant because it gives an obstruction to D being invertible. More precisely, we have the following result whose proof is an easy exercise in functional analysis.

THEOREM 1. *There exists a pseudodifferential operator R of order less than $m = \text{ord}(D)$ on the compact manifold M_0 such that $D + R : H^s \rightarrow H^{s-m}$ is an isomorphism if and only if $\text{ind}(D) = 0$. Moreover, if one such R exists, then it can be chosen to be of order $-\infty$, i.e., regularizing.*

If M_0 is non-compact, which is the case of main interest in our paper, then an elliptic differential operator D on M_0 need not be Fredholm in general (here, by ‘elliptic’ we mean that the principal symbol is invertible). The Fredholm index of D is therefore not defined and Theorem 1 is meaningless as it stands. We therefore introduce an extension of the Fredholm index on manifolds with poly-cylindrical ends. A *manifold with poly-cylindrical ends* is locally diffeomorphic at infinity with a product of manifolds with cylindrical ends (see Definition 1).

Received 8 December 2009, accepted in final form 10 December 2010, published online 9 November 2011.

2010 Mathematics Subject Classification 58B34 (primary), 35S05 (secondary).

Keywords: index theory, manifold with poly-cylindrical ends, manifold with corners, Lie groupoid.

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From now on, M_0 will denote a manifold with poly-cylindrical ends, usually non-compact, and M will denote its canonical compactification to a manifold with corners. Two of the main results of this paper are an extension of the Atiyah–Singer index theorem on the equality of the topological and analytic index and an extension of Theorem 1 to manifolds with poly-cylindrical ends (with the analytic index replacing the Fredholm index). Our work is motivated, in part, by recent work of Leichtnam and Piazza [LP05] and Nistor and Troitsky [NT04], who have showed that suitable generalizations of Theorem 1 are useful in geometric and topological applications. Extensions of Theorem 1 seem to be important for the study of boundary value problems on polyhedral domains [Nis05] using the method of layer potentials. In the process, we establish several extensions to non-compact manifolds with poly-cylindrical ends of the classical properties on the topological index of elliptic operators on compact manifolds [AS68, Car06].

Let us explain in a little more detail our main results. The tangent bundle TM_0 extends to a bundle A_M on M (which, we recall, is the canonical compactification of M_0 to a manifold with corners) defined by the property that its sections are the vector fields tangent to the faces of M . Let D be a differential (or pseudodifferential operator) on M_0 compatible with the structure at infinity on M_0 (i.e., $P \in \Psi_b^\infty(M)$, where $\Psi_b^\infty(M)$ denotes the algebra of pseudodifferential operators on M_0 compatible with the structure at infinity). Then the principal symbol of D extends to a symbol defined on A_M^* (see [Mel95]).

Assume that D is elliptic, in the sense that its principal symbol is invertible on A_M^* outside the zero section. Then D is invertible modulo regularizing operators. Let $C^*(M)$ be the norm closure of the algebra $\Psi_b^{-\infty}(M)$ of regularizing operators on M_0 compatible with the structure at infinity of M_0 . The obstruction of D to be invertible defines a map

$$\text{ind}_a = \text{ind}_a^M : K^0(A_M^*) \rightarrow K_0(C^*(M)). \tag{1}$$

This extension of the topological index is the *analytic index* map mentioned above. In particular, the principal symbol of D defines an element $\text{ind}_a(D) \in K_0(C^*(M))$. Theorem 1 then remains true for manifolds with poly-cylindrical ends if we use the analytic in place of the Fredholm index.

To be more precise, the invertibility of the principal symbol of a differential operator D on M_0 , i.e., the ellipticity of D , is not sufficient for the operator D to be Fredholm; but it is possible to add extra conditions on D , namely the invertibility of the restriction of the operator to the boundary, to obtain Fredholm operators. There are thus two types of index problems: one type involving the analytic index of (1) (with values in the K -group of a C^* -algebra) and the other type involving the usual Fredholm index (with values in \mathbb{Z}). In this article, we deal with the first type of index problem.

For manifolds with smooth, connected boundary, the structure of $C^*(M)$ is very simple (see [MN98, Mon03]), and that shows immediately that the groups $K_j(C^*(M))$ vanish, so our topological index is trivial. This has, however, the non-trivial consequence that any Dirac operator on a manifold with smooth boundary can be endowed with Fredholm boundary conditions. This fact, which goes back to Atiyah *et al.* [APS73], is of course non-trivial. The relation between the groups $K_j(C^*(M))$, for suitable M , and the existence of Fredholm boundary conditions was discussed also in [Nis03a, Nis03b].

To explain our generalization of the topological index theorem, consider an embedding $\iota : M \rightarrow X$ of manifolds with corners and let $\iota_!$ be the push-forward map in K -theory.

Then Theorem 12 states the commutativity of the diagram

$$\begin{CD}
 K_0(C^*(M)) @>\iota_*>> K_0(C^*(X)) \\
 @V\text{ind}_a^M VV @VV\text{ind}_a^X V \\
 K^0(A_M^*) @>\iota!>> K^0(A_X^*)
 \end{CD} \tag{2}$$

where the morphism ι_* is defined by Morita equivalence.

In order to interpret the above diagram as an index theorem in the usual way, we need to choose X such that $\iota_* : K_0(C^*(M)) \rightarrow K_0(C^*(X))$ and $\text{ind}_a^X : K^0(A_X^*) \rightarrow K_0(C^*(X))$ are isomorphisms. This would provide us with both an identification of the groups $K_0(C^*(M))$ and of the analytic index map ind_a^M .

To obtain a manifold X with the above mentioned properties, we shall proceed as follows. Let us assume M is compact with embedded faces (recall [Mel95] that this means that each face of M of maximal dimension has a defining function). To M we will associate a non-canonical manifold with embedded faces X_M and an embedding $i : M \rightarrow X_M$ such that the following properties hold.

- (i) Each open face of X_M is diffeomorphic to a Euclidean space.
- (ii) Each face of M is the transverse intersection of M and of a face of X_M .
- (iii) The map $F \rightarrow F \cap M$ establishes a bijection between the open faces of M and those of X_M .

We shall say that an embedding X_M with the above properties is a classifying space for M and prove that $i_K : K_0(C^*(M)) \rightarrow K_0(C^*(X_M))$ is an isomorphism, where j_K is a canonical morphism associated to any embedding of manifolds with corners j (see Lemma 6).

Let us now summarize the contents of the paper. In § 1, we review the definitions of manifolds with corners, of Lie groupoids, and of Lie algebroids. We also review and extend a result from [Nis00] on the integration of Lie algebroids. In § 2 we recall the definition of the analytic index using the tangent groupoid and then compare this definition with other possible definitions of an analytic index. In the process, we establish several technical results on tangent groupoids. As an application, in Theorem 7, we provide the generalization of Theorem 1 mentioned above. Section 3 contains the main properties of the analytic index. In that section, we also introduce the morphism j_K associated to an embedding of manifolds with corners j and we provide conditions for j_K and ind_a^M to be isomorphisms. The compatibility of the analytic index and of the shriek maps is established in § 4. This is then used in the following section to establish the equality of the analytic and topological indexes. In the final section we show that it is indeed possible to find a space X_M and an embedding $i : M \rightarrow X_M$ with the properties (i)–(iii) above (i.e., a classifying space for M exists). The computation of the K -groups of X_M can be obtained (at least in concrete situations) using standard algebraic topology techniques. However, this requires a different type of ideas, so is better left for another paper.

In addition to the works of Leichtnam–Piazza [LP05] and Nistor–Troitsky [NT04] mentioned above, many other related works have appeared before and after our paper was first circulated. While it is impractical to cite all these works, we do want to mention the following papers: the paper of Androulidakis and Skandalis [AS09], where they use the integration of Lie algebroids to study singular foliations and their index theory; the paper of Melo *et al.* [MSS06], where they prove an index theorem for manifolds with boundary in the framework of Boutet–de–Montvel calculus; the paper of Bunke [Bun09], in which analysis on families of manifolds with corners plays an important role; and the papers of Nazaïkinskiĭ *et al.* [NSS07], Melrose and his

collaborators [AM09, MP92] and Nistor [Nis03b], where possible generalizations of the Atiyah–Patodi–Singer boundary conditions were discussed.

1. Basic definitions

In this section we recall several basic definitions and constructions, including: manifolds with corners, manifolds with poly-cylindrical ends, groupoids, adapted pseudodifferential operators, and Lie algebroids. Good references to the special issues we deal with in this section are [LMN00, Mon99, Mon03, NWX99]. General references to the subject are [Con94, Mac87, Mel95, Ren80]. See [AR09, AJ08, BP09, Erp10, Vas09] for some recent applications.

1.1 Manifolds with corners

A manifold with corners M is a manifold modeled on $(\mathbb{R}_+)^n$ (which is denoted \mathbb{R}_+^n , and must not be understood as a half-space of \mathbb{R}^n , but as a quadrant). This means that any point $x \in M$ has a neighborhood of the form $\mathbb{R}^{n-k} \times \mathbb{R}_+^k$ (with x mapping to $0 = (0, 0, \dots, 0)$). We shall call k the *depth* of x .

The set of points of depth k is a union of connected components, each of which is a smooth submanifold in its own, called an *open face of codimension k* . The closure of an open face (of codimension k) is just called a *face* of M (of codimension k). A closed face is not necessarily a manifold with corners (think of the ‘tear-drop domain’ in the plane). A codimension one face of M is called a *hyperface* of M .

We shall sometimes require that each hyperface of M be an embedded submanifold of M . If this is the case, we shall call M a *manifold with embedded faces*, while in Melrose’s terminology it is just called a *manifold with corners*. A manifold M is a manifold with embedded faces if each hyperface has a *defining function*. Recall that a function $\rho : M \rightarrow \mathbb{R}_+$ is a defining function of the hyperface $H \subset M$ if ρ is smooth, if $\rho(x) = 0$ precisely when $x \in H$, and $d\rho$ does not vanish on H . Such a defining function provides us, in particular, with a trivialization of $NH = TM/TH$, the normal bundle of H .

If X and Y are two manifolds with corners, a map $f : X \rightarrow Y$ is called a *closed embedding* of manifolds with corners if the following properties hold.

- (a) It is differentiable, injective, with closed range.
- (b) The differential df is injective.
- (c) For each open face F of Y , $f(X)$ is transverse to F (recall that this means that $df(T_x X) + T_y F = T_y Y$, if $y = f(x) \in F$).
- (d) Each hyperface of X is a connected component of the inverse image of a hyperface of Y .

In particular, x and $f(x)$ will have the same depth.

A *submersion* $f : M \rightarrow N$ (between two manifolds with corners M and N) is a differentiable map f such that the following properties hold.

- (a) The differential $df(v)$ is an inward pointing tangent vector of N if and only if v is an inward pointing vector of M .
- (b) The differential df is surjective at all points.

If $i : X \rightarrow Y$ is a closed embedding, then there exists a *tubular neighborhood* $U \subset Y$, which means that there exists a vector bundle E over X , and an open neighborhood Z of the zero

section isomorphic to U , such that the following diagram is commutative.

$$\begin{array}{ccc} Z & \xrightarrow{\simeq} & U \\ \uparrow & & \downarrow \\ X & \xrightarrow{i} & Y \end{array}$$

The existence of such a tubular neighborhood is proved in [AIN06, Dou64]. For the benefit of the reader, we now provide a sketch of the proof of such a tubular neighborhood that formulated in our framework. Let us denote by $\mathcal{V}_b(X) \subset \Gamma(TX)$ the subspace of vector fields *tangent* to all faces of X . Then $\mathcal{V}_b(X)$ is a Lie algebra with respect to the Lie bracket and a $\mathcal{C}^\infty(X)$ -module. Moreover, there exists a vector bundle $A_X \rightarrow X$, uniquely determined up to isomorphism, such that

$$\mathcal{V}_b(X) \simeq \Gamma(A_X) \quad \text{as } \mathcal{C}^\infty(X)\text{-modules.}$$

If $Y \subset X$ is a closed, embedded submanifold with corners, then $\mathcal{V}_b(Y)$ consists of the restrictions to Y of the vector fields $V \in \mathcal{V}_b(M)$ with the property that V is tangent to Y . Fix an arbitrary open face F of Y . The assumption that every hyperface of Y be contained in a hyperface of X (even a connected component of the intersection of Y with a hyperface of X) implies that there exists an open face F' of X , of the same codimension as F , such that $F \subset F'$. Next, the definition of an embedded submanifold with corners implies that Y is transverse to F' and hence the natural map $T_x F' / T_x F \rightarrow T_x X / T_x Y$ is an isomorphism for any $x \in F$. It is then possible¹ to construct a connection whose associated exponential map $\exp : N_X^Y \rightarrow Y$ is such that its restrictions to each face is a map $\exp|_F : (N_X^Y)_F \rightarrow F'$ and is a local diffeomorphism around the zero section. Using this exponential map, one can define E , Z , and U with the required properties.

1.2 Manifolds with poly-cylindrical ends

We shall use the notation and terminology introduced in the previous subsection. Let us fix a metric on the bundle $A_M \rightarrow M$. This metric defines, in particular, a Riemannian metric on M_0 , the interior of M . The following definition is from [Mel95].

DEFINITION 1. A manifold with poly-cylindrical ends is a smooth Riemannian manifold M_0 that is diffeomorphic with the interior of a compact manifold with corners M such that the metric on M_0 is the restriction of a metric on A_M .

The interior M_0 of any compact manifold with corners M is therefore a manifold with poly-cylindrical ends, for any choice of a metric on $A_M \rightarrow M$. The compact manifold with corners M will be called the *compactification* of M_0 .

Let us denote the hyperfaces of M by H_i , $i = 1, \dots, N$, and fix, for any hyperface H_i , a defining function ρ_i , if that hyperface has such a function (i.e., if H_i is an embedded hyperface). Let h be a smooth metric on a manifold with embedded faces M . Then a typical example of a manifold with poly-cylindrical ends is provided by M_0 , the interior of M , with the metric

$$g = h + \sum_{i=1}^N (\rho_i^{-1} d\rho_i)^2. \tag{3}$$

For instance, if we take $M = [0, 1]^n$ with defining functions x_j and $1 - x_j$ and let $h = 0$, then the resulting manifold with poly-cylindrical ends is isometrically diffeomorphic with \mathbb{R}^n with the standard (flat) Euclidean metric.

¹ This was done in the general framework of Lie manifolds in [ALN04] and was used in [AIN06].

We shall denote by $\text{Diff}_b(M)$ the algebra of differential operators generated by $\mathcal{V}_b(M)$ together with multiplications by functions in $C^\infty(M)$. A simple but useful result [ALN04] states that all geometric operators (Laplace, Dirac, signature, . . .) associated to the Riemannian manifold with poly-cylindrical ends M_0 are in $\text{Diff}_b(M)$. We shall therefore restrict our study of differential operators on M_0 to differential operators in $\text{Diff}_b(M)$.

If each face of M has a defining function (i.e., M has embedded faces), then in [Mel95, MP92] an algebra $\Psi_b^\infty(M)$ of pseudodifferential operators on M was constructed. One of the main properties of $\Psi_b^\infty(M)$ is that a differential operator P is in $\Psi_b^\infty(M)$ precisely when $P \in \text{Diff}_b(M)$. This construction was generalized in [Mon99] to arbitrary manifolds with corners. For the purpose of this paper, it is convenient to introduce the algebra $\Psi_b^\infty(M)$ and related algebras using Lie groupoids. (In fact, our algebras are slightly smaller than the ones in [Mel95, MP92].) An operator $P \in \Psi_b^\infty(M)$ will be called *compatible with the structure at infinity* on M_0 .

1.3 Differentiable groupoids

A *small category* is a category whose class of morphisms is a set. The class of objects of a small category is then a set as well. By definition, a *groupoid* is a small category \mathcal{G} in which every morphism is invertible. See [Ren80] for general references on groupoids.

We shall follow the general notations: the set of objects (or *units*) of a groupoid \mathcal{G} is denoted by $\mathcal{G}^{(0)}$, and the set of morphisms (or *arrows*) is denoted, by abuse of notation, by \mathcal{G} instead of $\mathcal{G}^{(1)}$. A groupoid is endowed with two maps, the *domain* $d: \mathcal{G} \rightarrow \mathcal{G}^{(0)}$ and the *range* $r: \mathcal{G} \rightarrow \mathcal{G}^{(0)}$. The multiplication gh of $g, h \in \mathcal{G}^{(0)}$ is defined if, and only if, $d(g) = r(h)$. A groupoid \mathcal{G} is completely determined by the spaces $\mathcal{G}^{(0)}$ and \mathcal{G} and by the structural morphisms: d, r , multiplication, inversion, and the inclusion $\mathcal{G}^{(0)} \rightarrow \mathcal{G}$.

We shall consider *Lie groupoids* (\mathcal{G}, M) , that is, groupoids endowed with a differential structure such that the set of arrows, \mathcal{G} , and the set of units, M , are smooth manifolds with corners, all structural maps are differentiable, and d is a submersion of manifolds with corners. In particular, $d^{-1}(x)$ is a smooth manifold (without corners) for any $x \in M$ and $\mathcal{G}^{(0)}$ is an embedded submanifold with corners of \mathcal{G} . (The terminology ‘differentiable groupoid’ was used in [NWX99] instead of ‘Lie groupoid’, because the name ‘Lie groupoid’ was used in the past for differentiable groupoids with additional structures. This has changed, however, and the terminology ‘Lie groupoid’ better reflects the current use.)

A Lie groupoid \mathcal{G} is called *d-connected* if and only if all the sets $\mathcal{G}_x := d^{-1}(x)$ are connected (and hence also path connected). If we are given a Lie groupoid \mathcal{G} , let us define \mathcal{G}_0 to consist of all the path components of the units in the fibers \mathcal{G}_x . Then \mathcal{G}_0 is the an open subset of \mathcal{G} containing the units and is closed under the groupoid operations. We shall call \mathcal{G}_0 the *d-connected component* of the units in \mathcal{G} . It is a Lie groupoid on its own, and, as such, it is *d-connected*.

Examples 2. Let us include here some examples of Lie groupoids that will be needed later on.

(1) If X is a smooth manifold, $X \times X$, the *pair groupoid* has units X and is defined by $d(x, y) = y, r(x, y) = x$, and $(x, y)(y, z) = (x, z)$.

(2) Let $\pi: X \rightarrow M$ be a fibration with smooth fibers, and M a manifold with corners. Recall then that the *fiberwise product groupoid* $\mathcal{G} := X \times_M X$ also has units X , and is defined as $\mathcal{G} := \{(x_1, x_2) \in X^2, \pi(x_1) = \pi(x_2)\}$ with units $X, d(x_1, x_2) = x_2, r(x_1, x_2) = x_1$, and product $(x_1, x_2)(x_2, x_3) = (x_1, x_3)$. Thus \mathcal{G} is a subgroupoid of the pair groupoid $X \times X$. This example will be needed later on in the proof of Proposition 5.

- (3) If $\mathcal{G}_j, j = 1, 2$, are Lie groupoids, then $\mathcal{G}_1 \times \mathcal{G}_2$ is also a Lie groupoid.
- (4) If M is a manifold with corners, $\mathcal{G} = M$ is a Lie groupoid with only units.

1.4 Lie algebroids

This subsection may be skipped at a first reading. A *Lie algebroid* $\pi : A \rightarrow M$ over a smooth manifold with corners M is a smooth vector bundle A over M for which there is given a vector bundle map $\varrho : A \rightarrow TM$ satisfying the following conditions for any C^∞ -sections X and Y of A and any C^∞ -function f on M .

- (a) The space $\Gamma(A)$ is endowed with a Lie algebra structure.
- (b) We have $[\varrho(X), \varrho(Y)] = \varrho([X, Y])$.
- (c) We have $[X, fY] = (\varrho(X)f)Y + f[X, Y]$.

The simplest example of a Lie algebroid is the tangent bundle $TM \rightarrow M$, with the Lie algebra structure on $\Gamma(TM)$ being given by the Lie bracket. Similarly, the vector bundle $A_M \rightarrow M$ introduced above is also a Lie algebroid, the Lie algebra structure being again given by the Lie bracket.

If \mathcal{G} is a Lie groupoid, let us denote by

$$T_d\mathcal{G} := \bigcup_{g \in \mathcal{G}} T_g\mathcal{G}_{d(g)} \quad \text{and} \quad A(\mathcal{G}) := \bigcup_{x \in \mathcal{G}^{(0)}} T_x\mathcal{G}_x = T_d\mathcal{G}|_{\mathcal{G}^{(0)}}$$

the *d-vertical tangent bundle of \mathcal{G}* and, respectively, the *Lie algebroid of \mathcal{G}* . The sections of $T_d\mathcal{G}$ are vector fields tangent to the fibers of d , and hence the space of all these sections, $\Gamma(T_d\mathcal{G})$, is closed under the Lie bracket. Multiplication to the right by γ maps \mathcal{G}_y to \mathcal{G}_x , where x and y are the domain and the range of $\gamma \in \mathcal{G}$. A vector field $X \in \Gamma(T_d\mathcal{G})$ will be called right invariant if it is invariant under all these right multiplications. The space $\Gamma_R(T_d\mathcal{G})$ of right invariant vector fields is also closed under the Lie bracket because a local diffeomorphism preserves the Lie bracket. Since $A(\mathcal{G})$ is the restriction of the *d-vertical tangent bundle $T_d\mathcal{G}$* to the space of units M and r is a submersion, we have $\Gamma_R(T_d\mathcal{G}) \simeq \Gamma(A(\mathcal{G}))$, and hence the later is a Lie algebra. It turns out that $A(\mathcal{G})$ is, indeed, a Lie algebroid, where $\varrho := r_*$, the differential of the map r .

In this paper, we shall use the following result from [Nis00], formulated in the way it will be used in this paper. For any set S , we shall denote by S^c its complement (in a larger set that is understood from the context).

THEOREM 3. *Let $A \rightarrow M$ be a Lie algebroid with anchor map $\varrho : A \rightarrow TM$. Let $N \subset F$ be a closed submanifold (possibly with corners) of a face F of M . Assume that N is invariant, in the sense that $\varrho(X)$ is tangent to N for any $X \in \Gamma(A)$. Let \mathcal{G}_1 be a groupoid with units N and Lie algebroid $A|_N$. Also, let \mathcal{G}_2 be a groupoid with units $N^c = M \setminus N$ and Lie algebroid $A|_{N^c}$. Assume that both \mathcal{G}_1 and \mathcal{G}_2 are *d-connected*. Then the disjoint union $\mathcal{G} := \mathcal{G}_1 \cup \mathcal{G}_2$ has at most one differentiable structure compatible with the differentiable structures on \mathcal{G}_1 and \mathcal{G}_2 that makes it a Lie groupoid with Lie algebroid A .*

Let us remark that, by abstract set theory nonsense, it is enough to require in the above theorem that $A(\mathcal{G}_1) \simeq A|_N$ and $A(\mathcal{G}_2) \simeq A|_{N^c}$, but then we have to make sure that the isomorphism $A(\mathcal{G}) \simeq A$ is such that it is restricted to the given isomorphisms on N and N^c .

The differentiable structure, if there is one, is obtained using the exponential maps (see [Nis00] for details). It may happen, however, that there is no differentiable structure with the given

properties on \mathcal{G} or that the resulting manifold is non-Hausdorff. Variants of the above result for stratifications with finitely many strata can be obtained by induction.

A corollary of the above result is the following. We use the same notation as in the previous theorem.

COROLLARY 1. *Assume \mathcal{G} has a smooth structure that makes it a Lie groupoid. Let $\phi : \mathcal{G} \rightarrow \mathcal{G}$ be an isomorphism of groupoids that restricts to diffeomorphisms $\mathcal{G}_1 \rightarrow \mathcal{G}_1$ and $\mathcal{G}_2 \rightarrow \mathcal{G}_2$. If the induced map ϕ_* map defines a smooth isomorphism $A(\mathcal{G}) \simeq A(\mathcal{G})$, then ϕ is a diffeomorphism itself.*

Proof. Consider, on \mathcal{G} , the smooth structure coming from ϕ , and denote by \mathcal{G}_ϕ the resulting Lie groupoid. Then ϕ_* establishes an isomorphism of $A(\mathcal{G}_\phi)$ with $A(\mathcal{G})$. By the previous theorem, Theorem 3, the smooth structure \mathcal{G}_ϕ and the given smooth structure on \mathcal{G} are the same. Thus ϕ is differentiable. Similarly, ϕ^{-1} is differentiable. \square

2. The analytic index

We shall need to consider a special class of pseudodifferential operators on a manifold with poly-cylindrical ends M_0 with compactification M , very closely related to the b -calculus of Melrose. For us, it will be convenient to introduce this calculus using groupoids. For simplicity, we shall assume from now on that all our manifolds with corners have embedded faces.

2.1 Pseudodifferential operators on groupoids

If \mathcal{G} is a Lie groupoid with units M , then there is associated to it a pseudodifferential calculus (or algebra of pseudodifferential operators) $\Psi^\infty(\mathcal{G})$, whose operators of order m form a linear space denoted $\Psi^m(\mathcal{G})$, $m \in \mathbb{R}$, such that $\Psi^m(\mathcal{G})\Psi^{m'}(\mathcal{G}) \subset \Psi^{m+m'}(\mathcal{G})$. See [MP97, NWX99]. We shall need this construction only for Hausdorff groupoids, so we assume that \mathcal{G} is Hausdorff from now on. This calculus is defined as follows: $\Psi^m(\mathcal{G})$, $m \in \mathbb{Z}$ consists of smooth families of classical pseudodifferential operators (P_x) of order m , acting on $C_c^\infty(\mathcal{G}_x)$ for each $x \in M$, that are right invariant with respect to multiplication by elements of \mathcal{G} and are ‘uniformly supported’. To define what uniformly supported means, let us observe that the right invariance of the operators P_x implies that their distribution kernels K_{P_x} descend to a distribution $k_P \in I^m(\mathcal{G}, M)$ (see [Mon98, NWX99]). Then the family $P = (P_x)$ is called *uniformly supported* if, by definition, k_P has compact support in \mathcal{G} . The right invariance condition means, for $P = (P_x) \in \Psi^\infty(\mathcal{G})$, that right multiplication $\mathcal{G}_x \ni g' \mapsto g'g \in \mathcal{G}_y$ maps P_y to P_x , whenever $d(g) = y$ and $r(g) = x$.

We then have the following result (see [LMN00, Mon99, NWX99]).

THEOREM 4. *Let \mathcal{G} be a Lie groupoid with units M and Lie algebroid $A = A(\mathcal{G})$. The space $\Psi^\infty(\mathcal{G})$ is an algebra of pseudodifferential operators so that there exist surjective principal symbol maps $\sigma_{\mathcal{G}}^{(m)}$ with kernel $\Psi^{m-1}(\mathcal{G})$,*

$$\sigma_{\mathcal{G}}^{(m)} : \Psi^m(\mathcal{G}) \rightarrow S_{\text{cl}}^m(A^*)/S_{\text{cl}}^{m-1}(A^*).$$

Also, the algebra $\Psi^\infty(\mathcal{G})$ acts on $C^\infty(M)$ such that $\Psi^\infty(\mathcal{G})C_c^\infty(M_0) \subset C_c^\infty(M_0)$.

In order to define the algebra of pseudodifferential operators $\Psi_b^\infty(M)$ on our manifold with corners M (and acting on $C_c^\infty(M_0)$, where M_0 is our manifold with poly-cylindrical ends), we shall first define a Lie groupoid \mathcal{G} canonically associated to M , and then consider the associated pseudodifferential calculus $\Psi^\infty(\mathcal{G})$ of \mathcal{G} (see [LN01, Mon99, Mon03, NWX99]).

Let us denote the hyperfaces of M by H_i , $i = 1, \dots, N$, as above. Since M has embedded faces, any hyperface H_i has a defining function ρ_i , which we shall fix from now on. To M and $Y = \{\rho_i\}$ we associate the groupoid

$$\tilde{G}(M; Y) := \{(x, y, \lambda_1, \dots, \lambda_N) \in M \times M \times \mathbb{R}_+^{*N}, \rho_i(x) = \lambda_i \rho_i(y), \text{ for all } i\}. \tag{4}$$

DEFINITION 2. We define $G(M)$ to be the d -connected component of $\tilde{G}(M)$.

This definition is not canonical, since it depends on the choice of the defining functions (see [Mon98, Mon99] for a canonical definition). Then $G(M)$ is a Lie groupoid with units M . The operations are the ones induced from the pair groupoid on the first components and from the group structure on the last components, as follows:

$$d(x, y, \lambda_j) = y, \quad r(x, y, \lambda_j) = x \quad \text{and} \quad (x, y, \lambda_j)(y, z, \lambda'_j) = (x, z, \lambda_j \lambda'_j).$$

We can also consider more general systems Y of functions $\{\rho\}$ with the property that each function has a non-degenerate set of zeroes that is a *disjoint union* of hyperfaces of M . In particular, we have the following lemma.

LEMMA 1. Let $Y = \{\rho_i\}$ and let $Y' = \{\rho_1 \rho_2, \rho_3, \dots\}$, where the zero sets of ρ_1 and ρ_2 are disjoint. Then $\tilde{G}(M; Y)$ identifies, as a Lie groupoid, with an open subset of $\tilde{G}(M; Y')$.

Proof. The identification of $\tilde{G}(M; Y)$ with an open subset of $\tilde{G}(M; Y')$ is provided by $(x, y, \lambda_i) \rightarrow (x, y, \lambda_1 \lambda_2, \lambda_3, \dots)$. Let Z_j be the zero set of ρ_j . The difference between $\tilde{G}(M; Y)$ and $\tilde{G}(M; Y')$ is that $d^{-1}(Z_1 \cup Z_2)$ is $\simeq (Z_1 \cup Z_2)^2 \times \mathbb{R}$ in the larger groupoid, whereas $d^{-1}(Z_1 \cup Z_2) \cap \tilde{G}(M; Y) \simeq (Z_1^2 \cup Z_2^2) \times \mathbb{R}$. \square

We shall write $C^*(M) := C^*(G(M))$, for simplicity. This lemma leads immediately to the following corollary.

COROLLARY 2. Let $\Omega \subset M$ be an open subset. Fix a system of defining functions Y for M , then $\tilde{G}(\Omega; Y) \subset \tilde{G}(M; Y)$ as an open subset. Consequently, $G(\Omega)$ identifies canonically with an open subset of $G(M)$, and hence we have a natural inclusion $C^*(\Omega) \subset C^*(M) := C^*(G(M))$.

Proof. Equation (4) gives

$$\begin{aligned} \tilde{G}(\Omega; Y) &:= \{(x, y, \lambda_1, \dots, \lambda_N) \in \Omega \times \Omega \times \mathbb{R}_+^{*N}, \rho_i(x) = \lambda_i \rho_i(y), \text{ for all } i\} \\ &= d^{-1}(\Omega) \cap r^{-1}(\Omega), \end{aligned}$$

so $\tilde{G}(\Omega; Y)$ is indeed an open subset of $\tilde{G}(M; Y)$. Passing to the d -connected component preserves open inclusion. \square

Remark 5. We clearly have $G(\Omega) \subset d^{-1}(\Omega) \cap r^{-1}(\Omega)$, but we do not have equality unless there exists a bijection between the faces of Ω and those of M .

We now introduce the class of pseudodifferential operators we are interested in.

DEFINITION 3. Let M be a manifold with embedded faces (by our earlier assumption) and $G(M)$ be the Lie groupoid introduced in Definition 2, then we let

$$\Psi_b^\infty(M) := \Psi^\infty(G(M)).$$

This algebra is slightly smaller than the one constructed by Melrose [Mel95], but, for our purposes, it is as good. In fact, our algebra is the subalgebra of properly supported pseudodifferential operators in the Melrose’s algebra.

We also have that a differential operator D on M_0 is in $\Psi_b^\infty(M)$ if and only if it is in $\text{Diff}_b(M)$. We also obtain a principal symbol map

$$\sigma_b^{(m)} : \Psi_b^m(M) \rightarrow S_{\text{cl}}^m(A_M^*)/S_{\text{cl}}^{m-1}(A_M^*).$$

This new definition of the principal symbol recovers the usual principal symbol in the interior of M , but it also provides additional information at the boundary. Indeed, the *usual* principal symbol of a differential operator $P \in \text{Diff}_b(M)$ is never invertible at the boundary, so this differential operator can never be elliptic in the usual sense (unless, of course, it is actually a function). On the other hand, there are many differential operators $P \in \text{Diff}_b(M)$ whose principal symbol is invertible on A_M^* . An example is provided by $x\partial_x$ on the interval $[0, \infty)$. This operator has $x\xi$ as its usual principal symbol, where $(x, \xi) \in T^*[0, \infty) = [0, \infty) \times \mathbb{R}$, but in the calculus of $\Psi_b^\infty(M)$, it has principal symbol $\sigma_b^{(1)}(x\partial_x) = \xi$.

2.2 The adiabatic and tangent groupoids

For the definition and study of the analytic index, we shall need the adiabatic and tangent groupoids associated to a differentiable groupoid \mathcal{G} . We now recall their definition and establish a few elementary properties.

Let \mathcal{G} be a Lie groupoid with space of units M . We construct both the *adiabatic groupoid* ${}^{\text{ad}}\mathcal{G}$ and the *tangent groupoid* ${}^T\mathcal{G}$ (see [Con94, Lan03, LMN00, MP97, Ram99]). The space of units of ${}^{\text{ad}}\mathcal{G}$ is $M \times [0, \infty)$ and the tangent groupoid ${}^T\mathcal{G}$ will be defined as the restriction of ${}^{\text{ad}}\mathcal{G}$ to $M \times [0, 1]$.

The underlying set of the groupoid ${}^{\text{ad}}\mathcal{G}$ is the disjoint union

$${}^{\text{ad}}\mathcal{G} = A(\mathcal{G}) \times \{0\} \cup \mathcal{G} \times (0, \infty).$$

We endow $A(\mathcal{G}) \times \{0\}$ with the structure of commutative bundle of Lie groups induced by its vector bundle structure. We endow $\mathcal{G} \times (0, \infty)$ with the product (or pointwise) groupoid structure. Then the groupoid operations of ${}^{\text{ad}}\mathcal{G}$ are such that $A(\mathcal{G}) \times \{0\}$ and $\mathcal{G} \times (0, \infty)$ are subgroupoids with the induced structure. Now let us endow ${}^{\text{ad}}\mathcal{G}$ with a differentiable structure. The differentiable structure on ${}^{\text{ad}}\mathcal{G}$ is such that

$$\Gamma(A({}^{\text{ad}}\mathcal{G})) = t\Gamma(A(\mathcal{G} \times [0, \infty))). \tag{5}$$

More precisely, consider the product groupoid $\mathcal{G} \times [0, \infty)$ with pointwise operations. Then a section $X \in \Gamma(A(\mathcal{G} \times [0, \infty)))$ can be identified with a smooth function $[0, \infty) \ni t \rightarrow X(t) \in \Gamma(A(\mathcal{G}))$. We thus require $\Gamma(A({}^{\text{ad}}\mathcal{G})) = \{tX(t)\}$, with $X \in \Gamma(A(\mathcal{G} \times [0, \infty)))$.

Specifying the Lie algebroid of $A({}^{\text{ad}}\mathcal{G})$ completely determines its differentiable structure [Nis00]. For clarity, let us include also an explicit description of this differentiable structure. Let us consider an atlas (Ω_α) , consisting of domains of coordinate charts $\Omega_\alpha \subset {}^{\text{ad}}\mathcal{G}$ and diffeomorphisms $\phi_\alpha : \Omega_\alpha \rightarrow U_\alpha$, where U_α is an open subset of a Euclidean space.

Let $\Omega = \Omega_\alpha$ be a chart of \mathcal{G} , such that $\Omega \cap \mathcal{G}^{(0)} \neq \emptyset$; one can assume without loss of generality that $\Omega \simeq T \times U$ with respect to s , and that $\Omega \simeq T' \times U$ with respect to r . Let us denote by ϕ and ψ these diffeomorphisms. Thus, if $x \in U$, then $\mathcal{G}_x \simeq T$, and $A(\mathcal{G})_U \simeq \mathbb{R}^k \times U$. Let $(\Theta_x)_{x \in U}$ (respectively $(\Theta'_x)_{x \in U}$) be a smooth family of diffeomorphisms from \mathbb{R}^k to T (respectively T') such that $\iota(x) = \phi(\Theta_x(0), x)$ (respectively $\iota(x) = \psi(\Theta'_x(0), x)$), where ι denotes the inclusion of $\mathcal{G}^{(0)}$ into \mathcal{G} .

Then $\bar{\Omega} = A(\mathcal{G})_U \times \{0\} \cup \Omega \times (0, \infty)$ is an open subset of $\text{ad}\mathcal{G}$, homeomorphic to $\mathbb{R}^k \times U \times \mathbb{R}_+$ with respect to s and to r as follows:

$$\begin{aligned} \bar{\phi}(\xi, u, \alpha) &= \begin{cases} (\phi(\Theta_u(\alpha\xi), u), \alpha) & \text{if } \alpha \neq 0, \\ (\xi, u, 0) & \text{if } \alpha = 0, \end{cases} \\ \bar{\psi}(\xi, u, \alpha) &= \begin{cases} ((\phi(\Theta_u(\alpha\xi), u))^{-1}, \alpha) & \text{if } \alpha \neq 0, \\ (\xi, u, 0) & \text{if } \alpha = 0. \end{cases} \end{aligned}$$

This defines an atlas of $\text{ad}\mathcal{G}$, endowing it with a Lie groupoid structure.

Recall that a subset S of the units of a groupoid \mathcal{H} is called *invariant* if and only if $d^{-1}(S) = r^{-1}(S)$. Then we shall use the notation $\mathcal{H}_S := d^{-1}(S) = r^{-1}(S)$, and call it *the restriction of \mathcal{H} to S* . Then \mathcal{H}_S is also a groupoid precisely because S is invariant. We shall sometimes write $\mathcal{H}|_S$ instead of \mathcal{H}_S . For instance, ${}^T\mathcal{G}$, *the tangent groupoid of \mathcal{G}* is defined to be the restriction of $\text{ad}\mathcal{G}$ to $M \times [0, 1]$.

We have the following simple properties.

LEMMA 2. *Let \mathcal{G} be a Lie groupoid and $\mathcal{H} = \mathcal{G} \times \mathbb{R}^n$ be the product of \mathcal{G} with the Lie group \mathbb{R}^n with the induced product structure. Then $\text{ad}\mathcal{H} \simeq \text{ad}\mathcal{G} \times \mathbb{R}^n$*

Proof. Let us denote by M the set of units of \mathcal{G} . We have that $A(\mathcal{H}) = A(\mathcal{G}) \oplus \mathbb{R}^n$ as vector bundles, where the right copy of \mathbb{R}^n stands for the trivial bundle with fiber \mathbb{R}^n over M . Note that, as a set, $A(\mathcal{H}) = A(\mathcal{G}) \times \mathbb{R}^n$, so our notation will not lead to any confusion.

By definition,

$$\begin{aligned} \text{ad}\mathcal{H} &= A(\mathcal{H}) \times \{0\} \cup \mathcal{H} \times (0, \infty) = A(\mathcal{G}) \times \mathbb{R}^n \times \{0\} \cup \mathcal{G} \times \mathbb{R}^n \times (0, \infty) \\ &= [A(\mathcal{G}) \times \{0\} \cup \mathcal{G} \times (0, \infty)] \times \mathbb{R}^n = \text{ad}\mathcal{G} \times \mathbb{R}^n. \end{aligned}$$

So we can identify the underlying sets of $\text{ad}\mathcal{H}$ and $\text{ad}\mathcal{G} \times \mathbb{R}^n$. However, this identification is not the bijection we are looking for, because it does not preserve the differentiable structure. Instead, we define $\phi: \text{ad}\mathcal{H} \rightarrow \text{ad}\mathcal{G} \times \mathbb{R}^n$ as follows. Let $g \in \text{ad}\mathcal{G}$ and $\xi \in \mathbb{R}^n$, so that $(g, \xi) \in \text{ad}\mathcal{H}$, using the previous identification. Then $\phi(g, \xi) = (g, \xi)$ if $g \in A(\mathcal{H}) \times \{0\}$ and $\phi(g, \xi) = (g, t^{-1}\xi)$ if $(g, \xi) \in \mathcal{G} \times \{t\} \times \mathbb{R}^n$.

It remains to check that ϕ is differentiable with differentiable inverse. For this, it is enough to check that it induces an isomorphism at the level of Lie algebroids, because the smooth structures on both $\text{ad}\mathcal{H}$ and $\text{ad}\mathcal{G} \times \mathbb{R}^n$ are defined by their Lie algebroids (see Theorem 3 and Corollary 1). Indeed, (5) gives

$$\begin{aligned} \Gamma(A(\text{ad}\mathcal{H})) &= t\Gamma(A(\mathcal{H} \times [0, \infty))) = t[\Gamma(A(\mathcal{G} \times [0, \infty)))] \oplus \mathcal{C}^\infty(M \times [0, \infty))^n \\ &\xrightarrow{\phi_*} t[\Gamma(A(\mathcal{G} \times [0, \infty)))] \oplus \mathcal{C}^\infty(M \times [0, \infty))^n = \Gamma(A(\text{ad}\mathcal{G} \times \mathbb{R}^n)) \end{aligned}$$

where ϕ_* is an isomorphism. □

This gives the following corollary.

COROLLARY 3. *Let $\mathcal{H} = \mathcal{G} \times \mathbb{R}^n$, as above. We have that $C^*(\text{ad}\mathcal{H}) \simeq C^*(\text{ad}\mathcal{G}) \otimes \mathcal{C}_0(\mathbb{R}^n)$ and that $C^*({}^T\mathcal{H}) \simeq C^*({}^T\mathcal{G}) \otimes \mathcal{C}_0(\mathbb{R}^n)$, the tensor product being the (complete, maximal) C^* -tensor product.*

Proof. This corollary follows immediately from Lemma 2 and the relation

$$C^*(\mathcal{G}' \times \mathbb{R}^n) \simeq C^*(\mathcal{G}') \otimes \mathcal{C}_0(\mathbb{R}^n)$$

valid for any locally compact groupoid \mathcal{G}' . □

A similar argument using again Theorem 3 and Corollary 1 yields the following result.

LEMMA 3. *Let \mathcal{G} with a Lie groupoid with units M . Let $N \subset M$ be an invariant subset. Assume that $N \subset F$ is an embedded submanifold of a face F of M . Then the restriction operations and the formation of adiabatic and tangent groupoids commute, in the sense that we have*

$$(\mathcal{G}|_N)_{\text{ad}} \simeq {}^{\text{ad}}\mathcal{G}|_{N \times [0, \infty)} \quad \text{and} \quad (\mathcal{G}|_{N^c})_{\text{ad}} \simeq {}^{\text{ad}}\mathcal{G}|_{N^c \times [0, \infty)}.$$

A similar results holds for the tangent groupoids.

Proof. Again, it follows from the definition that $(\mathcal{G}|_N)_{\text{ad}}$ and ${}^{\text{ad}}\mathcal{G}|_{N \times [0, \infty)}$ have the same underlying set. Moreover, this canonical bijection is a groupoid isomorphism. From the definition of the Lie algebroid of the adiabatic groupoid, it follows that this canonical bijection is also differentiable with differentiable inverse, because it induces an isomorphism of the spaces of sections of the corresponding Lie algebroids. See Theorem 3 and, especially, Corollary 1. \square

We then obtain the following corollary.

COROLLARY 4. *With the notations of the above lemma we have a short exact sequence*

$$0 \rightarrow C^*(T\mathcal{G}|_{N^c}) \rightarrow C^*(T\mathcal{G}) \rightarrow C^*(T\mathcal{G}|_N) \rightarrow 0,$$

and a similar exact sequence for the adiabatic groupoid.

2.3 The analytic index

We now give two definitions of the analytic index. The first definition is based on a generalization of Connes’ tangent groupoid [Con94]. The second definition is based on the boundary map of the six-term exact sequence in K -theory induced by the symbol map. Both definitions will be needed in what follows.

For each $t \in [0, 1]$, $M \times \{t\}$ is a closed invariant subset of $M \times [0, \infty)$, and hence we obtain an *evaluation map*

$$e_t : C^*(T\mathcal{G}) \rightarrow C^*(T\mathcal{G}_{M \times \{t\}}).$$

By abuse of notation, we shall sometimes denote also by e_t the induced map in K -theory.

Let us also notice that the decomposition

$$M \times [0, 1] = M \times \{0\} \cup M \times (0, 1]$$

into open and closed invariant subspaces, gives rise to an exact sequence

$$0 \rightarrow C^*(T\mathcal{G}_{M \times (0, 1]}) \rightarrow C^*(T\mathcal{G}) \xrightarrow{e_0} C^*(A(\mathcal{G})) \rightarrow 0. \tag{6}$$

This leads to the following six-terms exact sequence in K -theory.

$$\begin{array}{ccccc} K_0(C^*(T\mathcal{G}_{M \times (0, 1]})) & \longrightarrow & K_0(C^*(T\mathcal{G})) & \longrightarrow & K_0(C^*(A(\mathcal{G}))) \\ & & & & \downarrow \\ & \uparrow & & & \\ K_1(C^*(A(\mathcal{G}))) & \longleftarrow & K_1(C^*(T\mathcal{G})) & \longleftarrow & K_1(C^*(T\mathcal{G}_{M \times (0, 1]})) \end{array}$$

We have $T\mathcal{G}_{M \times (0, 1]} = \mathcal{G} \times (0, 1]$ and hence $C^*(T\mathcal{G}_{M \times (0, 1]}) \simeq C^*(\mathcal{G}) \otimes \mathcal{C}_0((0, 1])$. In particular, $K_*(C^*(T\mathcal{G}_{M \times (0, 1]})) = K_*(C^*(\mathcal{G}) \otimes \mathcal{C}_0((0, 1])) = 0$. Thus the evaluation map e_0 is an isomorphism in K -theory.

The C^* -algebra $C^*(A(\mathcal{G}))$ is commutative, and we have $C^*(A(\mathcal{G})) \simeq \mathcal{C}_0(A^*(\mathcal{G}))$. Therefore $K_*(C^*(A(\mathcal{G}))) = K^*(A^*(\mathcal{G}))$. In turn, this isomorphism allows us to define the *analytic index* ind_a as the composition map

$$\text{ind}_a^{\mathcal{G}} = e_1 \circ e_0^{-1} : K^*(A^*(\mathcal{G})) \rightarrow K_*(C^*(\mathcal{G})), \tag{7}$$

where $e_1 : C^*(T\mathcal{G}) \rightarrow C^*(T\mathcal{G}_{M \times \{1\}}) = C^*(\mathcal{G})$ is defined by the restriction map to $M \times \{1\}$.

The definition of the analytic index gives the following proposition.

PROPOSITION 1. *Let \mathcal{G} be a Lie groupoid with Lie algebroid $\pi : A(\mathcal{G}) \rightarrow M$. Also, let $N \subset F \subset M$ be a closed, invariant subset which is an embedded submanifold of a face F of M . Then the analytic index defines a morphism of the six-term exact sequences associated to the pair $(A^*(\mathcal{G}), \pi^{-1}(N))$ and to the ideal $C^*(\mathcal{G}_{N^c}) \subset C^*(\mathcal{G})$, $N^c := M \setminus N$.*

$$\begin{array}{ccccccc} K^0(\pi^{-1}(N^c)) & \longrightarrow & K^0(A^*(\mathcal{G})) & \longrightarrow & K^0(\pi^{-1}(N)) & \longrightarrow & K^1(\pi^{-1}(N^c)) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K^0(C^*(\mathcal{G}_{N^c})) & \longrightarrow & K^0(C^*(\mathcal{G})) & \longrightarrow & K^0(C^*(\mathcal{G}_N)) & \longrightarrow & K^1(C^*(\mathcal{G}_{N^c})) \end{array}$$

Proof. The six-term, periodic long exact sequence in K -theory associated to the pair $(A^*(\mathcal{G}), \pi^{-1}(N))$ is naturally and canonically isomorphic to the six-term exact sequence in K -theory associated to the pair $\mathcal{C}_0(A^*_{M \setminus N}) \subset \mathcal{C}_0(A^*(\mathcal{G}))$ consisting of an algebra and an ideal in that algebra. Since e_0 and e_1 also induce morphisms of pairs (algebra, ideal), the result follows from Corollary 4, the naturality of the six-term exact sequence in K -theory, and the definition of the analytic index (8). □

For M a smooth manifold with embedded faces, we have $A(G(M)) = A_M$. Recall that $C^*(M) := C^*(G(M))$. Then the analytic index becomes the desired map

$$\text{ind}_a^M : K^*(A_M^*) \rightarrow K_*(C^*(M)). \tag{8}$$

Remark 6. Assume M has no corners (or boundary). Then $G(M) = M \times M$ is the product groupoid and hence $\Psi^\infty(G(M)) = \Psi^\infty(M)$. In particular, $C^*(M) := C^*(G(M)) \simeq \mathcal{K}$, the algebra of compact operators on M . In this case $K_0(C^*(M)) = \mathbb{Z}$, and ind_a is precisely the analytic index as introduced by [AS68]. This construction holds also for the case when M is not compact, but we have to use pseudodifferential operators of order zero that are ‘multiplication at infinity’, as in [Car06].

We now turn to the application mentioned in the introduction to obstructions to finding invertible perturbations by regularizing operators. Let $\mathfrak{A}(\mathcal{G})$ be the enveloping C^* -algebra of $\Psi^0(\mathcal{G})$ and let $S^*\mathcal{G} \subset A^*(\mathcal{G})$ be the subset of vectors of length one. The second method is based on the exact sequence

$$0 \rightarrow C^*(\mathcal{G}) \rightarrow \mathfrak{A}(\mathcal{G}) \xrightarrow{\sigma^{(0)}} C(S^*\mathcal{G}) \rightarrow 0. \tag{9}$$

The six-term exact sequence in K -theory associated to this exact sequence yields a boundary map

$$\text{ind}_a^{\mathcal{G}} := \partial : K^{*+1}(S^*\mathcal{G}) \rightarrow K_*(C^*(\mathcal{G})). \tag{10}$$

Considering the exact sequence

$$0 \rightarrow C_0(A(\mathcal{G})) \rightarrow C(B(\mathcal{G})) \rightarrow C(S(\mathcal{G})) \rightarrow 0,$$

where $B(\mathcal{G})$ is the ball bundle, we get a map $b : K^{*+1}(S^*\mathcal{G}) \rightarrow K^*(A^*M)$, and $\text{ind}_a^M \circ b = \text{ind}_a^{\mathcal{G}}$.

This second definition of the analytic index has the advantage that it leads to the following theorem. Let $\Psi_b^\infty(M)$ be as in Definition 3 and

$$\Psi_b^m(M; E_0, E_1) = e_1 M_N(\Psi_b^m(M)) e_0,$$

where e_j is the orthogonal projection onto the subbundle $E_j \subset M \times \mathbb{C}^N$, $j = 1, 2$.

THEOREM 7. *Assume M is a connected manifold with corners such that all its faces have positive dimension. Let $P \in \Psi_b^m(M; E_0, E_1)$ be an elliptic pseudodifferential operator acting between sections of two vector bundles $E_0, E_1 \rightarrow M$. Then there exists $R \in \Psi_b^{-\infty}(M; E_0, E_1)$ such that $P + R$ is invertible if, and only if, $\text{ind}_a^M(\sigma_P) = 0$, where σ_P is the principal symbol of P .*

Proof. If $E_0 = E_1$ is the trivial bundle, then the proof is the same as that of [NT04, Theorem 4.10].

Let us observe as in [LM03] that $E_0 \simeq E_1$ because TM has a non-zero section. By embedding E_0 into a trivial bundle, we can therefore assume that $P \in M_N(\Psi_b^m(M))$. The index $\text{ind}_a^M(\sigma_P) = \partial[\sigma_P] \in K_0(C^*(M))$ is therefore defined as in (10). Moreover, this definition is independent of the isomorphism $E_0 \simeq E_1$ and of the embedding of E_0 into a trivial bundle. This reduces the proof to the case of a trivial bundle. \square

3. Properties of the analytic index

We now prove some results on the analytic index whose definition was recalled in the previous section. In this and the following sections, we continue to assume that our manifolds with corners have embedded faces. Recall that $G(M)$ is the Lie groupoid associated to a manifold with corners M in Definition 2.

Also, recall that we have denoted $A_M := A(G(M))$ and $C^*(M) := C^*(G(M))$. In particular, $\Gamma(A_M) = \mathcal{V}_b$, the space of all vector fields tangent to the faces of M .

3.1 The role of faces

Let $F \subset M$ be a face of a manifold with embedded faces M . Recall that in our terminology, ‘face’ always means ‘closed face’. Then F is a closed, invariant subset of M (‘invariant’ here is with respect to the action of $G(M)$ on its units).

LEMMA 4. *Let $\pi : A_M \rightarrow M$ be the Lie algebroid of $G(M)$. Then, for any face $F \subset M$ of codimension k , we have isomorphisms*

$$\pi^{-1}(F) \simeq A_F \times \mathbb{R}^k \quad \text{and} \quad G(M)|_F \simeq G(F) \times \mathbb{R}^k.$$

Proof. We have that $\pi^{-1}(F) \simeq A_F \oplus \pi^{-1}(F)/A_F$. The choice of defining functions for the k hyperfaces containing F then gives an isomorphism $\pi^{-1}(F)/A_F \simeq F \times \mathbb{R}^k$. The last part follows from the definitions of $G(M)$ and $G(F)$, for the latter using the defining functions of M that are non-zero on F . \square

For the simplicity of the notation, let us denote by $\mathcal{G}_1 := G(M)|_F \simeq G(F) \times \mathbb{R}^k$. The analytic indices defined in (7) and (8) for this restriction groupoid are then identified by the following lemma.

LEMMA 5. For any face $F \subset M$ of codimension k we have a commutative diagram

$$\begin{CD} K^{*+k}(A^*(\mathcal{G}_1)) @>\text{ind}_a^{\mathcal{G}_1}>> K_{*+k}(C^*(\mathcal{G}_1)) \\ @AA\uparrow A @AA\uparrow A \\ K^*(A_F^*) @>\text{ind}_a^F>> K_*(C^*(F)) \end{CD}$$

where the vertical arrows are the periodicity isomorphisms.

Proof. Recall first that the periodicity isomorphism $K_*(\mathfrak{A}) \simeq K_{*+k}(\mathfrak{A} \otimes \mathcal{C}_0(\mathbb{R}^k))$ is natural in \mathfrak{A} , for any C^* -algebra \mathfrak{A} . The result then follows from this observation combined with Lemma 4 above and Corollary 3. \square

The following proposition is one of the main steps in the proof of our topological index Theorem 13.

PROPOSITION 2. Let X be a manifold with embedded faces such that each open face of X is diffeomorphic to a Euclidean space. Then the analytic index

$$\text{ind}_a^X : K^*(A_X^*) \rightarrow K_*(C^*(X))$$

(defined in (8)) is an isomorphism.

Proof. We shall proceed by induction on the number of faces of X . Let $G(X)$ be the groupoid of X , as before. Let $F \subset X$ be a face of minimal dimension. Then F is an invariant subset of X . Denote as above by \mathcal{G}_1 the restriction of $G(X)$ to F . In particular, F will be a smooth manifold without corners.

Note that our assumptions imply that $F \simeq \mathbb{R}^{n-k}$, where n is the dimension of M and k is the codimension of F . Therefore the analytic index ind_a^F is an isomorphism. In particular, our result is valid if X has exactly one face. Lemma 5 then shows that $\text{ind}_a^{\mathcal{G}_1}$ is an isomorphism as well.

We shall complete the proof using Proposition 1 with $N = F$ and $X = M$ as follows. Let us consider the six-term, exact sequence in K -theory associated to the pair $(A_X^*, A^*(\mathcal{G}_1))$, where $A^*(\mathcal{G}_1) = \pi^{-1}(F)$, with $\pi : A_X^* \rightarrow X$ the canonical projection. Also, let us consider the six-term, exact sequence in K -theory associated to the pair $C^*(F^c) \subset C^*(X)$, $F^c = X \setminus F$. Proposition 1 states that the analytic index defines a morphism of these two exact sequences. The analytic index for the quotient (i.e., $\text{ind}_a^{\mathcal{G}_1}$) has just been proved to be an isomorphism. The analytic index for F^c is an isomorphism by the induction hypothesis. The ‘Five Lemma’ then shows that the remaining two analytic index morphisms are also isomorphisms. The proof is now complete. \square

Remark 8. The above proposition can be regarded as a Baum–Connes isomorphism for manifolds with corners.

Another important step in the proof of our topological index theorem for manifolds with corners is the proof of the following proposition. We first need a lemma.

LEMMA 6. Let $\iota : M \rightarrow X$ be an embedding of manifolds with corners. Then ι defines a natural morphism

$$\iota_K : K_*(C^*(M)) \rightarrow K_*(C^*(X)). \tag{11}$$

Proof. The morphism ι_K will be defined by a Kasparov $C^*(M) - C^*(X)$ bimodule \mathcal{H} . Let us first identify M with a closed submanifold with corners of X . Let $X' \subset X$ be a small neighborhood of M in X such that every face of X' intersects M and the map $F \rightarrow F \cap M$ establishes a bijection between the faces of X' and those of M . For instance, X' could be a tubular neighborhood of M in X . Since $X' \subset X$ is open, there exists a natural inclusion $C^*(X') \rightarrow C^*(X)$ of C^* -algebras, by Corollary 2. It is enough then to construct a natural morphism $K_*(C^*(M)) \rightarrow K_*(C^*(X'))$. We can therefore replace X with X' and assume that there exists a bijection between the faces of M and those of X .

Let us introduce

$$\Omega := r^{-1}(M) \subset G(X),$$

that is, Ω is the set of elements of $G(X)$ whose range is in M . We endow Ω with the two maps $r : \Omega \rightarrow M$ and $d : \Omega \rightarrow X$ induced by the range and domain maps of the groupoid $G(X)$. These are continuous and open since $G(X)$ is a Lie groupoid. The groupoid $G(X)$ acts on the right on Ω by multiplication. We shall define \mathcal{H} as a Hilbert module completion of $\mathcal{C}_0(\Omega)$ as in [MRW87].

Next, we need to define the action of $C^*(G(M))$. This action will be by compact operators ('compact' here is used in the sense of Hilbert modules) and will come from an action of $G(M)$ on Ω . To define this action, we first prove that

$$G(M) = r^{-1}(M) \cap d^{-1}(M). \tag{12}$$

Indeed, our assumptions imply that, for any hyperface H of X with defining function $\rho = \rho_H$, the function $\rho|_M$ is a defining function of M . (We are, of course, using the fact that M and H intersect transversely.) In fact, our assumptions imply more: they imply that every defining function of M is obtained in this way. We can therefore establish a bijection between the defining functions ρ of (the hyperfaces of) X and the defining functions of (the hyperfaces of) M . Let us use these functions in the definition of $G(M)$, namely in (4). This, together with the fact that there is a bijection between the open faces of M and the open faces of X , then proves (12).

Equation (12) then allows us to define the action of $G(M)$ on Ω by left composition. Indeed, if $\gamma \in G(M)$ and $\gamma' \in \Omega$ with $r(\gamma') = d(\gamma)$, then $r(\gamma\gamma') = r(\gamma) \in M$ so that $\gamma\gamma' \in \Omega$. Also $G(X)$ acts on Ω by right composition: if $\gamma \in G(X)$ and $\gamma' \in \Omega$ with $d(\gamma') = r(\gamma)$, then $r(\gamma'\gamma) = r(\gamma') \in M$ so that $\gamma'\gamma \in \Omega$.

Then \mathcal{H} defines an element in $\Theta \in KK_0(C^*(M), C^*(X))$ (even, in fact, an imprimitivity module) and hence the Kasparov product with Θ defines the desired morphism $\iota_K : K_*(C^*(M)) \rightarrow K_*(C^*(X))$. \square

Remark 9. Let us spell out explicitly a conclusion of the above proof. If $M \subset X$ is an open subset and ι denotes the inclusion then $C^*(M) \subset C^*(X)$, by Corollary 2, and ι_K is simply the morphism associated to this inclusion of C^* -algebras.

PROPOSITION 3. *Let $\iota : M \rightarrow X$ be a closed embedding of manifold with corners. Assume that, for each open face F of X , the intersection $F \cap M$ is a non-empty open face of M and that every open face of M is obtained in this way. Then $\iota_K : K_*(C^*(M)) \rightarrow K_*(C^*(X))$ is an isomorphism.*

Proof. Recall from [MRW87] that two locally compact groupoids G and H are equivalent provided there exists a topological space Ω and two continuous, surjective open maps $r : \Omega \rightarrow G^{(0)}$ and $d : \Omega \rightarrow H^{(0)}$ together with a left (respectively right) action of G (respectively H) on Ω with respect to r (respectively d), such that r (respectively d) is a principal fibration of structural groupoid H (respectively G).

An important theorem of Muhly *et al.* states that if G and H are equivalent, then $K_*(C^*(G)) \simeq K_*(C^*(H))$ (see [MRW87]). More precisely, $\mathcal{C}_0(\Omega)$ has a completion to an Hilbert module that establishes a strong Morita equivalence between $C^*(G)$ and $C^*(H)$ (this is the imprimitivity module defining Θ in the proof of Lemma 6). This strong Morita equivalence is then known to imply the stated isomorphism $K_*(C^*(G)) \simeq K_*(C^*(H))$ (see [Rie82]).

To prove our result, it is therefore enough to show that the space $\Omega := r^{-1}(M)$ (considered also in the proof of Lemma 6) establishes an equivalence between $G(M)$ and $G(X)$.

Proving that r is a principal fibration of structural groupoid $G(X)$ amounts, by definition, to proving that, for any x in M , if ω and ω' are in $r^{-1}(x)$ in Ω , there exists $\gamma \in G(X)$ such that $\omega\gamma = \omega'$, and that the action of $G(X)$ is free and proper. The first condition is clear: $r(\omega) = r(\omega') = x$ so that $\gamma = \omega^{-1}\omega' \in G(X)$ exists. Besides, the action is free. Indeed, if $\omega\gamma = \omega$, then $r(\gamma) = d(\omega)$ (so they are composable) and $d(\gamma) = d(\omega\gamma) = d(\omega)$, so that γ is a unit. The action is proper. Indeed, the map

$$\phi : G(X) * \Omega \rightarrow \Omega \times \Omega, \quad (\gamma, \omega) \mapsto (\omega\gamma, \omega)$$

(where $G(X) * \Omega$ is the set of composable arrows) is a homeomorphism onto its image, which is the fibered product $\Omega \times_M \Omega$ with respect to r .

Similarly, let us check that s is a principal fibration with structural groupoid $G(M)$. Assume that $d(\omega) = d(\omega') \in M$; then $\omega' = \gamma\omega$, with $\gamma = \omega'\omega^{-1}$. Let us recall from the proof of Lemma 6 that $G(M) = d^{-1}(M) \cap r^{-1}(M)$. Hence $\gamma \in G(M)$. The proof is now complete. \square

Let M be a smooth compact manifold (without corners). Then the inclusion of a point $k : \text{pt} \hookrightarrow M$ satisfies the assumptions of the above proposition. The imprimitivity module in this case establishes the isomorphism $C^*(\text{pt}) \otimes \mathcal{K} \simeq C^*(M)$ and hence also the isomorphisms $K_*(C^*(M)) \simeq K_*(C^*(\text{pt})) = K_*(\mathbb{C}) \simeq \mathbb{Z}$.

More generally, let M be a compact manifold with corners and $\pi : X \rightarrow M$ be a smooth fiber bundle. Recall then that the *fiberwise product groupoid* $\mathcal{G} := X \times_M X$ was defined in the first section. The Lie algebroid of \mathcal{G} is $T_\pi X$, the vertical tangent bundle to $\pi : X \rightarrow M$. A simple calculation shows that $C^*(\mathcal{G})$ is a continuous field of C^* -algebras over M with fibers compact operators on L^2 of the fibers of π . Therefore $C^*(\mathcal{G})$ is canonically Morita equivalent to $C(M)$. If $X \rightarrow M$ has a cross-section (as in the cases when we shall use this construction in our paper), we can also obtain this Morita equivalence from the inclusion $M \subset X \times_M X$ given by this cross-section, which is an equivalence of groupoids. In any case, we obtain an isomorphism

$$K_*(C^*(\mathcal{G})) \simeq K_*(C(M)) = K^*(M), \quad \mathcal{G} := X \times_M X, \tag{13}$$

which we shall often use to identify these groups. For instance, the analytic index associated to $\mathcal{G} = X \times_M X$ becomes a map $\text{ind}_a^{\mathcal{G}} : K^*(T_\pi^* X) \rightarrow K^*(M) = K_*(C(M))$. Then we have the following well-known result, whose proof we sketch for the benefit of the reader.

PROPOSITION 4. *Let $\pi : X \rightarrow M$ be a smooth fiber bundle with X compact and $\mathcal{G} := X \times_M X$ be the fiberwise product groupoid. Let $[a] \in K^0(T_\pi^* X)$ be a K -theory class represented by an endomorphism $a : E \rightarrow F$ of vector bundles over $T_\pi^* X$ that are pull-backs of vector bundles on X and are such that a is an isomorphism outside a compact set. Assume that a is homogeneous of some positive order and let P_a be a family of (elliptic) pseudodifferential operators along the fibers of π with principal symbol a . Then $\text{ind}_a^{\mathcal{G}}([a]) \in K^0(M)$ coincides with the family index of P_a .*

Proof. Let B be a C^* -algebra. We shall denote by B^+ the algebra with an adjoint unit. For any $b \in B$, we shall denote by $p_b \in M_2(B^+)$ the graph projection associated to b , that is

$$p_b = \begin{bmatrix} 1 - e^{-B^*B} & \tau(B^*B)B^* \\ \tau(BB^*)B & e^{-BB^*} \end{bmatrix}, \tag{14}$$

where τ is a smooth, even function on \mathbb{R} satisfying $\tau(x^2)^2x^2 = e^{-x^2}(1 - e^{-x^2})$.

Let D be an elliptic differential operator of order m . Then $(1 + D^*D)^s$ establishes isomorphisms $H^r \rightarrow H^{r-m_s}$ of Sobolev spaces. This fact and the homotopy invariance of the index show that we can assume that a is polynomial, homogeneous of degree one, given by the principal symbol of a family D of first-order differential operators (the same argument was used in [AS68]). Let p_a be the graph projection associated to a , $p_a \in M_N(\mathcal{C}_0(T_\pi^*X)^+)$. Then we can extend the family a to the family $p_{tD} \in M_N(C^*(T\mathcal{G}))$. Therefore, the analytic index of $[a]$ is the class of $p_D \in K_0(C^*(\mathcal{G}))$. We can furthermore perturb tD with a regularizing family that vanishes in a neighborhood of $t=0$ without changing the class of p_D . We can hence assume that D has a kernel of constant dimension [AS68]. Then it is known that the class of p_D is $[\ker D] - [\ker D^*]$, that is, the family index of D . \square

We now prove a corollary that will be needed in the proof of Proposition 5. Let $\pi : U \rightarrow X$ be a vector bundle over a compact manifold with corners. Consider the fibered product groupoid $\mathcal{G} := U \times_X U$ as above. Then $T_\pi^*U = U \oplus U^*$ as vector bundles over X , and hence $i_! : K^*(X) \rightarrow K^*(T_\pi^*U)$ is defined.

COROLLARY 5. *We have that $\text{ind}_a^{\mathcal{G}} \circ i_!$ is the inverse of the isomorphism of (13) (which can be identified with the identity map of $K^*(X)$).*

Proof. Both $\text{ind}_a^{\mathcal{G}}$ and $i_!$ are $K^*(X)$ -linear. The same argument as in the proof of Proposition 4 (or by compactifying U fiberwise to a sphere bundle and then using Proposition 4), we obtain that $\text{ind}_a^{\mathcal{G}} \circ i_!(1)$ is the index of the family of Dirac operators on the fibers of U coupled with the potential given by Clifford multiplication with the independent variable. Since the equivariant index of the coupled Dirac operator is 1, we obtain that $\text{ind}_a^{\mathcal{G}} \circ i_!(1) = 1$ as in [AS68]. See [Hor85] for a simple proof of the facts needed about the coupled Dirac operator. \square

Remark 10. Let us take $X = \text{pt}$, to be reduced to a point, and let us identify U with \mathbb{R}^N , for some N . Then the above Corollary states, in particular, that $i_! : \mathbb{Z} = K^*(\text{pt}) \rightarrow K^*(T\mathbb{R}^n)$ and $\text{ind}_a^{\mathbb{R}^n} : K^*(T\mathbb{R}^n) \rightarrow \mathbb{Z} = K^*(\text{pt})$ are inverse to each other.

4. Commutativity of the diagram

In this section we shall prove a part of our topological index theorem, Theorem 13, involving an embedding $\iota : M \rightarrow X$ of our manifold with corners M into another manifold with corners X . This theorem amounts to the fact that the diagram (2) is commutative. In order to prove this, we shall first consider a tubular neighborhood

$$M \xrightarrow{k} U \xrightarrow{j} X \tag{15}$$

of M in X . The diagram (2) is then decomposed into the two diagrams below, and hence the proof of the commutativity of (2) reduces to the proof of the commutativity of the two diagrams below, whose morphisms are as follows: the morphisms k_K and j_K are defined by Lemma 6, the

morphism $k_!$ is the push-forward morphism, and j_* is the morphism in K -theory defined by an open embedding. Recall that $A_M = A(G(M))$ and $C^*(M) := C^*(G(M))$.

$$\begin{CD}
 K_*(C^*(M)) @>k_K>> K_*(C^*(U)) @>j_K>> K_*(C^*(X)) \\
 @V\text{ind}_a^M VV @V\text{ind}_a^U VV @V\text{ind}_a^X VV \\
 K^*(A_M^*) @>k_!>> K^*(A_U^*) @>j_*>> K^*(A_X^*)
 \end{CD} \tag{16}$$

The commutativity of the left diagram is part of the following proposition.

PROPOSITION 5. *Let $\pi : U \rightarrow M$ be a vector bundle over a manifold with corners M and let $k : M \rightarrow U$ be the ‘zero section’ embedding. Then the following diagram commutes.*

$$\begin{CD}
 K_*(C^*(M)) @>k_K>> K_*(C^*(U)) \\
 @V\text{ind}_a^M VV @V\text{ind}_a^U VV \\
 K^*(A_M^*) @>k_!>> K^*(A_U^*)
 \end{CD} \tag{17}$$

Proof. We shall prove this result using a double deformation groupoid \mathcal{G} , which is a Lie groupoid with units $U \times [0, 1]^2$. This groupoid is such that the projection $U \times [0, 1]^2 \rightarrow [0, 1]^2$ extends to a groupoid morphism $\mathcal{G} \rightarrow [0, 1]^2$ the latter being considered as a space, i.e., a groupoid equal to its units. In other words, if d and r are the domain and range of \mathcal{G} , then $d(g)$ and $r(g)$ have the same projection in $[0, 1]^2$.

In order to define \mathcal{G} , we give two different approaches: one based on a gluing of subgroupoids, using the Lie algebroid integration result, and one more direct. As a set, $\mathcal{G} := \mathcal{G}_1 \sqcup \mathcal{G}_2 \sqcup \mathcal{G}_3$ (\sqcup denotes the disjoint union), where

$$\begin{aligned}
 \mathcal{G}_1 &:= A_U \times \{0\} \times [0, 1], & \mathcal{G}_2 &:= A_M \times_M U \times_M U \times (0, 1] \times \{0\}, \\
 \mathcal{G}_3 &:= G(U) \times (0, 1] \times (0, 1].
 \end{aligned}$$

LEMMA 7. *The groupoid \mathcal{G} is a Lie groupoid.*

Proof. We prove this by applying Theorem 3 which states that there exists at most one differentiable structure on \mathcal{G} compatible with the groupoid structure (that is, making it a Lie groupoid). We define a Lie algebroid structure, and Lie groupoid structures for each of the subgroupoids $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$, and finally we show how to get a Lie groupoid structure compatible with these data.

Let us first describe $A(\mathcal{G})$, the Lie algebroid of the Lie groupoid that we want to construct. Recall that the Lie algebroid associated to U , A_U , or ‘compressed tangent bundle’, is such that $\Gamma(A_U)$ consists of the smooth vector fields on U that are tangent to all faces of U . Then $\pi : U \rightarrow M$ induces a map $\pi_* : A_U \rightarrow A_M$.

Let $A_U = A_v \oplus A_h$ be a decomposition of A_U into vertical and horizontal components, so that A_v is the kernel of $\pi_* : A_U \rightarrow A_M$ and $A_h \simeq \pi^*(A_M)$. Then we obtain the decomposition

$$\Gamma(U \times [0, 1]^2; A_U) = \Gamma(U \times [0, 1]^2; A_v) \oplus \Gamma(U \times [0, 1]^2; A_h).$$

Let us regard the above spaces of smooth sections X as families of sections $X(s, t) \in \Gamma(A_U)$ parameterized by $(s, t) \in [0, 1]^2$. Then $A(\mathcal{G})$, the Lie algebroid of the Lie groupoid that we want

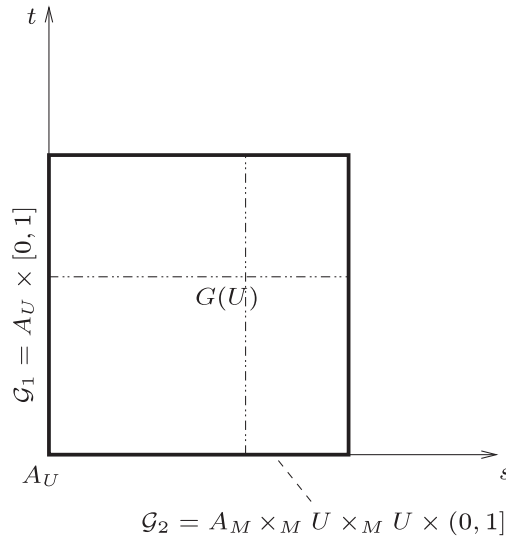


FIGURE 1. The groupoid \mathcal{G} .

to construct, is defined, as a set, by

$$\Gamma(A(\mathcal{G})) = s\Gamma(U \times [0, 1]^2; A_v) \oplus st\Gamma(U \times [0, 1]^2; A_h).$$

Then we define the Lie bracket to be the pointwise Lie bracket coming from the Lie bracket on sections of $\Gamma(A_U) : [X, Y](s, t) = [X(s, t), Y(s, t)]$.

Let us now define the Lie groupoid structure of each subgroupoid as follows. First, \mathcal{G}_1 is a commutative Lie groupoid, with operations defined by the vector bundle structure on A_U . It integrates the restriction of $A(\mathcal{G})$ to $\{s = 0\} = \{0\} \times [0, 1]$. The groupoid \mathcal{G}_2 is the fibered product of $A_M \rightarrow M$ and of the fiberwise pair groupoid $U \times_M U$ considered in Proposition 4 (in the case at hand, $X = U$). More precisely, let $\pi_M : A_M \rightarrow M$ be the canonical projection, then

$$\begin{aligned} &A_M \times_M U \times_M U \times (0, 1] \times \{0\} \\ &= \{(\xi, u_1, u_2, s, 0) \in A_M \times U \times U \times (0, 1] \times \{0\}, \pi_M(\xi) = \pi(u_1) = \pi(u_2)\}, \end{aligned}$$

with the product $(\xi, u_1, u_2, s, 0)(\xi', u_2, u_3, s, 0) = (\xi + \xi', u_1, u_3, s, 0)$. The factor $(0, 1] \times \{0\}$ therefore plays just the role of a space of parameters. The last groupoid, \mathcal{G}_3 is the product of the groupoid $G(U)$ associated to the manifold with corners U (Definition 2) with the space $(0, 1] \times (0, 1]$, which again plays just the role of a space of parameters. It integrates the restriction of $A(\mathcal{G})$ to $(0, 1] \times (0, 1]$.

Theorem 3 states that there exists at most one differentiable structure on \mathcal{G} compatible with the groupoid structure (that is, making it a Lie groupoid). The union $\mathcal{G}_1 \sqcup \mathcal{G}_3$ has the differentiable structure of the product of $(0, 1]$ with the tangent groupoid of $G(U)$. Due to the local structure of the deformations involved, in order to prove that \mathcal{G} does indeed have a smooth structure, it is enough to assume that $U \rightarrow M$ is a trivial bundle, in which case the resulting \mathcal{G} is seen to be smooth as follows.

Let $U = M \times \mathbb{R}^n$. Then $A_U = A_M \times T\mathbb{R}^n$ and $\mathcal{G}_2 = A_M \times (\mathbb{R}^n \times \mathbb{R}^n) \times (0, 1] \times \{0\}$, with $(\mathbb{R}^n \times \mathbb{R}^n)$ being the pair groupoid. Deforming just in the fiberwise direction means, in this case, that we deform the tangent space to \mathbb{R}^n to its tangent groupoid. We then define the topology

on $\mathcal{G}_1 \cup \mathcal{G}_2$, which is a groupoid with units $\{0\} \times [0, 1] \cup (0, 1] \times \{0\} \supset [0, 1] \times \{0\}$, such that the restriction to $[0, 1] \times \{0\}$ is $A_M \times {}^T G(\mathbb{R}^n)$, with ${}^T G(\mathbb{R}^n)$ denoting the tangent groupoid of the smooth manifold \mathbb{R}^n . This is possible since the tangent groupoid of \mathbb{R}^n at $t=0$ is $T\mathbb{R}^n$ and the restriction of \mathcal{G}_1 at $t=0$ is $A_M \times T\mathbb{R}^n$.

On the other hand, since $U = M \times \mathbb{R}^n$, the groupoid $G(U)$ is just the product of $G(M)$ with the pair groupoid $\mathbb{R}^n \times \mathbb{R}^n$. The union $\mathcal{G}_2 \sqcup \mathcal{G}_3$ is ${}^T G(M) \times (\mathbb{R}^n \times \mathbb{R}^n) \times (0, 1]$, with the factor $(0, 1]$ corresponding to the variable s . □

Let us now give an alternative, more direct definition of the groupoid used in the above proof.

The groupoid $G(U)$ is actually the pull-back of $G(M)$, $\pi^*(G(M)) = \{(u, v, \gamma) \in U \times U \times G(M), r(\gamma) = \pi(u), s(\gamma) = \pi(v)\}$. Consider the tangent groupoid of $G(M)$, ${}^T G(M)$, and then take its pull-back $\pi^*({}^T G(M))$. This is a groupoid which is different from ${}^T G(U)$, but which is also K -equivalent. Now consider the tangent groupoid of $\pi^*({}^T G(M))$, denoted by $\tilde{\mathcal{G}}$.

We can show that $\tilde{\mathcal{G}}$ is equal to \mathcal{G} . First of all,

$$\begin{aligned} \pi^*({}^T G(M)) &= \{(u, v, x, \xi) \in U \times U \times A_M\} \cup \pi^*(G(M)) \times (0, 1] \\ &= A_M \times_M U \times_M U \cup G(U) \times (0, 1]. \end{aligned}$$

The Lie algebroid of $A_M \times_M U \times_M U$ is $\pi^*(A_M) \times_M T_\pi U$, as explained before: this is thus A_U . This implies that the Lie algebroid of $\pi^*({}^T G(M))$ is $A_U \times [0, 1]$. As a consequence,

$$\tilde{\mathcal{G}} = A_U \times [0, 1] \times \{0\} \times \{0\} \cup A_M \times_M U \times_M U \times \{0\} \times (0, 1] \cup G(U) \times (0, 1] \times (0, 1] = \mathcal{G}.$$

With this definition of \mathcal{G} , the Lie groupoid structure on \mathcal{G} is straightforward.

Let us now denote by $e_{i,j}$ (for $i, j = 0, 1$) the various K -theory morphisms induced by the restriction maps $\mathcal{G} \rightarrow \mathcal{G}_{s=i,t=j}$, where, for instance, $\mathcal{G}_{s=1,t=0} = A_M \times_M U \times_M U$ is the lower right corner of the figure above. Let $C^*(M, U) = C^*(\mathcal{G}_{s=1,t=0})$.

LEMMA 8. *The evaluation maps $e_{0,0}$ and $e_{1,0}$ are isomorphisms, and the following properties hold.*

- The analytic index of U is $\text{ind}_a^U = e_{1,1} \circ e_{0,0}^{-1}$.
- The maps

$$\text{ind}_a^{(1)} := e_{1,0} \circ e_{0,0}^{-1} : K^*(A_U^*) \rightarrow K^*(C^*(M, U))$$

and

$$\text{ind}_a^{(2)} := e_{1,1} \circ e_{1,0}^{-1} : K_*(C^*(M, U)) \rightarrow K_*(C^*(U))$$

are such that

$$\text{ind}_a^U = \text{ind}_a^{(2)} \circ \text{ind}_a^{(1)}. \tag{18}$$

Proof. We shall define several successive decompositions (Figure 2). Let us notice that the above construction of \mathcal{G} is such that $\mathcal{G}' := \mathcal{G}_1 \cup \mathcal{G}_2$ is a closed subgroupoid of \mathcal{G} , with complement $\mathcal{G}_3 = G(U) \times (0, 1] \times (0, 1]$. This induces an exact sequence

$$0 \rightarrow C^*(\mathcal{G}_3) \rightarrow C^*(\mathcal{G}) \rightarrow C^*(\mathcal{G}_1 \cup \mathcal{G}_2) \rightarrow 0;$$

but $K_*(C^*(\mathcal{G}_3)) = 0$, so $K_*(C^*(\mathcal{G})) \simeq K_*(C^*(\mathcal{G}_1 \cup \mathcal{G}_2))$. Now \mathcal{G}_2 is an open subgroupoid of $\mathcal{G}_1 \cup \mathcal{G}_2$, with vanishing K -theory groups. A similar exact sequence argument then shows that $K_*(C^*(\mathcal{G}_1 \cup \mathcal{G}_2)) \simeq K_*(C^*(\mathcal{G}_1)) \simeq K_*(A_U)$. Thus $e_{0,0} : K_*(C^*(\mathcal{G})) \rightarrow K_*(C_0(A_U^*))$ is an isomorphism. Since the restriction of \mathcal{G} to the diagonal $s = t$ of $[0, 1]^2$ is the tangent groupoid of U , we obtain that $\text{ind}_a^U = e_{1,1} \circ e_{0,0}^{-1}$.

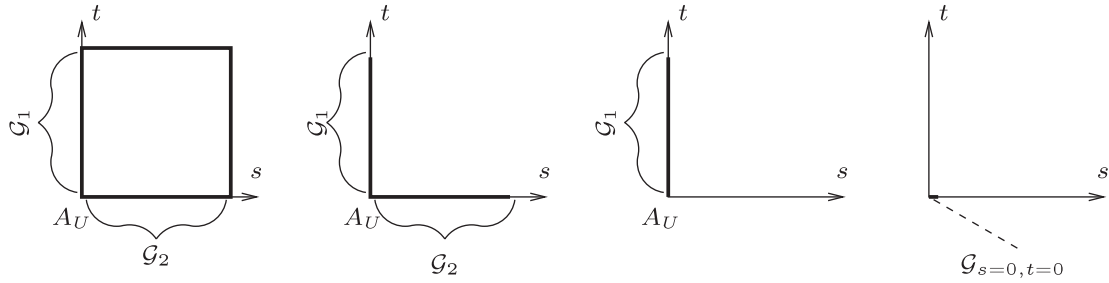


FIGURE 2. Successive restrictions of \mathcal{G} .

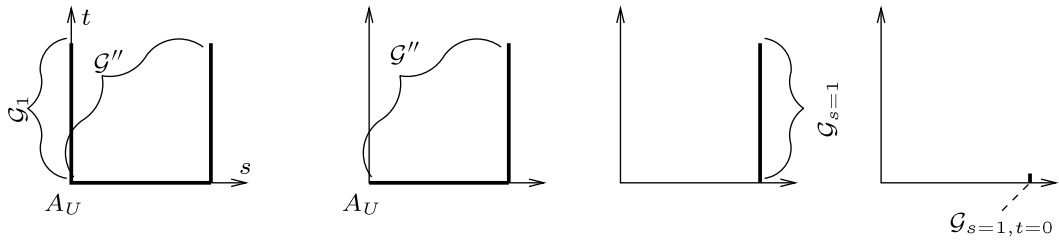


FIGURE 3. Other successive restrictions of \mathcal{G} .

We aim at factorizing the index map ind_a^U in two maps defined along the lower and right sides of the square. The first map is

$$\text{ind}_a^{(1)} := e_{1,0} \circ e_{0,0}^{-1} : K^*(A_U^*) \rightarrow K_*(C^*(M, U)).$$

For the second map, we shall define other successive decompositions (Figure 3).

For the second map, we consider the subgroupoid $\mathcal{G}'' := \mathcal{G}_L$ where $L := \{s = 1\} \cup \{t = 0\}$. Using a short exact sequence argument as above, we obtain that $\mathcal{G}'' \cup \mathcal{G}_1$ has the same K -theory as \mathcal{G} because its complement in \mathcal{G} is $G_U \times (0, 1) \times (0, 1]$. Then, similarly, $\mathcal{G}'' \cup \mathcal{G}_1$ turns out to have the same K -theory as \mathcal{G}'' since the complement of \mathcal{G}'' in $\mathcal{G}'' \cup \mathcal{G}_1$ is $T_U \times (0, 1]$. Furthermore, the decomposition $\mathcal{G}'' = \mathcal{G}_{s < 1, t = 0} \cup \mathcal{G}_{s = 1}$ induces a K -isomorphism between \mathcal{G}'' and $\mathcal{G}_{s = 1}$. Indeed, still considering that $U = M \times \mathbb{R}^n$, we showed previously that $\mathcal{G}_{t = 0} = A_M \times {}^T G(\mathbb{R}^n)$. However, the K -theory of ${}^T G(\mathbb{R}^n)_{s < 1}$ is isomorphic to that of the adiabatic groupoid, which is zero since the analytic index for a Euclidean space is an isomorphism.

The last step is that the evaluation at $t = 0$ in the groupoid $\mathcal{G}_{s = 1}$ is an isomorphism, using the same arguments as before. In conclusion, $e_{1,0} : K_*(C^*(\mathcal{G})) \rightarrow K_*(C^*(M, U))$ is an isomorphism.

The map $\text{ind}_a^{(2)}$ is thus well defined, and the equality

$$\text{ind}_a^U = \text{ind}_a^{(2)} \circ \text{ind}_a^{(1)}$$

is straightforward. □

Remark 11. In the proof above we used \mathcal{G}'' since we shall need later to consider the groupoid $\mathcal{H} = \mathcal{G}_{\{s = 1\}}$. However, the proof could also have been handled considering the groupoid \mathcal{G}' .

To end the proof of Proposition 5, let us consider now the commutative diagram, in which the morphism Θ_0 is the isomorphism of (13) (its definition will be recalled below as part of a

more general construction).

$$\begin{CD}
 K^*(A_M^*) @= K^*(A_M^*) @>\text{ind}_a^M>> K_*(C^*(M)) \\
 @V k! VV @V \Theta_0 \downarrow \simeq VV @V k_K \downarrow \simeq VV \\
 K^*(A_U^*) @>\text{ind}_a^{(1)}>> K_*(C^*(M, U)) @>\text{ind}_a^{(2)}>> K_*(C^*(U))
 \end{CD} \tag{19}$$

Equation (18) shows that it is enough to prove that the above diagram is commutative.

Let X be the fiberwise one point compactification of A_M^* . Then $K^*(A_M^*) \subset K^*(X)$, since $X \setminus A_M^*$ is a retract of X . The commutativity of the left diagram then follows from Corollary 5 (after we lift the bundle U to X).

We need to define the morphism Θ_0 . Let us consider the groupoid \mathcal{H} defined as the restriction of \mathcal{G} to $\{1\} \times [0, 1]$ used also to define the morphism $\text{ind}_a^{(2)}$. It has units $U \times [0, 1]$. Let $\Omega = r^{-1}(M \times [0, 1])$, as in the proof of Lemma 6. As in the proof of that lemma, Θ defines an imprimitivity module between \mathcal{H} and $r^{-1}(M) \cap d^{-1}(M) = {}^T\mathcal{G}_M$. This imprimitivity module induces imprimitivity modules Θ_t for $t \in [0, 1]$ (the parameter of the deformation). By the proof of Proposition 3, the isomorphism k_K is defined by Θ_1 . The isomorphism defined by Θ_0 was also denoted by Θ_0 . The commutativity of the right rectangle in the above diagram then follows from the compatibility of the isomorphisms defined by Θ with restriction morphisms. \square

The commutativity of the second square in the diagram (16) follows from the naturality of the tangent groupoid construction. Here are the details.

PROPOSITION 6. *Let $j : U \rightarrow X$ be the inclusion of the open subset U . Then the diagram below commutes.*

$$\begin{CD}
 K_*(C^*(U)) @>j_K>> K_*(C^*(X)) \\
 @V \text{ind}_a^U VV @V \text{ind}_a^X VV \\
 K^*(A_U^*) @>j^* \simeq>> K^*(A_X^*)
 \end{CD}$$

Proof. As U is open in X , the groupoid ${}^T\mathcal{G}(U)$ identifies with the restriction of ${}^T\mathcal{G}(X)$ to U . This induces a map ${}^T C^*(\mathcal{G}(U)) \rightarrow C^*({}^T\mathcal{G}(X))$. Hence we get the following commutative diagram, in which the vertical arrows are inclusions.

$$\begin{CD}
 C^*(U) @>>> C^*(X) \\
 @V e_1 VV @V e_1 VV \\
 C^*({}^T\mathcal{G}(U)) @>>> C^*({}^T\mathcal{G}(X)) \\
 @V e_0 VV @V e_0 VV \\
 \mathcal{C}_0(A_U^*) @>>> \mathcal{C}_0(A_X^*)
 \end{CD}$$

This commutative diagram, together with Lemma 6 and Remark 9, give the analogue commutative diagram in K -theory.

$$\begin{array}{ccc}
 K_*(C^*(G(U))) & \longrightarrow & K_*(C^*(G(X))) \\
 \uparrow e_1 & & \uparrow e_1 \\
 K_*(C^*(TG(U))) & \longrightarrow & K_*(C^*(TG(X))) \\
 \downarrow e_0 & & \downarrow e_0 \\
 K^*(A^*(G(U))) & \longrightarrow & K^*(A^*(G(X)))
 \end{array}$$

The commutativity of the above diagram proves our result. □

We are ready now to prove one of our main results by putting together what he have proved in the previous two propositions, as explained in the beginning of this section.

THEOREM 12. *Let $M \xrightarrow{\iota} X$ be a closed embedding of manifolds with corners. Then the diagram*

$$\begin{array}{ccc}
 K_*(C^*(M)) & \xrightarrow{\iota_K} & K_*(C^*(X)) \\
 \uparrow \text{ind}_a^M & & \text{ind}_a^X \uparrow \\
 K^*(A_M^*) & \xrightarrow{\iota!} & K^*(A_X^*)
 \end{array} \tag{20}$$

is commutative.

Proof. As $i(M)$ is a closed submanifold of X , there exists a tubular neighborhood U of $i(M)$ in X , along with a fibration $\pi : U \rightarrow M$. Let $k : M \rightarrow U$ be the embedding of M into U as the zero section and $j : U \rightarrow X$ be the embedding of U as an open subset of M . The result then follows from the commutativity of the diagrams in Propositions 5 and 6 and from $i_! = j_* \circ k_!$ and $i_K = j_K \circ k_K$. (The diagram (16) explains this reasoning.) □

5. An Atiyah–Singer type theorem

Motivated by Theorem 12 and by the results of § 3 (see Propositions 2 and 3) we introduce the following definition.

DEFINITION 4. A classifying manifold X_M of M is a compact manifold with corners X_M , together with a closed embedding $\iota : M \rightarrow X_M$ with the following properties.

- (i) Each open face of X_M is diffeomorphic to a Euclidean space.
- (ii) The map $F \rightarrow F \cap M$ induces a bijection between the open faces of X_M and M .

Note that if $M \subset X_M$ are as in the above definition, then each face of M is the transverse intersection of M with a face of X_M . As a consequence we obtain the following result, which generalizes the main theorem of [AS68].

LEMMA 9. *Let M be a manifold with embedded faces, and $\iota : M \hookrightarrow X_M$ be a classifying space of M . Then the maps ι_K and ind_a^X of Theorem 12 are isomorphisms.*

Proof. This was proved in Propositions 2 and 3. □

Let $\iota : M \rightarrow X_M$ be a classifying space for M . The above lemma then allows us to define (see (20))

$$\text{ind}_t^M := \iota_K^{-1} \circ \text{ind}_a^X \circ \iota_! : K^*(A_M^*) \rightarrow K_*(C^*(M)).$$

If M is a smooth compact manifold (so, in particular, $\partial M = \emptyset$), then $C^*(M) = \mathcal{K}$, the algebra of compact operators on $L^2(M)$ and hence $K_0(C^*(M)) = \mathbb{Z}$. Any embedding $\iota : M \hookrightarrow \mathbb{R}^N$ will then be a classifying space for M . Moreover, as explained in Remark 10, for $X = \mathbb{R}^n$, the map $\iota_K^{-1} \circ \text{ind}_a^X : K^*(TX) \rightarrow \mathbb{Z}$ is the inverse of $j_! : K^0(\text{pt}) \rightarrow K^0(T\mathbb{R}^N)$ and hence $\text{ind}_t^{\mathbb{R}^N} = (j_!)^{-1} \iota_!$, which is the definition of the topological index from [AS68]. In view of this fact, we shall also call the map ind_t^M the *topological index* associated to M .

THEOREM 13. *The topological index map ind_t^M depends only on M , that is, it is independent of the classifying space X_M , and we have*

$$\text{ind}_t^M = \text{ind}_a^M : K^*(A_M^*) \rightarrow K_*(C^*(M)).$$

Proof. This follows immediately from Theorem 12. □

If M is a smooth compact manifold (without boundary), this recovers the Atiyah–Singer index theorem on the equality of the analytic and topological index [AS68].

Remark 14. Let us also mention that $K_*(C^*(M)) \simeq K^*(X_M)$ provides us with a way of determining $K_*(C^*(M))$, which is a non-trivial problem. The structure of $C^*(M)$ has been studied in [MN98, Mon03]. It is given through a composition series whose subquotients are easy to compute. We shall use our results in order to determine $K_*(C^*(M))$ in a joint work with Etienne Fieux.

6. Construction of the classifying space X_M

We now show that a classifying manifold X_M of M exists (Definition 4). The choice of X_M is not canonical, in general.

Let M be a compact manifold with embedded faces, and let $(H_i)_{1 \leq i \leq r}$ be the set of hyperfaces of M . For each closed hyperface H_i , we shall fix a defining function ρ_i (these could be, for example, the defining functions used in the definition of $G(M)$, Definition 2). Also, let us choose an embedding of ϕ of M into some \mathbb{R}^N . The map

$$\psi = (\phi, \rho_1, \dots, \rho_r) : M \rightarrow \mathbb{R}^N \times [0, \infty)^r$$

is thus an embedding of manifolds with corners, which, however, does not induce a bijection of the faces. To fix this problem, we need to add extra coordinates which will disconnect the faces of $\mathbb{R}^N \times [0, \infty)^r$ putting them in bijection with the faces of M . If $J \subset \{1, \dots, r\}$, define

$$F_J = \bigcap_{j \in J} H_j.$$

If nonempty, this is a disjoint union of closed faces of codimension $|J|$, the number of elements of J . Assume F_J is not empty and let f_J be a continuous function on F_J with values in $\{1, \dots, n_J\}$ where n_J is the number of connected components of F_J . We chose f_J to take different values on different connected components of F_J . This function can then be extended, thanks to Tietze’s theorem, to a smooth function still denoted by $f_J : M \rightarrow \mathbb{R}$. Let us denote by \mathcal{J} the set of nonempty subsets of $\{1, \dots, r\}$ for which F_J is not empty, and $l = |\mathcal{J}|$ the number of its elements.

Then we obtain an embedding

$$\Psi := (\phi, \rho_1, \dots, \rho_r, f_J) : M \rightarrow X_0 := \mathbb{R}^N \times [0, \infty)^r \times \mathbb{R}^l, \quad J \in \mathcal{J}.$$

We still need to disconnect the faces of X_0 whose inverse image in M is disconnected. To this end, for any $J \in \mathcal{J}$, let $Y_J \subset \mathbb{R}^N \times [0, \infty)^r \times \mathbb{R}^l$ be the closed subset defined by

$$Y_J := \{(z, x_1, \dots, x_r, y_J), y_J - \frac{1}{2} \in \mathbb{Z} \text{ and } x_j = 0 \text{ for all } j \in J\}.$$

Then $\Psi(M) \cap Y_J = \emptyset$, by the construction of f_J . Let $X_1 := X_0 \setminus \bigcup_{J \in \mathcal{J}} Y_J$. Finally, remove from X_1 all the faces that do not intersect M and call what is left X_M :

$$X_M := X_1 \setminus \bigcup F, \quad F \subset X_1 \text{ are faces such that } F \cap M = \emptyset. \tag{21}$$

Naturally this creates many more faces than we have in M , so the last step is to take X to be the complement in X_1 of the open faces which do not intersect $\Psi(M)$.

PROPOSITION 7. *The manifold $X_M \supset M$ of (21) is a classifying space for M .*

Proof. We need to prove that the following properties are true.

- (a) Each open face of X_M is diffeomorphic to a Euclidean space.
- (b) The map $F \rightarrow F \cap M$ defines a bijection between the set of open faces of M and the set of open faces of X_M .
- (c) The set M is a closed, embedded submanifold of X_M .

The open faces of $X_0 := \mathbb{R}^N \times [0, \infty)^r \times \mathbb{R}^l$ are in one-to-one correspondence with the subsets of $\{1, 2, \dots, r\}$. More precisely,

$$G_I = \{(z, x_1, \dots, x_r, y_J) \in X_0, x_j = 0 \Leftrightarrow j \in I\},$$

$I \subset \{1, 2, \dots, r\}$, are all the open faces of X_0 .

Fix $I \subset \{1, 2, \dots, r\}$. The open faces $F_1 \subset X_1$ contained in G_I are the connected components of

$$G_I \setminus \bigcup_J Y_J = G_I \setminus \bigcup_J (G_I \cap Y_J), \quad J \in \mathcal{J}.$$

Since $G_I \cap Y_J = \emptyset$ for any $J \in \mathcal{J}$ that is not contained in I , it is enough to consider only $J \subset I$, $J \in \mathcal{J}$. Fix a face $F_1 \subset X_1$. Then $y_J \in (m_J - 1/2, m_J + 1/2)$ on F_1 , for some $m_J \in \mathbb{Z}$. This shows that

$$F_1 = \{(z, \dots, y_J) \in G_I, m_J - 1/2 < y_J < m_J + 1/2, J \subset I, J \in \mathcal{J}\}.$$

In particular, $F_1 \simeq \mathbb{R}^N \times (0, \infty)^a \times (-1/2, 1/2)^b \times \mathbb{R}^{l-b}$, $a = r - |I|$. This verifies condition (a) above.

Let F_1 be the face fixed above. Then $F_1 \cap M$ is the set of points $x \in M$ satisfying $\rho_i(x) = 0$, for all $i \in I$, and $f_J(x) \in (m_J - 1/2, m_J + 1/2)$, for all $J \subset I$ such that $J \in \mathcal{J}$; therefore

$$F_1 \cap M = \{x \in M, \rho_i(x) = 0 \text{ if } i \in I, f_I(x) \in (m_I - 1/2, m_I + 1/2)\},$$

which is either empty or a connected component of F_J , by the construction of f_J . In other words, for any open face $F_1 \subset X_M$, the intersection $F_1 \cap M$ is either empty or an open face of M . This verifies condition (b) above.

Finally, let us notice that each open face F of M is contained in exactly one face F_1 of X_M and each open face F_1 of X_1 is contained in an open face G_I of X_0 . This shows that M intersects F_1

transversely, because dx_j , $j \in J$, are linearly independent on $F_J := \bigcap_{j \in J} H_j$. Also, M is a closed submanifold of X_M because $\phi : M \rightarrow \mathbb{R}^N$ (the first component of Ψ) is an embedding. This verifies condition (c) above and thus completes the proof. \square

Remark 15. In the case M is a smooth manifold, our construction is such that X_M is a Euclidean space.

ACKNOWLEDGEMENTS

We thank Catarina Carvalho, Sergiu Moroianu, and Georges Skandalis for useful discussions. We also wish to thank the referee who helped us improve this article. Nistor was partially supported by the NSF Grant DMS 0555831.

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