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Connections on a Parabolic Principal Bundle Over a Curve

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Abstract. The aim here is to define connections on a parabolic principal bundle. Some applications are given.

1 Introduction

The notion of a parabolic vector bundle was introduced in [MS]. Let X be a connected Riemann surface and $D \subset X$ a finite subset. A parabolic vector bundle over X with parabolic structure over D is a usual vector bundle E over X together with an extra structure over D. For each point $p \in D$, the extra structure over p is a strictly decreasing filtration of linear subspaces of the fiber E_p together with a nonnegative rational number associated with each term of the filtration. The sequence of numbers is strictly increasing and is strictly bounded by 1. See Section 2 or [MS, MY] for the details.

In [BBN1, BBN2], the notion of a parabolic vector bundle was extended to the more general context of where the structure group is a general connected linear algebraic group defined over the field of complex numbers (as opposed to $\mathrm{GL}(r, \mathbb{C})$ which corresponds to parabolic vector bundles of rank r).

Let G be a complex connected linear algebraic group. A parabolic G-bundle over X is given by the following data: a connected complex manifold E_G with a projection $\psi\colon E_G \to X$ and an action of G on the right of E_G such that

(1) $X = E_G/G$;
(2) the projection ψ and the action of G make $\psi^{-1}(X \setminus D)$ a principal G-bundle over the complement $X \setminus D$;
(3) for any point $z \in \psi^{-1}(D)$ the isotropy subgroup, for the action of G on E_G, is a finite cyclic group.

(See Section 2 for the details.)

The aim here is to extend the notion of a connection on a principal bundle to the context of parabolic G-bundles. It turns out that one of the descriptions of usual connections, namely as a $\mathfrak{g}$-valued one-form on the total space, where $\mathfrak{g}$ is the Lie algebra of G, is well suited for parabolic G-bundles. It may be pointed out that the more standard description (see [At]) of a connection on a usual principal bundle as a splitting of the Atiyah exact sequence is not suited for parabolic G-bundles.

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A theorem due to Atiyah and Weil says that a holomorphic vector bundle E over a compact Riemann surface admits a holomorphic connection if and only if each indecomposable component of E is of degree zero (see [At, We]). In [AB] this was generalized to give a criterion for a holomorphic principal G-bundle over a compact Riemann surface to admit a holomorphic connection, where G is a complex reductive group.

In Theorem 4.1 we give a criterion for a parabolic G-bundle to admit a holomorphic connection, where G as before is a complex reductive group.

Let G be a complex semisimple group. In Theorem 5.2 we prove that a parabolic G-bundle E_G admits a flat unitary connection if and only if E_G is polystable.

2 Preliminaries

Let X be a connected smooth projective curve defined over the field of complex numbers, or equivalently, a connected compact Riemann surface. Let

$$D = \{p_1, \dots, p_l\} \subset X$$

be a reduced effective divisor on X; so $\{p_i\}_{i=1}^{l}$ are distinct points.

Let E be an algebraic vector bundle over X. A *quasiparabolic* structure on E over D is a filtration of subspaces

$$E|_{p_i} = F_1^i \supset F_2^i \supset F_3^i \supset \cdots \supset F_{m_i}^i \supset F_{m_i+1}^i = 0 \tag{2.1}$$

of the fiber of E over p_i, where $i \in [1, l]$. For a quasiparabolic structure as above, *parabolic weights* are a collection of rational numbers

$$0 \leq \lambda_1^{(i)} < \lambda_2^{(i)} < \lambda_3^{(i)} < \cdots < \lambda_{m_i}^{(i)} < 1 \tag{2.2}$$

where $i \in [1, l]$. The parabolic weight $\lambda_j^{(i)}$ corresponds to F_j^i in (2.1). A *parabolic structure* on E is a quasiparabolic structure with parabolic weights. A vector bundle equipped with a parabolic structure on it is also called a *parabolic vector bundle*. See [MS, MY] for the details.

For notational convenience, a parabolic vector bundle defined as above will be denoted by E_*. The divisor D is called the *parabolic divisor* for E_*. We recall that

$$\text{par-deg}(E_*) := \text{degree}(E) + \sum_{i=1}^{l} \sum_{j=1}^{m_i} \lambda_j^{(i)} \dim(F_j^i / F_{j+1}^i) \tag{2.3}$$

is called the *parabolic degree* of the above defined parabolic vector bundle E_* (see [MS, Definition 1.11] and [MY]).

Let G be a connected complex linear algebraic group. We will recall the definition of a parabolic G-bundle over X (see [BBN2] for the details).

A *parabolic G-bundle* over X with parabolic structure over D is a connected smooth complex variety E_G on which G acts algebraically on the right, that is, the map

$$f\colon E_G \times G \to E_G$$

defining the action of G on E_G is an algebraic morphism, together with a surjective algebraic map

$$\psi\colon E_G \to X \tag{2.4}$$

satisfying the following conditions:

(1) $\psi \circ f = \psi \circ p_1$, where p_1 is the natural projection of $E_G \times G$ to E_G, that is, the map ψ is equivariant for the actions of G with G acting trivially on X;
(2) for each point $x \in X$, the action of G on the reduced fiber $\psi^{-1}(x)_{\mathrm{red}}$ is transitive;
(3) the restriction of ψ to $\psi^{-1}(X \setminus D)$ makes $\psi^{-1}(X \setminus D)$ a principal G-bundle over $X \setminus D$, that is, the map ψ is smooth over $\psi^{-1}(X \setminus D)$ and the map to the fiber product

$$\psi^{-1}(X \setminus D) \times G \to \psi^{-1}(X \setminus D) \times_{X\setminus D} \psi^{-1}(X \setminus D)$$

defined by $(z, g) \mapsto (z, f(z, g))$ is an isomorphism.
(4) for any closed point $z \in \psi^{-1}(D)$, the isotropy group $G_z \subset G$, for the action of G on E_G, is a finite cyclic group that acts faithfully on the line $T_zE_G/T_z\psi^{-1}(D)_{\mathrm{red}}$.

Note that since ψ is equivariant for the action of G, the isotropy subgroup G_z preserves the subspace $T_z\psi^{-1}(D)_{\mathrm{red}} \subset T_zE_G$. Therefore, there is an induced action of G_z on the fiber $T_zE_G/T_z\psi^{-1}(D)_{\mathrm{red}}$ of the normal bundle. For any $p_i \in D$ the subvariety $\psi^{-1}(p_i)_{\mathrm{red}} \subset E_G$ is clearly a smooth divisor.

Consider the special case of $G = \mathrm{GL}(n, \mathbb{C})$. There is a bijective correspondence between the parabolic $\mathrm{GL}(n, \mathbb{C})$-bundles and the parabolic vector bundles of rank n defined earlier (see [BBN1, BBN2]). Let E_{GL} be a parabolic $\mathrm{GL}(n, \mathbb{C})$-bundle over X. So E_{GL} restricts to a usual $\mathrm{GL}(n, \mathbb{C})$-bundle over $X \setminus D$. Therefore, using the standard action of $\mathrm{GL}(n, \mathbb{C})$ on $\mathbb{C}^n$, the $\mathrm{GL}(n, \mathbb{C})$-bundle $E_{\mathrm{GL}}|_{X\setminus D}$ gives a vector bundle of rank n over $X \setminus D$. This vector bundle has a natural extension to X, constructed using E_{GL}, that carries the parabolic structure of the parabolic vector bundle corresponding to E_{GL}. Take a point $x \in D$. Take a point $z \in E_{\mathrm{GL}}$ over x (that is, $\psi(z) = x$), and let $G_z \subset \mathrm{GL}(n, \mathbb{C})$ be the isotropy subgroup for z for the action of $\mathrm{GL}(n, \mathbb{C})$ on E_{GL}. Since the finite cyclic group G_z acts faithfully on the line $T_zE_{\mathrm{GL}}/T_z\psi^{-1}(D)$, the group G_z has a natural (unique) generator $\gamma_z \in G_z$ defined by the following condition: the action of γ_z on $T_zE_{\mathrm{GL}}/T_z\psi^{-1}(D)$ is multiplication by $\exp(2\pi\sqrt{-1}/(\#G_z))$, where $\#G_z$ is the order of G_z. The parabolic structure, of the parabolic vector bundle associated to E_{GL}, over the parabolic point x is constructed as follows:

Consider the standard action of $\gamma_z \in \mathrm{GL}(n, \mathbb{C})$ on $\mathbb{C}^n$. Note that all the eigenvalues of it are of the form $\exp(2\pi\sqrt{-1}k/(\#G_z))$, where $k \in [0, \#G_z - 1]$. If $\exp(2\pi\sqrt{-1}k/(\#G_z))$ is an eigenvalue, where $k \in [0, \#G_z - 1]$, then $k/\#G_z$ is a parabolic weight at x for the parabolic vector bundle associated to E_{GL}, and the multiplicity of a parabolic weight $k/\#G_z$ coincides with the multiplicity of the eigenvalue $\exp(2\pi\sqrt{-1}k/(\#G_z))$ for the action of γ_z on $\mathbb{C}^n$. The details are given in [BBN2].

We now return to the case of general G. Let V be a finite dimensional complex left G-module, where G is a connected complex linear algebraic group. Let E_G be a parabolic G-bundle over X. Consider the quotient space

$$E_{\mathrm{GL}} := \frac{E_G \times \mathrm{GL}(V)}{G}, \tag{2.5}$$

where the quotient is for the "twisted" diagonal action of G; the action of any $g \in G$ sends any $(z, T) \in E_G \times \mathrm{GL}(V)$ to $(zg, g^{-1}T)$. It is easy to see that E_{GL} is a parabolic $\mathrm{GL}(V)$-bundle. Hence E_{GL} gives a parabolic vector bundle E^V_*.

Let $\mathrm{Rep}(G)$ denote the category of finite dimensional complex left G-modules, and let $\mathrm{PVect}(X)$ be the category of parabolic vector bundles over X with parabolic structure over D. Both these categories are equipped with the direct sum, tensor product and dualization operations.

The parabolic G-bundle E_G defines a functor from $\mathrm{Rep}(G)$ to $\mathrm{PVect}(X)$ by sending any G-module V to the parabolic vector bundle E^V_* constructed as above from V. This way, E_G defines a parabolic G-bundle in the sense of [BBN1]; in [BBN1], following [No1, No2], parabolic G-bundles were defined as a functor from $\mathrm{Rep}(G)$ to $\mathrm{PVect}(X)$ satisfying certain conditions. Such a functor is in particular compatible with the three operations of taking dual, direct sum and tensor product.

Let G_1 be a connected complex linear algebraic group and

$$\rho\colon G \to G_1 \tag{2.6}$$

a homomorphism of algebraic groups. Let E_G be a parabolic G-bundle over X. Consider the quotient

$$E_G(G_1) = \frac{E_G \times G_1}{G} \tag{2.7}$$

for the twisted diagonal action of G on $E_G \times G_1$. The action of any $g \in G$ sends any $(z, g_1) \in E_G \times G_1$ to $(zg, \rho(g^{-1})g_1)$. It is easy to see that $E_G(G_1)$ is a parabolic G_1-bundle over X.

We will call $E_G(G_1)$ as the parabolic G_1-bundle obtained by *extending the structure group* of E_G to G_1 using the homomorphism ρ.

Therefore, E_{GL} in (2.5) is the extension of structure of E_G to $\mathrm{GL}(V)$ using the homomorphism $G \to \mathrm{GL}(V)$ defined by the G-module V.

Let E_G be a parabolic G-bundle over X. Let $H \subset G$ be a connected closed algebraic subgroup. Consider the quotient

$$q_H\colon E_G \to E_G/H \tag{2.8}$$

for the action of G on E_G. Let

$$f_H\colon E_G/H \to X \tag{2.9}$$

be the natural projection. Take any (closed) point $z \in E_G/H$. Take any (closed) point $\widehat{z} \in E_G$ such that $q_H(\widehat{z}) = z$, where q_H is defined in (2.8). It is easy to see that the following two conditions are equivalent:

(1) the projection f_H in (2.9) is smooth at z, that is, the differential

$$\mathrm{d}f_H(z)\colon T_z E_G/H \to T_{f_H(z)}X$$

is surjective.

(2) The isotropy subgroup $G_{\widehat{z}} \subset G$ is contained in H.

Since H acts transitively on $q_H^{-1}(z)$, if the condition $G_{z'} \subset G$ holds for one point $z' \in q_H^{-1}(z)$, then we have $G_{z''} \subset G$ for every point $z'' \in q_H^{-1}(z)$.

By a section of f_H we will mean a morphism $\sigma\colon X \to E_G/H$ such that $f_H \circ \sigma$ is the identity map of X.

Definition 2.1 A *reduction of structure group of E_G to H* is a section

$$\sigma\colon X \to E_G/H$$

of f_H (defined in (2.9)) such that f_H is smooth on the image of σ (in other words, for each point $x \in X$ the differential

$$\mathrm{d} f_H(\sigma(x))\colon T_{\sigma(x)}E_G/H \to T_xX$$

is surjective).

A section $\sigma\colon X \to E_G/H$ of the projection f_H defines a reduction of structure group of E_G to H if and only if

$$q_H^{-1}(\sigma(X)) \subset E_G$$

(where q_H is the projection in (2.8)) is a parabolic H-bundle over X (the action of H on $q_H^{-1}(\sigma(X))$ is induced by the action of G on E_G) satisfying the condition that $G_z \subset H$ for each point $z \in q_H^{-1}(\sigma(X))$ (recall that $G_z \subset G$ is the isotropy subgroup at z for the action of G on E_G).

We noted earlier that for any $p_i \in D$, if $G_z \subset H$ for one point $z \in q_H^{-1}(\sigma(p_i))$, then $G_{z'} \subset H$ for each point $z' \in q_H^{-1}(\sigma(p_i))$.

Let $E_H \subset E_G$ be a reduction of structure group of a parabolic G-bundle E_G to a subgroup $H \subset G$, defined by a section σ as in Definition 2.1. Let E_G^1 be the parabolic G-bundle obtained by extending the structure group of the parabolic H-bundle E_H using the inclusion homomorphism $H \hookrightarrow G$. It is easy to see that the two parabolic G-bundles E_G and E_G^1 are canonically identified. Indeed, consider the morphism $E_H \times G \to E_G$ defined by the action of G on E_G (recall that $E_H \subset E_G$). This morphism factors through the quotient $(E_H \times G)/H$ and defines an isomorphism of E_G^1 with E_G.

Consider $E_G(G_1)$ constructed in (2.7). Note that there is a natural morphism

$$r\colon E_G \to E_G(G_1) \tag{2.10}$$

that sends any $z \in E_G$ to the equivalence class for $(z, e) \in E_G \times G_1$, where $e \in G_1$ is the identity element. It is easy to see that r is G-equivariant; the action of G on $E_G(G_1)$ is through $\rho(G)$, where ρ is the homomorphism on (2.6).

If G is a closed subgroup of G_1 and ρ (in (2.6)) the inclusion map, then the image of the map r, constructed in (2.10), defines a reduction of structure group to G of the parabolic G_1-bundle $E_G(G_1)$. Indeed, this is an immediate consequence of the above definition of a reduction of structure group.

3 Holomorphic Connections

Let $\psi\colon E_G \to X$ be a parabolic G-bundle over X. So for any closed point $x \in X$, the reduced fiber

$$(E_G)_x := \psi^{-1}(x)$$

is an orbit for the action of G on E_G.

Let $\mathfrak{g}$ be the Lie algebra of G. The action of G on $(E_G)_x$ gives a homomorphism of vector bundles

$$\omega'_x\colon \underline{\mathfrak{g}} \to T(E_G)_x$$

over $(E_G)_x$, where $\underline{\mathfrak{g}}$ denotes the trivial vector bundle over $(E_G)_x$ with fiber $\mathfrak{g}$ and $T(E_G)_x$ is the tangent bundle. The given condition that the isotropy G_z of any point $z \in (E_G)_x$ is a finite subgroup of G immediately implies that the above defined homomorphism ω'_x is in fact an isomorphism. Consequently, the inverse $(\omega'_x)^{-1}$ exists and it defines an algebraic one-form

$$\omega_x \in H^0((E_G)_x, \Omega^1_{(E_G)_x} \otimes_{\mathbb{C}} \mathfrak{g}) \tag{3.1}$$

with values in $\mathfrak{g}$. This form ω_x is also known as the *Maurer–Cartan form*.

Definition 3.1 A *holomorphic connection* on the parabolic G-bundle E_G is an algebraic one-form θ on E_G with values in $\mathfrak{g}$

$$\theta \in H^0(E_G, \Omega^1_{E_G} \otimes_{\mathbb{C}} \mathfrak{g})$$

such that

(1) for each point $x \in X$ the restriction $\theta|_{(E_G)_x}$ coincides with the Maurer–Cartan form ω_x defined in (3.1), and
(2) θ intertwines the action of G on E_G and the adjoint action of G on $\mathfrak{g}$, or in other words, θ is equivariant for the actions of G.

When E_G is a usual principal G-bundle, the above definition of a holomorphic connection coincides with one given in [At].

Proposition 3.2 *Let E_G be a parabolic G-bundle equipped with a holomorphic connection θ (as defined in Definition 3.1).*

Let $\rho\colon G \to G_1$ be a homomorphism of connected algebraic groups. Then θ induces a holomorphic connection on the parabolic G_1-bundle $E_G(G_1)$ obtained by extending the structure group of E_G using ρ.

Let $H \subset G$ be a closed connected algebraic subgroup and $E_H \subset E_G$ a reduction of structure group of E_G to H. Let

$$\beta\colon \mathfrak{g} \to \mathfrak{h}$$

be an H-equivariant splitting of H-modules, where $\mathfrak{g}$ (respectively, $\mathfrak{h}$) is the Lie algebra of G (respectively, H); both $\mathfrak{g}$ and $\mathfrak{h}$ are considered as H-modules for the adjoint action. Then $\beta \circ \theta$ gives a holomorphic connection on the parabolic H-bundle E_H.

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Proof Let $d\rho\colon \mathfrak{g} \to \mathfrak{g}_1$ be the differential of ρ, where $\mathfrak{g}_1$ is the Lie algebra of G_1. Consider the map r defined in (2.10). From the definition of a form defining a holomorphic connection it follows immediately that

$$d\rho \circ \theta\colon \ \mathrm{Tr}(E_G) \longrightarrow \mathfrak{g}_1$$

is a G-equivariant $\mathfrak{g}_1$-valued one-form on the image of the map r. Now, this form extends uniquely to a G_1-equivariant $\mathfrak{g}_1$-valued one-form on the total space of the parabolic G_1-bundle $E_G(G_1)$. It is easy to check that this extended form defines a connection on the parabolic G_1-bundle $E_G(G_1)$.

Take any β as in the statement of the proposition. So $\beta \circ \iota$ is the identity map of $\mathfrak{h}$, where $\iota\colon \mathfrak{h} \hookrightarrow \mathfrak{g}$ is the inclusion map. Consider the $\mathfrak{h}$-valued one-form $\beta \circ \theta$ on the subvariety $E_H \subset E_G$. Since the form θ is G-equivariant and β is H-equivariant, it follows immediately that $\beta \circ \theta$ is H-equivariant. The restriction of $\beta \circ \theta$ to a reduced fiber of the projection $E_H \to X$ clearly coincides with the Maurer–Cartan form (recall that β is a splitting of the inclusion of $\mathfrak{h}$ in $\mathfrak{g}$). Therefore, $\beta \circ \theta$ defines a holomorphic connection on the parabolic H-bundle E_H. This completes the proof of the proposition. ∎

Let Y be a connected smooth projective curve defined over $\mathbb{C}$ and

$$\Gamma \subset \mathrm{Aut}(Y)$$

a finite subgroup of the automorphism group of Y. So Γ acts naturally on the right of Y.

A *Γ-linearized principal G-bundle* over Y is a (usual) principal G-bundle

$$\psi'\colon E'_G \longrightarrow Y \tag{3.2}$$

over Y together with an algebraic right action of the finite group Γ on the total space E'_G such that

(1) the action of G on the principal G-bundle E'_G commutes with the action of Γ on E'_G, and
(2) the projection ψ' in (3.2) commutes with the actions of Γ on E'_G and Y.

Let Y_Γ denote the quotient Y/Γ. So Y_Γ is also a connected smooth projective curve over $\mathbb{C}$. Consider the quotient E'_G/Γ, where E'_G is a Γ-linearized principal G-bundle over Y. Since the projection of E'_G to Y commutes with the actions of Γ (condition (2)), we have an induced projection

$$\psi\colon E_G := E'_G/\Gamma \longrightarrow Y_\Gamma$$

induced by ψ'.

Since the actions of G and Γ on E'_G commute, the quotient E_G has an induced action of G

$$f\colon E_G \times G \longrightarrow E_G.$$

It is easy to see that the triple (E_G, ψ, f) defines a parabolic G-bundle over Y_Γ. The parabolic divisor is the divisor of Y_Γ over which the quotient map $Y \longrightarrow Y_\Gamma$ is ramified. All parabolic G-bundles arise as quotients of the above type [BBN2, Theorem 3.7]. It should be clarified that given a parabolic G-bundle E_G over X the Galois covering Y of X (such that E_G is the quotient of a principal G-bundle over Y) depends on E_G. The content of the above mentioned theorem of [BBN2] is that given a parabolic G-bundle E_G over X, there exists a Galois (ramified) covering of Y of X and a Γ-linearized principal G-bundle E'_G over Y, where $\Gamma = \mathrm{Gal}(Y/X)$ is the Galois group, such that $E_G = E'_G/\Gamma$.

Let E'_G be a principal G-bundle over Y. We recall that a holomorphic connection on E'_G is a G-equivariant algebraic one-form

$$\theta' \in H^0(E'_G, \Omega^1_{E'_G} \otimes_{\mathbb{C}} \mathfrak{g}) \tag{3.3}$$

whose restriction to each fiber coincides with the Maurer–Cartan form. Equivalently, a holomorphic connection on E'_G is a holomorphic splitting of the Atiyah exact sequence

$$0 \to \mathrm{ad}(E'_G) \to \mathrm{At}(E'_G) \to TY \to 0 \tag{3.4}$$

over Y, where $\mathrm{ad}(E'_G)$ is the adjoint bundle and $\mathrm{At}(E'_G)$ the Atiyah bundle (see [At] for the details).

Now assume that E'_G is Γ-linearized, where $\Gamma \subset \mathrm{Aut}(Y)$ is a finite subgroup.

Definition 3.3 A holomorphic connection θ' (as in (3.3)) on the G-bundle E'_G will be called a Γ-*connection* if the action of Γ on E'_G leaves the form θ' invariant (the action of Γ on $\mathfrak{g}$ is the trivial action).

Note that the actions of Γ on X and E'_G induce actions of Γ on all the three vector bundles in the Atiyah exact sequence (3.4), and the homomorphisms in (3.4) commute with the actions of Γ. If the connection θ' in (3.3) corresponds to the holomorphic splitting

$$D\colon TY \to \mathrm{At}(E'_G)$$

of (3.4), then θ' is a Γ-connection if and only if the splitting homomorphism D commutes with the actions of Γ on TY and $\mathrm{At}(E'_G)$.

Let $\psi\colon E_G \to X$ be a parabolic G-bundle over X (as in (2.4)). We noted earlier that by [BBN2, Theorem 3.7] there is an irreducible smooth projective curve Y, a finite subgroup $\Gamma \subset \mathrm{Aut}(Y)$ and a Γ-linearized principal G-bundle E'_G over Y such that $X = Y/\Gamma$ and $E_G = E'_G/\Gamma$.

Proposition 3.4 *There is a natural bijective correspondence between the holomorphic connections on the parabolic G-bundle E_G and the Γ-connections on the principal G-bundle E'_G.*

Proof Let

$$q\colon E'_G \to E'_G/\Gamma = E_G \tag{3.5}$$

be the quotient map. Given a $\mathfrak{g}$-valued one-form θ on E_G, the pullback

$$\theta' := q^*\theta \tag{3.6}$$

is a $\mathfrak{g}$-valued one-form on E'_G. Assume that θ satisfies the two conditions in Definition 3.1; in other words, θ is a holomorphic connection on E_G. Then, as the projection q commutes with the actions of G on E'_G and E_G, the second condition in Definition 3.1 that θ is G-equivariant implies that θ' is G-equivariant.

Let $q'\colon Y \to Y/\Gamma = X$ be the quotient map. For any closed point $y \in Y$ the restriction

$$q|_{(E'_G)_y}\colon (E'_G)_y \to (E_G)_{q'(y)}$$

is a finite unramified covering map, where $(E'_G)_y$ is the fiber E_G over y and $(E_G)_{q'(y)}$ as before is the reduced inverse image $\psi^{-1}(q'(y))$. Since $q|_{(E'_G)_y}$ is a finite unramified covering map, the given condition that the restriction of θ to $(E_G)_{q'(y)}$ coincides with the Maurer–Cartan form implies that the restriction of θ' to $(E'_G)_y$ coincides with the Maurer–Cartan form on $(E'_G)_y$. Therefore, the form θ' defines a holomorphic connection on the G-bundle E'_G.

Since θ' is a pullback of a form from E'_G/Γ, we conclude that the action of Γ on E'_G leaves θ' invariant. Consequently, θ' defines a Γ-connection on E'_G.

For the converse direction, let θ' be a Γ-invariant $\mathfrak{g}$-valued one-form on the total space of E'_G defining a Γ-connection on the principal G-bundle E'_G. Since θ' is Γ-invariant, it descends to a $\mathfrak{g}$-valued one-form on the quotient $E_G = E'_G/\Gamma$. In other words, there is a $\mathfrak{g}$-valued one-form θ on E_G such that $q^*\theta = \theta'$, where q is the projection defined in (3.5). To prove the existence of such a form θ, note that for any point $y \in E'_G$, the map q around z is holomorphically isomorphic to a map of the form

$$(z_1, \dots, z_{n-1}, z_n) \to (z_1, \dots, z_{n-1}, z_n^k)$$

where k is a positive integer. Therefore, it suffices to show that any holomorphic one-form ω defined on the unit disk $\mathbb{D} \subset \mathbb{C}$ and invariant under the multiplication action (on $\mathbb{D}$) of μ_k, the cyclic group defined by the k-th roots of unity, is a pullback of a form on the quotient space $\mathbb{D}/\mu_n$. But this is clear as ω must vanish at $0 \in \mathbb{D}$ of order at least $k-1$.

Now it is easy to check that the descended $\mathfrak{g}$-valued one-form θ on E_G satisfies the two conditions in Definition 3.1. Therefore, any form on E'_G defining a Γ-connection is the pullback of a form on E_G defining a holomorphic connection on the parabolic G-bundle E_G. This completes the proof of the proposition. ∎

Let $\psi\colon E_G \to X$ be a parabolic G-bundle over X. Recall from the definition of a parabolic G-bundle that $\psi^{-1}(X \setminus D)$ is a principal G-bundle over the complement $X \setminus D$. Let

$$\theta \in H^0(E_G, \Omega^1_{E_G} \otimes_{\mathbb{C}} \mathfrak{g})$$

be a holomorphic connection on E_G. So the restriction $\theta|_{\psi^{-1}(X\setminus D)}$ is a holomorphic connection on the principal G-bundle $\psi^{-1}(X \setminus D)$ over $X \setminus D$. Any holomorphic connection on a curve is flat as there are no nonzero holomorphic two-forms on a Riemann surface.

Therefore, $\theta|_{\psi^{-1}(X\setminus D)}$ is a flat connection on the principal G-bundle $\psi^{-1}(X \setminus D)$ over $X \setminus D$. Take any point $p_i \in D$. Since θ' in (3.6) is a flat connection on E'_G over Y, it follows immediately that the monodromy of the connection on $\psi^{-1}(X\setminus D)$ (defined by $\theta|_{\psi^{-1}(X\setminus D)}$) along a loop in $X \setminus D$ around p_i and contractible in X is of finite order.

Let E'_G be a Γ-linearized principal G-bundle over Y. Let $E_G = E'_G/\Gamma$ be the corresponding parabolic G-bundle over $X = Y/\Gamma$. As in (3.5), the quotient map $E'_G \to E_G$ will be denoted by q. Let $E_H \subset E_G$ be a reduction of structure group to a subgroup $H \subset G$. The inverse image

$$q^{-1}(E_H) \subset E'_G$$

is clearly a reduction of structure group of the principal G-bundle E'_G to H which is left invariant by the action of Γ on E'_G. It is also straightforward to check that if

$$E'_H \subset E'_G$$

is a reduction of structure group of E'_G to $H \subset G$ with the property that the action of Γ on E'_G leaves the subvariety $E'_H \subset E'_G$ invariant, then

$$E'_H/\Gamma \subset E'_G/\Gamma$$

is a reduction of structure group to H of the parabolic G-bundle $E_G = E'_G/\Gamma$.

Let θ be a holomorphic connection on the parabolic G-bundle E_G. Let θ' be the corresponding Γ-connection on E'_G constructed in Proposition 3.4. Let H and β be as in Proposition 3.2. So using Proposition 3.2, the holomorphic connection θ induces a holomorphic connection

$$\theta_H := \beta \circ \theta|_{E_H}$$

on the parabolic H-bundle E_H (we are using the notation of Proposition 3.2). On the other hand, using β, the Γ-connection θ' on E'_G induces a Γ-connection on the Γ-linearized principal H-bundle $E'_H := q^{-1}(E_H)$, where q is the projection defined in (3.5); the construction of this connection is identical to the construction of the connection in the second part of Proposition 3.2. It is easy to see that the holomorphic connection on $E_H = E'_H/\Gamma$ corresponding to this Γ-connection on E'_H coincides, by the correspondence in Proposition 3.4, to the holomorphic connection θ_H on E_H constructed above.

Let $\rho\colon G \to G_1$ be a homomorphism as in (2.6). Let $E'_G(G_1) = (E'_G \times G_1)/G$ be the Γ-linearized principal G_1-bundle over Y obtained by extending the structure group of the G-bundle E'_G using ρ. Clearly we have

$$E'_G(G_1)/\Gamma = E_G(G_1),$$

where $E_G(G_1)$ is the parabolic G-bundle defined in (2.7). Using this it is straightforward to check that the correspondence of connections given by Proposition 3.4 is compatible with the construction given in Proposition 3.2 of induced connection on an extension of structure group. In other words, if θ is a holomorphic connection on E_G and θ' the corresponding Γ-connection on E'_G, then the holomorphic connection on $E_G(G_1)$ induced by θ corresponds to the Γ-connection on $E'_G(G_1)$ induced by θ'.

In the next section we will give a criterion for a parabolic G-bundle, where G is reductive, to admit a holomorphic connection.

4 Criterion for Existence of a Connection

Let $\sigma\colon X \to E_G/H$, as in Definition 2.1, be a reduction of structure group to H of the parabolic G-bundle E_G over X. Let

$$E_H := q_H^{-1}(\sigma(X)) \subset E_G$$

be the corresponding parabolic H-bundle, where q_H is the projection in (2.8). Take a character

$$\chi\colon H \to \mathbb{G}_m = \mathbb{C}^*$$

of H. Let $E_H(\mathbb{C}^*)$ be the parabolic $\mathbb{C}^*$-bundle obtained by extending the structure group of E_H using the homomorphism χ. We noted earlier that there is a natural bijective correspondence between parabolic $\mathrm{GL}(n,\mathbb{C})$-bundles and parabolic vector bundles of rank n. Let

$$E^{\chi}_* = E_H(\mathbb{C}^*)(\mathbb{C})_* \tag{4.1}$$

be the parabolic line bundle associated to the parabolic $\mathbb{C}^*$-bundle $E_H(\mathbb{C}^*)$.

Henceforth, G will be assumed to be a reductive group.

A closed connected subgroup P of G is called a *parabolic subgroup* if G/P is complete. Note that we allow G to be a parabolic subgroup of itself. The unipotent radical of P will be denoted by $R_u(P)$. A *Levi subgroup* of G is a connected reductive subgroup $H \subset G$ such that

(1) H is contained in some parabolic subgroup P of G, and
(2) H projects isomorphically onto the Levi quotient $P/R_u(P)$ of the above parabolic subgroup P.

So Levi subgroups are precisely the centralizers of tori contained in G.

Theorem 4.1 *Let E_G be a parabolic G-bundle over the curve X, where G is a complex reductive group. The parabolic G-bundle E_G admits a holomorphic connection if and only if for every Levi subgroup $H \subset G$, for every holomorphic reduction of structure group $E_H \subset E_G$ to H, and for every character χ of H the following holds:*

$$\text{par-deg}(E^{\chi}_*) = 0,$$

where the parabolic line bundle E^{χ}_* is defined in (4.1) *and the parabolic degree is defined in* (2.3).

Proof Let θ be a holomorphic connection on the parabolic G-bundle E_G. Take a Γ-linearized principal G-bundle E'_G on Y, as in Proposition 3.4, that corresponds to E_G. Let θ' be the Γ-connection on E'_G corresponding to θ by Proposition 3.4.

Take a Levi subgroup $H \subset G$ and a reduction of structure group $E_H \subset E_G$ as in the statement of the theorem. Let

$$E'_H = q^{-1}(E_H) \subset E'_G$$

be the corresponding reduction of structure group of E'_G to H, where q is the projection defined in (3.5).

Since H is a complex reductive group, any exact sequence of finite dimensional complex H-modules splits. Consider the inclusion of H-modules $\mathfrak{h} \subset \mathfrak{g}$, where $\mathfrak{g}$ (respectively, $\mathfrak{h}$) is the Lie algebra of G (respectively, H), and $\mathfrak{h}$, $\mathfrak{g}$ are considered as H-modules using the adjoint action of H. Fix a splitting

$$\beta \colon \mathfrak{g} \to \mathfrak{h}$$

of this inclusion of H-modules. The connection θ' and β combine together to give a Γ-connection of E'_H; the connection on E'_H is constructed exactly as done in the second part of Proposition 3.2. Let θ'_H denote this connection on E'_H.

As before, $E_H(\mathbb{C}^*)$ denotes the parabolic $\mathbb{C}^*$-bundle obtained by extending the structure group of E_H using the character χ of H. Note that $E_H(\mathbb{C}^*)$ corresponds to the Γ-linearized principal $\mathbb{C}^*$-bundle $E'_H(\mathbb{C}^*)$ over Y obtained by extending the structure group of the principal H-bundle E'_H using the character χ of H. The parabolic line bundle E^{χ}_* defined in (4.1) corresponds to the Γ-linearized line bundle $E'_H(\mathbb{C})$ over Y associated to the principal $\mathbb{C}^*$-bundle $E'_H(\mathbb{C}^*)$ for the standard action of $\mathbb{C}^*$ on $\mathbb{C}$. (See [Bi] for the correspondence between parabolic vector bundles and the Γ-linearized vector bundles.)

The holomorphic connection θ'_H on E'_H constructed above induces a connection on $E'_H(\mathbb{C}^*)$ which in turn induces a holomorphic connection on the line bundle $E'_H(\mathbb{C})$ over Y. We conclude that

$$\text{degree}(E'_H(\mathbb{C})) = 0 \tag{4.2}$$

as $E'_H(\mathbb{C})$ admits a holomorphic connection; see [At].

On the other hand, we have

$$\#\Gamma \cdot \text{par-deg}(E^{\chi}_*) = \text{degree}(E'_H(\mathbb{C}))$$

[Bi, (3.12)], where $\#\Gamma$ is the order of the finite group Γ. Therefore, (4.2) gives that $\text{par-deg}(E^{\chi}_*) = 0$.

To prove the converse, let E_G be a parabolic G-bundle over X satisfying the condition that $\text{par-deg}(E^{\chi}_*) = 0$ for all H, E_H and χ as in the statement of the theorem.

Let E'_G be a Γ-linearized principal G-bundle over Y as in Proposition 3.4 such that $E_G = E'_G/\Gamma$. Using Proposition 3.4 it suffices to show that E'_G admits a Γ-connection.

Assume that E'_G admits a holomorphic connection. Let

$$\theta' \in H^0(E'_G, \Omega^1_{E'_G} \otimes_{\mathbb{C}} \mathfrak{g})$$

be a form as in (3.3) defining a holomorphic connection on E'_G. Consider the $\mathfrak{g}$-valued one-form

$$\theta'' = \frac{\sum_{\gamma\in\Gamma}\gamma^*\theta}{\#\Gamma}$$

on the total space of E'_G, where $\gamma^*\theta$ is the pullback of θ by the automorphism of E'_G defined by the action of γ on E'_G. It is easy to see that the form θ'' is left invariant by the action of Γ on E'_G. Hence θ'' defines a Γ-connection on E'_G.

Therefore, to complete the proof of the theorem it suffices to prove the following lemma.

Lemma 4.2 *Let E'_G be a Γ-linearized principal G-bundle over Y such that for every Levi subgroup $H \subset G$, for every Γ-invariant holomorphic reduction of structure group $E'_H \subset E'_G$ to H, and for every character χ of H the following holds:*

$$\mathrm{degree}(E'_H(\mathbb{C})) = 0,$$

where $E'_H(\mathbb{C}) = (E'_H \times \mathbb{C})/H$ is the line bundle over Y associated to the principal H-bundle E'_H for the character χ. Then the principal G-bundle E'_G admits a holomorphic connection.

Proof To prove the lemma, we first recall that E'_G admits a holomorphic connection if and only if the Atiyah exact sequence (3.4) splits holomorphically. Let

$$\tau \in H^1(Y, \mathrm{ad}(E'_G) \otimes K_Y) \tag{4.3}$$

be the obstruction class for holomorphic splitting of (3.4), where K_Y is the holomorphic cotangent bundle of Y.

Since G is reductive, its Lie algebra $\mathfrak{g}$ admits a nondegenerate symmetric bilinear form which is left invariant by the adjoint action of G on $\mathfrak{g}$. In other words, $\mathfrak{g} \cong \mathfrak{g}^*$ as G-modules. Fix such a G-invariant bilinear form. This gives an isomorphism of vector bundles $\mathrm{ad}(E'_G) \cong \mathrm{ad}(E'_G)^*$. Now using Serre duality, the cohomology class τ in (4.3) corresponds to an element

$$\tau' \in H^0(Y, \mathrm{ad}(E'_G))^*.$$

Since the G-bundle E'_G is Γ-linearized, we conclude that

$$\tau' \in \left(H^0(Y, \mathrm{ad}(E'_G))^*\right)^{\Gamma} \subset H^0(Y, \mathrm{ad}(E'_G))^*, \tag{4.4}$$

where $\left(H^0(Y, \mathrm{ad}(E'_G))^*\right)^{\Gamma} \subset H^0(Y, \mathrm{ad}(E'_G))^*$ is the space of invariants for the induced action of Γ on $H^0(Y, \mathrm{ad}(E'_G))^*$.

Note that for any finite dimensional complex Γ-module V, the restriction homomorphism

$$f_V\colon (V^*)^{\Gamma} \to (V^{\Gamma})^*$$

is an isomorphism, where $V^{\Gamma} \subset V$ (respectively, $(V^*)^{\Gamma} \subset V^*$) is the space of all Γ-invariants. Indeed, for any nontrivial irreducible Γ-submodule $V_1 \subset V$ we have

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$\omega(V_1) = 0$, where $\omega \in (V^*)^\Gamma$; hence f_V is injective. To prove that f_V is surjective, extend any functional $\omega \in (V^\Gamma)^*$ to V by defining it to be zero on any nontrivial irreducible Γ-submodule of V.

Therefore, we have

$$\tau' \in \left(H^0(Y, \mathrm{ad}(E'_G))^\Gamma\right)^* = \left(H^0(Y, \mathrm{ad}(E'_G))^*\right)^\Gamma \subset H^0(Y, \mathrm{ad}(E'_G))^*, \tag{4.5}$$

where τ' is constructed in (4.4).

Take any invariant section

$$\phi \in H^0(Y, \mathrm{ad}(E'_G))^\Gamma. \tag{4.6}$$

The fibers of the Lie algebra bundle $\mathrm{ad}(E'_G)$ over Y are isomorphic to the Lie algebra $\mathfrak{g}$ of G. Consider the Jordan decomposition

$$\phi = \phi_s + \phi_n \tag{4.7}$$

of ϕ in (4.6). So

$$\phi_s, \phi_n \in H^0(Y, \mathrm{ad}(E'_G)),$$

and for any closed point $y \in Y$, the element

$$\phi_s(y) \in \mathrm{ad}(E'_G)_y$$

(respectively, $\phi_n(y) \in \mathrm{ad}(E'_G)_y$) is semisimple (respectively, nilpotent) with

$$[\phi_s(y), \phi_n(y)] = 0;$$

see [Bo, 4.4] for Jordan decomposition.

Note that from the uniqueness of the Jordan decomposition it follows immediately that

$$\phi_n, \phi_s \in H^0(Y, \mathrm{ad}(E'_G))^\Gamma$$

(recall that $\phi \in H^0(Y, \mathrm{ad}(E'_G))^\Gamma$).

Proposition 3.9 of [AB] says that

$$\tau'(\phi_n) = 0,$$

where τ' is constructed in (4.5).

So to prove the lemma it is enough to show that

$$\tau'(\phi_s) = 0. \tag{4.8}$$

If $H \subset G$ is a Levi subgroup and $H_1 \subset H$ a Levi subgroup of the reductive group H, then H_1 is a Levi subgroup of G. Indeed, this follows from the fact that if $Z_0(H_1)$ is the connected component of the center of H_1 containing the identity element, then the centralizer of $Z_0(H_1)$ in H coincides with the intersection of H and the centralizer of $Z_0(H_1)$ in G.

Using this property, we may assume that the Γ-linearized principal G-bundle E'_G does not admit any Γ-invariant reduction of structure group to any proper Levi subgroup of G. Indeed, if E'_G admits a Γ-invariant reduction of structure group $E_{G_1} \subset E_G$ to a proper Levi subgroup $G_1 \subset G$, then we can replace G by G_1 and E_G by E_{G_1} in the lemma; note that a holomorphic connection on E_{G_1} induces a holomorphic connection on E_G. Repeating this inductively we finally obtain a Γ-invariant reduction of structure group to a Levi subgroup which does not admit any further Γ-invariant reduction to a proper Levi subgroup.

Therefore, we assume that the Γ-linearized principal G-bundle E'_G does not admit any Γ-invariant reduction of structure group to any proper Levi subgroup of G.

Let $\mathfrak{z}(\mathfrak{g}) \subset \mathfrak{g}$ be the center of the Lie algebra. Since G acts trivially on $\mathfrak{z}(\mathfrak{g})$, the adjoint vector bundle $\mathrm{ad}(E'_G)$ has a trivial subbundle with fibers identified with $\mathfrak{z}(\mathfrak{g})$. Therefore, there is a natural injective homomorphism

$$\delta\colon \mathfrak{z}(\mathfrak{g}) \to H^0(Y, \mathrm{ad}(E'_G))^\Gamma. \tag{4.9}$$

We will show that the given condition that the Γ-linearized principal G-bundle E'_G does not admit any Γ-invariant reduction of structure group to any proper Levi subgroup of G implies that the section ϕ_s (in (4.7)) is in the image of the homomorphism δ constructed in (4.9).

Let $q_{\mathfrak{g}}\colon E'_G \times \mathfrak{g} \to \mathrm{ad}(E'_G)$ be the natural quotient map. Let

$$p^1\colon E'_G \times \mathfrak{g} \to E'_G$$

be the projection to the first factor of the Cartesian product. For the section ϕ_s in (4.7) consider

$$Z(\phi_s) := p^1\left(q_{\mathfrak{g}}^{-1}(\phi_s(Y))\right) \subset E'_G,$$

where $q_{\mathfrak{g}}$ and p^1 are defined above. It is easy to see that this subvariety $Z(\phi_s) \subset E'_G$ defines a reduction of structure group of E'_G to a Levi subgroup of G, and furthermore, the Levi subgroup is proper if

$$\phi_s \notin \delta(\mathfrak{z}(\mathfrak{g})),$$

where δ is constructed in (4.9) (see [BP] for the details).

Therefore, there is $\omega \in \mathfrak{z}(\mathfrak{g})$ such that

$$\phi_s = \delta(\omega). \tag{4.10}$$

Let

$$\phi_s^* = H^0(Y, \mathrm{ad}(E'_G)^*)^\Gamma \tag{4.11}$$

be the section defined by ϕ_s using the isomorphism $\mathrm{ad}(E'_G)^* \cong \mathrm{ad}(E'_G)$ (recall that by fixing a G-invariant bilinear form on $\mathfrak{g}$ we obtained an isomorphism of $\mathrm{ad}(E'_G)$ with $\mathrm{ad}(E'_G)^*$).

For a character $\chi'\colon G \to \mathbb{C}^*$ of G, let

$$\mathrm{d}\chi'\colon \mathfrak{g} \to \mathbb{C} \tag{4.12}$$

be the homomorphism of Lie algebras defined by the differential of χ' at $e \in G$.

From (4.10) if follows that there are characters $\chi_1, \chi_2, \dots, \chi_k$ of G such that

$$\phi_s^* = \sum_{i=1}^{k} \lambda_i \mathrm{d}\chi_i, \tag{4.13}$$

where ϕ_s^* (respectively, $\mathrm{d}\chi_i$) is defined in (4.11) (respectively, (4.12)) and $\lambda_i \in \mathbb{C}$; the above integer k can be taken to be $\dim_{\mathbb{C}} \mathfrak{z}(\mathfrak{g})$. Indeed, this follows immediately from the fact that the Lie algebra $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{z}(\mathfrak{g}).$$

Take any character $\chi' \colon G \to \mathbb{C}^*$ of G. Let

$$E'_G(\chi') = \frac{E'_G \times \mathbb{C}_{\chi'}}{G}$$

be the line bundle over Y associated to the G-bundle E'_G for the G-module $\mathbb{C}_{\chi'}$ (the G-module defined by the action of G on $\mathbb{C}$ through χ' is denoted by $\mathbb{C}_{\chi'}$). The Atiyah obstruction class τ in (4.3) is compatible with the extension of structure group of a principal bundle. In other words, for any homomorphism $\rho \colon G \to G'$ of algebraic groups, the Atiyah obstruction class for the principal G'-bundle $E'_G(G')$ obtained by extending the structure group of E'_G using ρ coincides with the image of τ (defined in (4.3)) in $H^1(Y, \mathrm{ad}(E'_G(G')) \otimes K_Y)$ by the homomorphism

$$H^1(Y, \mathrm{ad}(E'_G) \otimes K_Y) \to H^1(Y, \mathrm{ad}(E'_G(G')) \otimes K_Y)$$

constructed using ρ. Using this observation it is straightforward to check that the following identity holds:

$$2\pi\sqrt{-1} \cdot \mathrm{degree}(E'_G(\chi')) = \langle \mathrm{d}\chi', \tau \rangle, \tag{4.14}$$

where $\mathrm{d}\chi'$ (respectively, τ) is defined in (4.12) (respectively, (4.3)) and $\langle -, - \rangle$ is the Serre duality pairing $H^0(Y, \mathrm{ad}(E'_G)^*) \otimes H^1(Y, \mathrm{ad}(E'_G) \otimes K_Y) \to \mathbb{C}$.

Since the given condition in the statement of the lemma says that

$$\mathrm{degree}(E'_G(\chi')) = 0$$

for all character χ', the equality (4.8) follows from (4.13) and (4.14). This completes the proof of the lemma. ∎

We already noted that Lemma 4.2 completes the proof of Theorem 4.1. Therefore, the proof of Theorem 4.1 is complete. ∎

5 Einstein–Hermitian Connection on Stable Parabolic Bundles

As before, $\psi\colon E_G \to X$ is a parabolic G-bundle with G a reductive group.

A complex connection on E_G is a C^∞ form, with values in $\mathfrak{g}$, of Hodge type $(1,1)$ on the total space of E_G

$$\theta \in C^\infty(E_G, \Omega^{1,0}_{E_G} \otimes_{\mathbb{C}} \mathfrak{g})$$

such that

(1) for each point $x \in X$ the restriction $\theta|_{(E_G)_x}$ to the fiber $(E_G)_x$ coincides with the Maurer–Cartan form ω_x defined in (3.1), and

(2) θ intertwines the action of G on E_G and the adjoint action of G on $\mathfrak{g}$, or, in other words, θ is equivariant for the actions of G.

(See [KN, p. 64, Proposition 1.1].)

Note that if the above form θ defining a complex connection is a holomorphic form, then it defines a holomorphic connection on E_G. In other words, holomorphic connections are a special case of complex connections.

Given a complex connection form θ, the $\mathfrak{g}$-valued two-form

$$\Omega(\theta) := \mathrm{d}\theta + \frac{1}{2}[\theta, \theta] \tag{5.1}$$

is known as the *curvature* of θ (see [KN, p. 77, Theorem 5.2]).

Since θ is of Hodge type $(1,0)$, it follows immediately that the curvature form $\Omega(\theta)$ is a sum of a $(2,0)$-form and a $(1,1)$-form. We will show that $\Omega(\theta)$ is of Hodge type $(1,1)$, that is, the $(2,0)$ Hodge type component vanishes.

Since the connection form θ is G-equivariant, it follows immediately that the curvature form $\Omega(\theta)$ defined in (5.1) is also G-equivariant. Since a Maurer–Cartan form ω satisfies the identity

$$\mathrm{d}\omega + \frac{1}{2}[\omega, \omega] = 0$$

it follows that the curvature form $\Omega(\theta)$ is given by a G-equivariant smooth section of $(\bigwedge^2 N) \otimes_{\mathbb{C}} \mathfrak{g}$, where N is the normal bundle to the orbits for the action of G on E_G. Over the complement $\psi^{-1}(X \setminus D) \subset E_G$, the normal bundle N is identified with $\psi^* T^{\mathbb{R}}X$, where $T^{\mathbb{R}}X$ is the real tangent bundle over X; the isomorphism is given by the differential of ψ. Now, the projection ψ is holomorphic and X does not admit any nonzero form of Hodge type $(2,0)$. Consequently, $\Omega(\theta)$ is a $\mathfrak{g}$-valued form on E_G of Hodge type $(1,1)$. Therefore, we have

$$\Omega(\theta) = \overline{\partial}\theta. \tag{5.2}$$

A complex connection θ is called *flat* if the curvature $\Omega(\theta)$ defined in (5.1) vanishes identically.

From (5.2) it follows immediately that the vanishing of $\Omega(\theta)$ is equivalent to form θ being holomorphic. Therefore, a flat complex connection on E_G is the same as a holomorphic connection on E_G.

Fix a maximal compact subgroup

$$K(G) \subset G.$$

As in (2.8), let $q_{K(G)} \colon E_G/K(G) \to X$ be the quotient map for the action of G on E_G. Take a smooth section σ of $q_{K(G)}$. So

$$\sigma \colon X \to E_G/K(G)$$

is a C^∞ map and $q_{K(G)} \circ \sigma$ is the identity map of X. As in Definition 2.1, such a section σ will be called a C^∞ *reduction of structure group of* E_G *to* $K(G)$ provided for each point $x \in X$ the projection $q_{K(G)}$ is a submersion at $\sigma(x) \in E_G/K(G)$.

For a section σ of the projection $q_{K(G)}$ consider

$$q_{K(G)}^{-1}(\sigma(X)) \subset E_G.$$

The section σ gives a reduction of structure group of E_G to $K(G)$ if and only if for each point $p_i \in D$ and any point $z \in q_{K(G)}^{-1}(\sigma(p_i))$, the isotropy subgroup $G_z \subset G$ (for the action of G on E_G) is contained in the compact subgroup $K(G)$. (See the comments following Definition 2.1.)

Take a reduction of structure group $\sigma \colon X \to E_G/K(G)$ of E_G to $K(G)$. Set

$$Z(K(G)) := q_{K(G)}^{-1}(\sigma(X)) \subset E_G.$$

There is a unique complex connection θ on E_G satisfying the following condition: for each point $z \in Z(K(G))$, the kernel of the homomorphism

$$\theta(z) \colon T_z E_G \to \mathfrak{g}$$

is contained in the subspace

$$T_z^{\mathbb{R}} Z(K(G)) - \sqrt{-1}\, J(z) T_z^{\mathbb{R}} Z(K(G)) \subset T_z E_G,$$

where $T^{\mathbb{R}}$ is the real tangent space and $J(z)$ is the almost complex structure of E_G at the point z; here $T_z^{\mathbb{R}} Z(K(G)) - \sqrt{-1}\, J(z) T_z^{\mathbb{R}} Z(K(G))$ denotes the space of all tangent vectors of $(1,0)$ type, that is, tangent vectors of the form $w - \sqrt{-1}\, J(z)(w)$, where $w \in T_z^{\mathbb{R}} Z(K(G))$.

The above assertion is a reformulation of [Ko, Proposition 4.9, p. 11].

Definition 5.1 *A unitary connection* on a parabolic G-bundle E_G is a complex connection θ on E_G such that there is a reduction σ of structure group of E_G to the maximal compact subgroup $K(G)$ with the property that the complex connection on E_G corresponding to σ coincides with θ.

We will now recall from [BBN2] the definition of a (semi)stable parabolic G-bundle.

A parabolic G-bundle E_G over X is called *stable* (respectively, *semistable*) if for any reduction of structure group E_H of E_G to any proper parabolic subgroup $H \subset G$ and for every nontrivial antidominant character χ of H trivial on the center of G the following inequality holds:

$$\text{par-deg}(E_*^{\chi}) > 0$$

(respectively, par-deg$(E_*^\chi) \geq 0$), where E_*^χ is the parabolic line bundle over X constructed in (4.1) from E_H and χ.

A semistable parabolic G-bundle E_G is called *polystable* if there is a Levi subgroup $H \subset G$ and a reduction of structure group $E_H \subset E_G$ of E_G to H such that

(1) the parabolic H-bundle E_H is stable, and
(2) for every character χ of H trivial on the center of G the associated parabolic line bundle E_*^χ (constructed in (4.1)) is of parabolic degree zero.

See [BBN2, p. 134, Definition 3.13] for the above definitions. These definitions were modeled on [Ra].

Theorem 5.2 *Let E_G be a parabolic G-bundle over X, where G is a connected semisimple linear algebraic group over $\mathbb{C}$. The parabolic G-bundle E_G admits a flat unitary connection if and only if E_G is polystable.*

Proof Assume that E_G is polystable. Take a Γ-linearized principal G-bundle E'_G over Y such that E_G corresponds to E'_G (see [BBN2, Theorem 3.7]). From [BBN2, Theorem 3.14] and [BBN1, Proposition 4.1] it follows that the given condition that E_G is polystable implies that E'_G is polystable. Consequently, the G-bundle E'_G admits a unitary holomorphic Γ-connection [BBN1, Proposition 4.7]; recall that a holomorphic connection on a bundle over a curve is the same as a flat complex connection. Now using Proposition 3.4 it follows immediately that E_G admits a flat unitary connection.

For the converse direction, assume that E_G admits a flat unitary connection. Therefore, the G-bundle E'_G over Y admits a unitary holomorphic connection (Proposition 3.4). Hence E'_G is polystable [RS, Theorem 1]. From this it follows that E_G is polystable (see [BBN2, Theorem 3.14]). This completes the proof of the theorem. ∎

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