Canad. Math. Bull. Vol. **28** (3), 1985

A FUNCTIONAL EQUATION FOR DEGREE TWO LOCAL FACTORS

BY

PAUL GÉRARDIN AND WEN-CH'ING WINNIE LI

Dedicated to the memory of Robert Arnold Smith

ABSTRACT. We show that the Fourier transforms of the admissible irreducible representations of the group GL_2 over a nonarchimedian local field F are characterized by a functional equation (MF). We also prove that the functions satisfying (MF) and having at most one pole are exactly the Fourier transforms of the irreducible representations of the quaternion group H over F. The Jacquet-Langlands correspondence between irreducible representations of H and discrete series of GL_2 then follows immediately from our criteria.

Introduction. In this paper the base field F is a nonarchimedean local field with q elements in its residue field. Once and for all, fix a character ψ of the additive group F which is trivial on the group $\mathbb{O}$ of integers but not on any larger $\mathbb{O}$-module. In the group $\mathcal{A}(F^\times)$ of characters of the multiplicative group $F^\times$ of F, the subgroup of unramified characters is identified with $\mathbb{C}^\times$ by $Z \longmapsto (t \longmapsto Z^{\operatorname{ord} t})$, where $\operatorname{ord}: F^\times \to \mathbb{Z}$ is the order in the usual absolute value $|t| = q^{-\operatorname{ord} t}$. Put on $\mathcal{A}(F^\times)$ the analytic structure so that the connected components are the cosets of $\mathbb{C}^\times$ and each connected component $\chi\mathbb{C}^\times$ is endowed with the usual analytic structure on $\mathbb{C}^\times$. We complete each component $\chi\mathbb{C}^\times$ by adding the origin $\chi 0$. Clearly the components of $\mathcal{A}(F^\times)$ are parametrized by the discrete group $\hat{\mathbb{O}}^\times$ of characters of the group $\mathbb{O}^\times$ of units. A function on $\mathcal{A}(F^\times)$ is said to be *rational* (resp. *homogeneous*) if it is a rational function (resp. a monomial) on each component $\chi\mathbb{C}^\times$; in particular, it has a finite order at the origin $\chi 0$. For a ramified $\chi \in \mathcal{A}(F^\times)$ denote by $f(\chi)$ (the exponent of) its conductor; set $f(\chi) = 1$ for χ unramified.

Recall that the gamma function Γ is a rational function on $\mathcal{A}(F^\times)$ defined by the principal value of the integral

$$\Gamma(\chi) = \int_{F^\times} \chi(t)\psi(t)\, d^\times t$$

as well as the analytic continuation on the component of unramified characters. Here $d^\times t$ is the Haar measure on $F^\times$ with $\operatorname{vol}(\mathbb{O}^\times) = 1$. More precisely,

---

Received by the editors August 30, 1984.

Research supported in part by N.S.F. grant no. DMS-8404083.

AMS Subject Classification: 12B27, 12B30.

https://doi.org/10.4153/CMB-1985-043-0 Published online by Cambridge University Press

$$\Gamma(Z) = (1 - q^{-1})^{-1} \frac{1 - q^{-1}Z^{-1}}{1 - Z} \text{ on the identity component}$$

and

$$\Gamma(\chi Z) = Z^{-f(\chi)}\Gamma(\chi)(\neq 0) \text{ for } \chi \text{ ramified and } Z \in \mathbb{C}^{\times}.$$

Introduce also the modified function

$$\gamma^F(\chi) = (1 - q^{-1})\Gamma(\chi q^{-1/2}),$$

then it satisfies the functional equation

$$\gamma^F(\chi)\gamma^F(\chi^{-1}) = \chi(-1).$$

The Fourier transform γ_π of an admissible irreducible representation π of the group $G = \mathrm{GL}_2(F)$ is the scalar in the following functional equation in $\chi \in \mathcal{A}(F^\times)$ of operators: for any compactly supported locally constant function f on $M_2(F)$,

$$\int_G \pi(x)\chi(\det x)\hat{f}(x)\, d^*x = \gamma_\pi(\chi) \int_G \pi^{-1}(x)\chi^{-1}(\det x) f(x)\, d^*x,$$

where $d^*x = |\det x|^{-1}\, dx$ with dx being the Haar measure on $M_2(F)$ such that $M_2(\mathcal{O})$ has volume 1 and $\hat{f}$ is the Fourier transform of f with respect to $\psi \circ \mathrm{tr}$. See [3], Theorem 3.3, for more details. It was proved by Jacquet and Langlands in [4], Theorem 2.18, Propositions 3.5, 3.6, and p. 84, that if π is a principal series $\pi(\mu, \nu)$ for $\mu, \nu \in \mathcal{A}(F^\times)$ (possibly one-dimensional) or a special representation $\sigma(\mu, \nu)$ for $\mu\nu^{-1} = q^{\pm 1}$ (with the notations of [4], p. 103–104), then

$$\gamma_\pi(\chi) = \gamma^F(\chi\mu)\gamma^F(\chi\nu);$$

otherwise, that is, when π is supercuspidal, γ_π is homogeneous on $\mathcal{A}(F^\times)$. In case π is infinite-dimensional, the Fourier transform γ_π also arises from the action of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in the Kirillov model of π, and it determines the class of π. A criterion for γ_π was given in [6], Theorem E, as a rational function γ on $\mathcal{A}(F^\times)$ satisfying the following complement and multiplicative formulae for the same character ω. (In fact, ω is the central character of the representation.)

Complement Formula. There is a character ω of $F^\times$ such that

$$\gamma(\chi)\gamma(\chi^{-1}\omega^{-1}) = \omega(-1).$$

In order to describe the multiplicative formula, we need some notation. On each component $\chi\mathbb{C}^\times$ of $\mathcal{A}(F^\times)$, choose a simple closed rectifiable curve C_χ, positively oriented around the origin $\chi 0$, such that the poles of γ on this component lie outside C_χ. Denote by C_m the union of these C_χ's with $f(\chi) \leq m$. Choose the differential form $d\chi$ on $\mathcal{A}(F^\times)$ to be $(2\pi i)^{-1}Z^{-1}dZ$ at each point χZ.

(MF) Multiplicative Formula. There is a character ω of F^* such that, given any component of $\mathcal{A}(F^\times) \times \mathcal{A}(F^\times)$, we have, for m sufficiently large,

$$\oint_{C_m} \Gamma(\alpha\chi^{-1})\Gamma(\beta\chi^{-1})\gamma(\chi)\,d\chi = (\alpha\beta\omega)(-1)\Gamma(\alpha^{-1}\beta^{-1}\omega^{-1})\gamma(\alpha)\gamma(\beta)$$

$$+\begin{cases} 0 & \text{if } \alpha\beta\omega \text{ is ramified,} \\ \omega(-1)\dfrac{Z^{-m}}{1-Z} & \text{if } \alpha\beta\omega = Z \text{ is unramified,} \end{cases}$$

for all points (α, β) on the component with α inside C_α and β inside C_β. Here $\oint_{C_m} g(\chi)d\chi$ means the sum of $1/(2\pi i)\oint_{C_\chi} g(\chi Z)\, Z^{-1}\, dZ$ as C_χ runs through all components of C_m.

Evaluating the above contour integral by Cauchy's residue theorem and noting that $\Gamma(\alpha\chi^{-1})$ has residue 1 at $\chi = \alpha$, we can reformulate (MF) as follows:

(ALTERNATE MF) There is a character ω of F^* such that, given any component of $\mathcal{A}(F^\times) \times \mathcal{A}(F^\times)$, we have, for m sufficiently large and (α, β) in the component,

$$(\alpha\beta\omega)(-1)\Gamma(\alpha^{-1}\beta^{-1}\omega^{-1})\gamma(\alpha)\gamma(\beta) - \Gamma(\beta\alpha^{-1})\gamma(\alpha) - \Gamma(\alpha\beta^{-1})\gamma(\beta)$$

$$= \sum_{f(\chi)<m} \mathrm{Res}_{\chi 0}\Gamma(\alpha\chi^{-1})\Gamma(\beta\chi^{-1})\gamma(\chi)\,d\chi$$

$$+\begin{cases} 0 & \text{if } \alpha\beta\omega \text{ is ramified,} \\ \omega(-1)\dfrac{Z^{-m+1}}{Z-1} & \text{if } \alpha\beta\omega = Z \text{ is unramified.} \end{cases}$$

Here $\Sigma_{f(\chi)<m}$ means that the sum over components of characters with conductor $< m$.

REMARK. As a result of Theorem 4 and Lemma 3 below, the last term in the above formula can be written as $\Sigma_{n\geq m}\ \Sigma_{f(\chi)=n}\ \mathrm{Res}_{\chi 0}\Gamma(\alpha\chi^{-1})\Gamma(\beta\chi^{-1})\gamma(\chi)\,d\chi$ if $\alpha\beta\omega$ is ramified or $\alpha\beta\omega = Z$ with $|Z| > 1$. In this case, the identity reads

$$(\alpha\beta\omega)(-1)\Gamma(\alpha^{-1}\beta^{-1}\omega^{-1})\gamma(\alpha)\gamma(\beta) - \Gamma(\beta\alpha^{-1})\gamma(\alpha) - \Gamma(\alpha\beta^{-1})\gamma(\beta)$$

$$= \sum_{\hat{\mathcal{O}}^\times} \mathrm{Res}_{\chi 0}\Gamma(\alpha\chi^{-1})\Gamma(\beta\chi^{-1})\gamma(\chi)\,d\chi,$$

where $\Sigma_{\hat{\mathcal{O}}^\times}$ means principal value, that is, $\Sigma_{n\geq 1}\ \Sigma_{f(\chi)=n}$.

The purpose of this paper is to study properties of a rational function on $\mathcal{A}(F^\times)$ satisfying the multiplicative formula. We shall show

THEOREM 1. *The multiplicative formula implies the complement formula with the same* ω.

An immediate consequence of this theorem is the following improvement of the criterion for Fourier transforms of representations of G given in Theorem E of [6].

THEOREM G. *Let* γ *be a rational function on* $\mathcal{A}(F^\times)$. *Then* $\gamma = \gamma_\pi$ *for some admissible irreducible representation* π *of* G *if and only if it satisfies the multiplicative formula.*

Theorem G asserts that the rational functions satisfying the multiplicative formula are exactly the Fourier transforms of representations of G. In particular, $\gamma(\chi) = \gamma^F(\chi\mu)\gamma^F(\chi\nu)$ satisfies (MF) with $\omega = \mu\nu$ as discussed earlier. We shall give a direct proof of this fact in Section 3.

THEOREM 2. *For any characters μ, ν of $F^\times$, the function*

$$\gamma(\chi) = \gamma^F(\chi\mu)\gamma^F(\chi\nu), \qquad \chi \in \mathcal{A}(F^\times),$$

satisfies the multiplicative formula with $\omega = \mu\nu$.

We also prove the following statements without using representation theory.

THEOREM 3. *The inhomogeneous rational functions on $\mathcal{A}(F^\times)$ satisfying the multiplicative formula are exactly those given by Theorem 2.*

PROPOSITION 1. *Let γ be a rational function satisfying* (MF) *for the character ω. Denote by c the order at zero of γ on the identity component. Then $c \leq -1 - f(\omega)$.*

THEOREM 4. *Let γ and c be as in Proposition 1. Then for all characters $\chi \in \mathcal{A}(F^\times)$ with conductor $\geq -c$, we have*

(DEEP TWIST) $$\gamma(\chi) = \gamma^F(\chi)\gamma^F(\chi\omega).$$

REMARK. Theorem 4 together with Lemma 2.2 of [5] implies that the character ω in the multiplicative formula is determined by γ. Another proof of this fact will be given in Section 5, Corollary 2.

Let H be the multiplicative group of a quaternion algebra over F (unique up to isomorphism). For an irreducible representation σ of H, its Fourier transform γ_σ is given by, for $\chi \in \mathcal{A}(F^\times)$,

$$\gamma_\sigma(\chi)Id = -\int_H \sigma(x)\chi(Nx)\psi(Tx)d^*x,$$

where N(resp. T) is the reduced norm (resp. trace), and $d^*x = |Nx|^{-1}dx$ with dx being the Haar measure of the quaternion algebra assigning volume q^{-1} to the ring of integers. If σ has degree >1, then γ_σ is homogeneous; if σ has degree 1, then $\sigma = \lambda \circ N$ for some character λ of $F^\times$, in which case

$$\gamma_\sigma(\chi) = \gamma^F(\chi\lambda q^{1/2})\gamma^F(\chi\lambda q^{-1/2})$$

has a simple pole at $\chi = \lambda^{-1}q$ and is holomorphic elsewhere. The authors showed in [2], Corollary 2.2.9, that the Fourier transform γ_σ determines the reduced character of σ and hence the class of the representation. Moreover, γ_σ satisfies the multiplicative formula, the deep twist condition (Theorem 4), and the

(CUSPIDAL CONDITION) There exists a character ω of $F^\times$ such that for any $(\nu, \tau) \in F^\times \times F$ with $\tau = a + b$ and $\nu = ab$ for two distinct elements a, b in $F^\times$, we have

$$\int_{\mathfrak{P}^{-n}} \left(\oint_{C_m} \gamma(\chi)\omega^{-1}\chi^{-2}(t)\chi(\nu)^{-1}d\chi\right)\psi(-t\tau)|t|^{-1}d^\times t = 0$$

for m, n large enough, where $\mathcal{P}^{-n}$ is the set of elements in F with order $\geq -n$.

It was also shown in [2], Main Theorem 3.1.1, that these three conditions (with the same ω) characterize the Fourier transforms of representations of H. In section 6 we prove

THEOREM 5. *Let γ be a rational function on $\mathcal{A}(F^\times)$ satisfying the multiplicative formula* (MF). *Then it satisfies the cuspidal condition for the same ω if and only if it has at most one pole.*

As a consequence of Theorems 4 and 5, we have the following strengthened criterion for Fourier transforms of representations of H.

THEOREM H. *Let γ be a rational function on $\mathcal{A}(F^\times)$. Then $\gamma = \gamma_\sigma$ for some irreducible representation σ of H if and only if γ satisfies the multiplicative formula and has at most one simple pole on $\mathcal{A}(F^\times)$.*

Observe that the Fourier transform γ_π of an infinite-dimensional admissible irreducible representation π of G has two simple poles or one double pole if π is a principal series, it has one simple pole if π is a special representation and is holomorphic if π is supercuspidal. Thus γ_π has at most one simple pole on $\mathcal{A}(F^\times)$ if and only if π is a discrete series. Theorems G and H then give immediately the following local correspondence of Jacquet and Langlands (Theorem 15.1 of [4]):

THEOREM HG. *There is a bijection between representations of* H *and discrete series representations of* G *such that the corresponding representations have the same Fourier transform.*

This result was also proved in [2], Theorem 4.3.5, using orbital integrals. The approach in this paper avoids completely analysis in representation theory.

1. **Three lemmata**. We begin by a simple observation.

LEMMA 1. *Let $\lambda \in \mathcal{A}(F^\times)$. If γ is a rational function on $\mathcal{A}(F^\times)$ satisfying the multiplicative formula for the character ω, then the same is true for the function $\chi \mapsto \gamma(\lambda\chi)$ with $\lambda^2\omega$ replacing ω.*

For the second lemma, we take a rational function γ on $\mathcal{A}(F^\times)$ and set, for $\alpha, \beta \in \mathcal{A}(F^\times)$ and $m > 0$,

$$R_{m,\gamma}(\alpha, \beta) = \sum_{f(\chi)<m} \operatorname{Res}_{\chi 0} \Gamma(\alpha\chi^{-1})\Gamma(\beta\chi^{-1})\gamma(\chi)\,d\chi.$$

LEMMA 2. *Viewed as a function in two variables A and B in $\mathbb{C}^\times$, $R_{m,\gamma}(\alpha A, \beta B)$ is a finite linear combination of $A^{-a}B^{-b}$ with positive integers a and b.*

PROOF. It suffices to show that on any connected component $\chi\mathbb{C}^\times$ of $\mathcal{A}(F^\times)$, the function in A, B given by

$$\operatorname{Res}_{\chi 0} \Gamma(\alpha A\chi^{-1})\Gamma(\beta B\chi^{-1})\gamma(\chi)\,d\chi$$

is as described. If α is not in $\chi\mathbb{C}^\times$, then $\Gamma(\alpha A\chi^{-1}) = \Gamma(\alpha\chi^{-1})A^{-f(\alpha\chi^{-1})}$; if α is in $\chi\mathbb{C}^\times$, then near $T = 0$, $\Gamma(\alpha A\chi^{-1}T^{-1})$ is a series in positive powers of $A^{-1}T$. Hence near $T = 0$ we have $\Gamma(\alpha A\chi^{-1}T^{-1})\Gamma(\beta B\chi^{-1}T^{-1}) = \Sigma_{a,b>0} A^{-a}B^{-b} c_{a,b}(\chi)T^{a+b}$ for some constants $c_{a,b}(\chi)$. As $\gamma(\chi T)$ is rational in T, the residue at $T = 0$ of $\Gamma(\alpha A\chi^{-1}T^{-1})\Gamma(\beta B\chi^{-1}T^{-1})\gamma(\chi T)T^{-1}$ is a finite linear combination of $A^{-a}B^{-b}$ with $a, b > 0$. This proves the lemma.

The following expression of (Alternate MF) at (α, β) will be of frequent use:

$$\text{(MF)}' \quad \begin{aligned} &\Gamma(\beta\alpha^{-1})\gamma(\alpha) + \Gamma(\alpha\beta^{-1})\gamma(\beta) + R_{m,\gamma}(\alpha,\beta) \\ &\quad = (\alpha\beta\omega)(-1)\Gamma(\alpha^{-1}\beta^{-1}\omega^{-1})\gamma(\alpha)\gamma(\beta) \\ &\qquad + \begin{cases} 0 & \text{if } \alpha\beta\omega \text{ is ramified,} \\ \omega(-1)\dfrac{Z^{-m+1}}{1-Z} & \text{if } \alpha\beta\omega = Z \text{ is unramified.} \end{cases} \end{aligned}$$

The next lemma gives an example of computation of residues.

LEMMA 3. *Let α, β, μ, ν be characters of $F^\times$. If m is an integer satisfying $m \geq 2f(\alpha), 2f(\beta), 2f(\mu), 2f(\nu)$, then*

$$\sum_{f(\chi)=m} \operatorname{Res}_{\chi 0}\Gamma(\alpha\chi^{-1})\Gamma(\beta\chi^{-1})\gamma^F(\chi\mu)\gamma^F(\chi\nu)$$

$$= \begin{cases} 0 & \text{if } \alpha\beta\mu\nu \textit{ is ramified,} \\ (\mu\nu)(-1)Z^{-m} & \textit{if } \alpha\beta\mu\nu = Z \textit{ is unramified,} \end{cases}$$

where the summation is over all components of characters of conductor equal to m.

PROOF. This is Lemma 3.2.8 of [2] if we observe (cf. [1], 1.4) that $\gamma^F(\chi\mu) \times \gamma^F(\chi\nu) = \gamma^F(\chi)\gamma^F(\chi\omega)$ with $\omega = \mu\nu$, using the hypothesis $m \geq 2f(\mu), 2f(\nu)$.

2. **Complement formula**. Let γ be a rational function on $\mathscr{A}(F^\times)$ satisfying the multiplicative formula (MF) for the character ω. We prove Theorem 1, that is, for all α in $\mathscr{A}(F^\times)$,

$$\gamma(\alpha)\gamma(\alpha^{-1}\omega^{-1}) = \omega(-1).$$

We may assume that α and $\alpha^{-1}\omega^{-1}$ are not poles of γ. Then (MF)$'$ at $(\alpha A, \alpha^{-1}\omega^{-1})$ with m large reads

$$\Gamma(A^{-1})\gamma(\alpha A)\gamma(\alpha^{-1}\omega^{-1}) - \Gamma(\omega^{-1}\alpha^{-2}A^{-1})\gamma(\alpha A) - \Gamma(\omega\alpha^2 A)\gamma(\alpha^{-1}\omega^{-1})$$

$$= R_{m,\gamma}(\alpha A, \alpha^{-1}\omega^{-1}) + \frac{A^{-m+1}}{A-1}\omega(-1).$$

By Lemma 2, the right hand side has a simple pole at $A = 1$ with residue $\omega(-1)$. On the left hand side, the first term has a simple pole at $A = 1$ with residue $\gamma(\alpha)\gamma(\alpha^{-1}\omega^{-1})$, while the sum of the remaining two terms is holomorphic at $A = 1$

https://doi.org/10.4153/CMB-1985-043-0 Published online by Cambridge University Press

(which is obvious when $\alpha^2\omega \neq 1$ and in case $\alpha^2\omega = 1$ the residues at 1 of $\Gamma(A^{-1})\gamma(\alpha A)$ and $\Gamma(A)\gamma(\alpha)$ are $\gamma(\alpha)$ and $-\gamma(\alpha)$, respectively). This proves the theorem.

COROLLARY 1. *A rational function satisfying the multiplicative formula does not vanish identically on any component of* $\mathcal{A}(F^\times)$.

3. **The multiplicative formula for inhomogeneous γ.** In this section we show that for any characters μ, ν of $F^\times$, the function $\gamma(\chi) = \gamma^F(\chi\mu)\gamma^F(\chi\nu)$ satisfies the multiplicative formula (MF) for $\omega = \mu\nu$. We take the contour integral form. By analyticity in α, β, μ, ν of both sides, it is sufficient to prove it for $(\alpha, \beta, \mu, \nu)$ in an open set meeting every component of $\mathcal{A}(F^\times)^4$. For each character χ of $F^\times$, we define $|\chi|$ to be the positive number given by $|\chi(t)| = |\chi|^{\mathrm{ord} t}$, $t \in F^\times$. For $|\chi| < 1$, the integral $\int_{\mathcal{P}^{-n}} \chi(t)\psi(t)d^\times t$ is absolutely convergent, and is equal to $\Gamma(\chi)$ provided $n \geq f(\chi)$. If $|\alpha\beta\omega| > 1$, we can pass to the limit $m \to \infty$ in (MF) and, by Lemma 3, our γ will satisfy (MF) for $\omega = \mu\nu$ if and only if

$$(3.1)\qquad \oint_C \Gamma(\alpha\chi^{-1})\Gamma(\beta\chi^{-1})\gamma(\chi)\,d\chi = (\alpha\beta\omega)(-1)\Gamma(\alpha^{-1}\beta^{-1}\omega^{-1})\gamma(\alpha)\gamma(\beta),$$

where C is the union of C_m.

We introduce the algebra $E = F \times F$. We identify F with the diagonal of E. We call $\bar{\ }$, N, T the involution, norm, trace on E: if $x = (u, v)$, then $\bar{x} = (v, u)$, $Nx = x\bar{x} = uv$, $Tx = x + \bar{x} = u + v$; the kernel E_1 of the norm $N: E^\times \to F^\times$ is the set of (t, t^{-1}) for $t \in F^\times$. We have Haar measures dx and $d^\times x$ on E and $E^\times$ given by $du\,dv$ and $d^\times u\,d^\times v$, respectively, where du and dv assign volume 1 to $\mathcal{O}$; if $d^*u = |u|^{-1/2}du$, we write also d^*x for $d^*u\,d^*v$, that is, $d^*x = |Nx|^{-1/2}dx$. For any integer a, we write $\mathcal{P}_E^a$ for $\mathcal{P}^a \times \mathcal{P}^a$. Observe that each $t \in F^\times$ is the norm of an $x = (u, v) \in E^\times$ with $[\mathrm{ord} t/2] \leq \mathrm{ord}\, u$ and $\mathrm{ord}\, v \leq \lceil \mathrm{ord} t/2 \rceil$. The characters of $E^\times$ are the tensor products of two characters of $F^\times$. For $\alpha \in \mathcal{A}(F^\times)$ with $|\alpha\mu|$ and $|\alpha\nu| < q^{1/2}$, our γ function at α can be written as $\int_{\mathcal{P}_E^{-a}} \theta(x)\alpha(Nx)\psi(Tx)\,d^*x$, $a \geq f(\alpha\mu), f(\alpha\nu)$, where $\theta = \mu \otimes \nu$. (For simplicity, we wrote only $\mathcal{P}_E^{-a}$ instead of $\mathcal{P}_E^{-a} \cap E^\times$.) This integral is the Fourier transform at α of the character θ of $E^\times$. It factors through $\int_{w \in E_1, wx \in \mathcal{P}_E^{-a}} \theta(wx)\psi \circ T(wx)\,d^\times w = \theta(x) \int_{-a-\mathrm{ord}\, u \leq \mathrm{ord}\, t \leq a + \mathrm{ord}\, v} (\mu\nu^{-1})(t)\psi(tu + t^{-1}v)\,d^\times t$ if $x = (u, v)$; this is also equal to

$$(3.2)\qquad \mu(Nx) \int_{-a-\mathrm{ord} Nx \leq \mathrm{ord}\, t \leq a} (\mu\nu^{-1})(t)\psi(tNx + t^{-1})\,d^\times t,$$

which is stable for $a \geq f(\mu\nu^{-1})$ and $-\mathrm{ord} Nx$. Denote by $J_\theta(Nx)$ the stabilized value.

Let $r > 1$. We take α, β, μ, ν satisfying the inequalities $|\alpha|$ and $|\beta| < r < |\mu|^{-1}q^{1/2}$ and $|\nu|^{-1}q^{1/2}$, $|\alpha\beta\mu\nu| > 1$. Let C be the curve in $\mathcal{A}(F^\times)$ given in each component by the circle of radius r centered at the origin. Then each of the involved Γ-functions in (MF) is represented by its integral form for $\chi \in C$; moreover the poles $\mu^{-1}q^{1/2}$ and $\nu^{-1}q^{1/2}$ of γ are outside C. Take $m \geq f(\alpha), f(\beta), f(\mu), f(\nu)$. Then, for $a, b, c \geq m$ we have, by Fubini theorem,

https://doi.org/10.4153/CMB-1985-043-0 Published online by Cambridge University Press

$$\oint_{C_m} \Gamma(\alpha\chi^{-1})\Gamma(\beta\chi^{-1})\gamma(\chi)\,d\chi = \int_{\mathcal{P}^{-a}\times\mathcal{P}^{-b}\times\mathcal{P}^{-c}} (\alpha\psi)(u)(\beta\psi)(v)(\theta\psi\circ T)(z) \times \left(\oint \chi\left(\frac{Nz}{uv}\right)d\chi\right) d^\times u\, d^\times v\, d^\times z$$

The inner integral against χ is 0 unless $Nz \in uv(1+\mathcal{P}^m)$, in which case it is $\mathrm{Card}\hat{\mathcal{O}}_m^\times$; writing $v = u^{-1}(Nz)(1+w)$ with $w \in \mathcal{P}^m$, we have $(\beta\psi)(v) = (\beta\psi)(u^{-1}Nz)\psi(u^{-1}wNz)$. The integral $\int_{\mathcal{P}^m}\psi(u^{-1}wNz)\,dw$ is 0 unless $Nz \in u\mathcal{P}^{-m}$, in which case it is $\mathrm{meas}(1+\mathcal{P}^m) = (\mathrm{Card}\hat{\mathcal{O}}_m^\times)^{-1}$; so, for a and $c \geq m \geq f(\alpha), f(\beta), f(\mu), f(\nu)$, we have

$$\oint_{C_m} \Gamma(\alpha\chi^{-1})\Gamma(\beta\chi^{-1})\gamma(\chi)\,d\chi = \int_{u\in\mathcal{P}^{-a}} (\alpha\psi)(u) \times \left(\int_{z\in\mathcal{P}_E^{-c},\, Nz\in\mathcal{P}^{-m}} (\beta\psi)(u^{-1}Nz)(\theta\psi\circ T)(z)\,d^*z\right) d^*u.$$

For each nonzero $u \in \mathcal{P}^{-a}$, we choose $x \in E$ with norm u and in the integral in z we make the change of variable $z = xy$; we then factor the integral in y through the norm map to make appear $J_\theta(N(xy))$ with J_θ defined by (3.2); this leads to

$$\oint_{C_m} \Gamma(\alpha\chi^{-1})\Gamma(\beta\chi^{-1})\gamma(\chi)\,d\chi = \int_{\mathcal{P}^{-a}} (\alpha\psi)(u)\left(\int_{\mathcal{P}^{-m}}(\beta\psi)(v)\,J_\theta(uv)\,d^*v\right)d^*u,$$

which is stable as $a \to \infty$. By Lemma 3, we know that this sequence in m is convergent (we have assumed $|\alpha\beta\mu\nu| > 1$), so, finally, we have:

$$(3.3)\qquad \oint_{C} \Gamma(\alpha\chi^{-1})\Gamma(\beta\chi^{-1})\gamma(\chi)\,d\chi = \lim_{m\to\infty}\lim_{a\to\infty} \int_{\mathcal{P}^{-a}} (\alpha\psi)(u) \times \left(\int_{\mathcal{P}^{-m}} (\beta\psi)(v)\,J_\theta(uv)\,d^*v\right) d^*u.$$

On the other hand, for $e \geq f(\alpha\mu)$ and $f(\alpha\nu)$, $f \geq f(\beta\mu)$ and $f(\beta\nu)$, $d \geq f(\alpha\beta\omega)$, we have an absolutely convergent triple integral:

$$(\alpha\beta\omega)(-1)\Gamma(\alpha^{-1}\beta^{-1}\omega^{-1})\gamma(\alpha)\gamma(\beta) = \int_{t\in\mathcal{P}^{-d},\, x\in\mathcal{P}_E^{-e},\, y\in\mathcal{P}_E^{-f}} (\alpha\beta\omega)(t)^{-1}\alpha(Nx) \times \beta(Ny)\theta(x)\theta(y)\psi(Tx+Ty-t)\,d^*t\,d^*x\,d^*y$$

which is also the limit as s goes to infinity of the truncated integral with $d \leq \mathrm{ord}\, t < s$. For each $t \in \mathcal{P}^{-d}$, $t \notin \mathcal{P}^s$, choose $z \in \mathcal{P}_E^{-D}$ $D = \lceil d/2 \rceil$, with no component in $\mathcal{P}^S$, $S = \lceil s/2 \rceil$, such that $Nz = t$. Then under the change of variables $x \to zx$, $y \to \bar{z}y$, the integrand becomes

$$(3.4)\qquad (\alpha\psi)(Nx)(\beta\psi)(Ny)(\theta\psi\circ T)(xy)\bar{\psi}\circ N(z-\bar{x}-y)$$

and the measure becomes $d^*x\,d^*y\,|Nz|\,d^\times Nz$; by stability in e and f, we may integrate over the set $x \in \mathcal{P}_E^{-e-s}$, $y \in \mathcal{P}_E^{-f-s}$, $Nz \in \mathcal{P}^{-d}$, $Nz \notin \mathcal{P}^s$. The integral in y gives a function of $(x, z) \in E^\times \times E^\times$ which is invariant by the action of $w \in E_1 : (x, z) \mapsto (xw, \bar{w}z)$. We can write $(E^\times \times E^\times)/E_1$ either as $E^\times \times F^\times$ by $(x, z) \mapsto (x, Nz)$ or as $F^\times \times E^\times$ by $(x, z) \mapsto (Nx, z)$: we see that the measures $d^*x\, |Nz| d^\times z$ and $|Nx|^{1/2} d^\times Nz\, dz$ correspond. So we rewrite our integral with a domain in (Nx, y, z) where $x \in \mathcal{P}_E^{-e-s}$, $y \in \mathcal{P}_E^{-f-s}$, $z \in \mathcal{P}_E^{-D} Nz \notin \mathcal{P}^s$. This integral with the condition $Nz \notin \mathcal{P}^s$ replaced by $Nz \in \mathcal{P}^s$ is dominated by

$$\int_{N\mathcal{P}_E^{-e-s}} \left(\frac{r}{q^{1/2}}\right)^{\operatorname{ord} u} d^\times u \int_{\mathcal{P}_E^{-f-s}} r^{\operatorname{ord} Ny}\, d^*y \int_{\mathcal{P}_E^{-D}, Nz \in \mathcal{P}^s} dz.$$

The last integral can be computed explicitly: it is $((2D + s)(1 - q^{-1}) + 1)q^{-s}$; the contribution of the two first is, up to a constant, $(r^2/q)^{-s}$, so the product is $O(sr^{-s})$ when s goes to infinity. Hence, the condition $r > 1$ allows us to consider only the integral of (3.4) on the set of (u, y, z) such that $u = Nx$, $x \in \mathcal{P}_E^{-e-s}$, $y \in \mathcal{P}_E^{-f-s}$, $z \in \mathcal{P}_E^{-D}$. The integral in z gives the term $\int_{\mathcal{P}_E^{-D}} \bar{\psi} \circ N(z - \bar{x} - y)dz$. From the definition $E = F \times F$, we see that

$$\int_{\mathcal{P}_E^{-D}} \psi \circ N(z + x)\, dz = \begin{cases} 1 \text{ if } x \in \mathcal{P}_E^{-D}, \\ 0 \text{ otherwise.} \end{cases} \tag{3.5}$$

Thus, this integration in z picks up the condition $\bar{x} + y \in \mathcal{P}_E^{-D}$. Now, the integral in y is, for $x \in \mathcal{P}_E^{-e-S}$:

$$\int_{\mathcal{P}_E^{-f-S}, \bar{x}+y \in \mathcal{P}_E^{-D}} (\beta\psi)(Ny)(\theta\psi \circ T)(xy)\, d^*y.$$

For D large enough, the condition $\bar{x} + y \in \mathcal{P}_E^{-D}$ is satisfied; for f large enough we make appear $J_\theta(N(xy))$ as above and the integral of (3.4) is equal to

$$\int_{N\mathcal{P}_E^{-e-S}} (\alpha\psi)(u) \int_{N\mathcal{P}_E^{-f-S}} (\alpha\psi)(v)\, J_\theta(uv)\, d^*v\, d^*u.$$

If we take now $e \geqslant f$ and let s go to infinity, we get

$$(\alpha\beta\omega)(-1)\Gamma(\alpha^{-1}\beta^{-1}\omega^{-1})\gamma(\alpha)\gamma(\beta) = \lim_{m\to\infty} \lim_{n\to\infty} \int_{\mathcal{P}^{-a}} (\alpha\psi)(u) \times \left(\int_{\mathcal{P}^{-m}} (\beta\psi)(v)\, J_\theta(uv) d^*v\right) d^*u.$$

Comparing with (3.3), we have the multiplicative formula (MF). This shows that the function of χ given by $\gamma^F(\chi\mu)\gamma^F(\chi\nu)$ satisfied it for $\omega = \mu\nu$.

This proof is similar to that of Theorem 3.3.1 of [2] for the Fourier transform of an irreducible representation of the quaternion group; there, the decisive relation corresponding to (3.5) is $\int_H \psi \circ N(z)\, dz = -1$. We could also have taken a similar viewpoint, starting with a function θ on $E^\times$ of type ω under $F^\times$ and fixed by some $1 + \mathcal{P}_E^a$; then its Fourier transform $\gamma(\chi) = \int_{E^\times} \theta(x)\chi(Nx)\psi(Tx)\, d^*x$ satisfies (MF) if θ is a character of $E^\times$.

4. **Determining inhomogeneous γ.** Let γ be a rational function on $\mathscr{A}(F^\times)$ satisfying the multiplicative formula for the character ω. We prove first

PROPOSITION 2. *Let α be a character of $F^\times$. Then on the connected component of α, either $\gamma(\alpha A)$ is a monomial in A of degree at most $-1 - f(\alpha^2\omega)$ or $\gamma(\alpha A) = \gamma^F(\alpha\mu A)\gamma^F(\alpha\nu A)$ for some characters μ, ν of $F^\times$* with $\omega = \mu\nu$.

PROOF. In view of Lemma 1, we may assume $\alpha = 1$. We examine the multiplicative formula (MF) at (A, B) for m sufficiently large. The identity (MF)$'$ reads

$$\Gamma(BA^{-1})\gamma(A) + \Gamma(AB^{-1})\gamma(B) + R_{m,\gamma}(A,B) \tag{4.1}$$

$$= \omega(-1)\Gamma(\omega^{-1}A^{-1}B^{-1})\gamma(A)\gamma(B) + \begin{cases} 0 & \text{if } \omega \text{ is ramified,} \\ \omega(-1)\dfrac{(ZAB)^{-m+1}}{1 - ZAB} & \text{if } \omega = Z. \end{cases}$$

Here $R_{m,\gamma}(A,B)$ is a polynomial in A^{-1} and B^{-1} with degrees in A and B both $\leqslant -1$ by Lemma 2. As $\gamma(A)$ is not identically zero by Corollary 1, for A with small absolute value, we can express $\gamma(A)$ in series: $\gamma(A) = \Sigma_{n\geqslant c}\gamma_n A^n$, where c is the order of $\gamma(A)$ at $A = 0$. Hence we shall regard (4.1) as an equation in the two variables A and B. We distinguish two cases according to the ramification of ω.

CASE 1. ω is ramified. For brevity we write f for $f(\omega)$. Noticing

$$\Gamma(Z) = (1 - q^{-1})^{-1}\frac{1 - q^{-1}Z^{-1}}{1 - Z}$$

and

$$\Gamma(Z^{-1}) = (1 - q^{-1})^{-1}\frac{Z - q^{-1}Z^2}{Z - 1}$$

for $Z \in \mathbb{C}^\times$, we multiply both sides of (4.1) by $(1 - q^{-1})(1 - BA^{-1})$ to arrive at

$$\begin{aligned} (1 - q^{-1}AB^{-1})\gamma(A) - (BA^{-1} - q^{-1}B^2A^{-2})\gamma(B) \\ + \mathrm{R}_{m,\gamma}(A,B)(1 - BA^{-1})(1 - q^{-1}) = \gamma(A)\gamma(B)\Gamma(\omega^{-1})\omega(-1) \\ \times A^fB^f(1 - AB^{-1})(1 - q^{-1}) \end{aligned} \tag{4.2}$$

Observe that the monomials A^aB^b with $a \geqslant 0$ and $b \in \mathbb{Z}$ do not occur in the last two terms of the left hand side of (4.2), therefore we obtain information about $\gamma(A)$ by comparing the coefficients of these terms. The absence of A^aB^b for $a \geqslant 0$ and $b \geqslant 1$ on the left side of (4.2) yields, from the right side of (4.2),

$$\gamma_a\gamma_{b+1} = \gamma_{a+1}\gamma_b \quad \text{for} \quad a, b \geqslant -f,$$

which in turn implies

(i) There is a complex number M such that $\gamma_n = \gamma_{-f}M^{n+f}$ for $n \geqslant -f + 1$.
Equating the coefficients of A^0B^0 and AB^{-1} gives rise to

(ii) $\gamma_0 = (1 - q^{-1})\Gamma(\omega^{-1})\omega(-1)(\gamma_{-f}^2 - \gamma_{-f+1}\gamma_{-f-1})$
and
(iii) $\gamma_0 = -q(1 - q^{-1})\Gamma(\omega^{-1})\omega(-1)(\gamma_{-f+1}\gamma_{-f-1} - \gamma_{-f+2}\gamma_{-f-2})$
respectively. Finally, comparing the coefficients of A^aB^{c+f} for $a \geqslant 0$ leads to
(iv) If $c < -f - 1$, then $\gamma(A)$ is a finite series in A with degree $\leqslant -f - 1$.
Thus if $\gamma(A)$ is a finite series, then it has degree $\leqslant -f$ by (i) and hence $\leqslant -f - 1$ by (ii). If $\gamma(A)$ is an infinite series, then $c \geqslant -f - 1$ by (iv) (so that $\gamma_{-f-2} = 0$), and, by (i), $\gamma_n = \gamma_{-f}M^{n+f} \neq 0$ for $n \geqslant -f$. Equating (ii) and (iii) yields

$$\gamma_{-f-1} = (1 - q)^{-1}\gamma_{-f}M^{-1}.$$

We can then obtain the value of γ_{-f} in terms of M from (iii):

$$\gamma_{-f} = \omega(-1)M^f/\Gamma(\omega^{-1}).$$

Summarizing the above discussion, we have

$$\begin{aligned}\gamma(A) = \sum_{n\geqslant -f-1} \gamma_nA^n &= \omega(-1)\frac{\Gamma(\mu A)}{\Gamma(\omega^{-1}MA)}\\ &= \omega(-1)\frac{\gamma^F(MA)}{\gamma^F(\omega^{-1}\mu A)} = \gamma^F(\mu A)\gamma^F(\nu A),\end{aligned}$$

where $\mu = Mq^{1/2}$ and $\mu\nu = \omega$. Note that in this case γ has a zero on the identity component.

Similar conclusion holds for $\gamma(\omega^{-1}A)$. In particular, if $\gamma(A)$ is a finite series in A and is not a monomial, then it has zero(s) and no pole on the identity component. The complement formula $\gamma(A)\gamma(\omega^{-1}A^{-1}) = \omega(-1)$ then implies that $\gamma(\omega^{-1}A)$ has pole(s) and no zero on the component of ω^{-1}, a contradiction. This proves the proposition for ramified ω.

CASE 2. $\omega = Z$ is unramified. Multiply both sides of (4.1) by $(1 - BA^{-1})$ $(1 - ZAB)(1 - q^{-1})$ to get

$$\begin{aligned}(4.3)\quad (1 - q^{-1}AB^{-1})(1 - ZAB)\gamma(A) - (BA^{-1} - q^{-1}B^2A^{-2})(1 - ZAB)\gamma(B)\\ + R'(A,B)(1 - BA^{-1}) = \gamma(A)\gamma(B)\\ \times(-ZAB + q^{-1}Z^2A^2B^2)(1 - BA^{-1}),\end{aligned}$$

where $R'(A,B) = (1 - q^{-1})R_{m,\gamma}(A,B)(1 - ZAB) - \omega(-1)(1 - q^{-1})(ZAB)^{-m+1}$ is a finite series in A and B with degree in $A \leqslant 0$ and degree in $B \leqslant 0$. This time note that the monomial A^aB^b with $a \geqslant 1$ and $b \in \mathbb{Z}$ do not occur in the last two terms of the left hand side of (4.3). The vanishing of A^aB^b with $a \geqslant 1$, $b \geqslant 2$ on the left and hence right hand side of (4.3) yields

$$\text{(v)}\qquad \gamma_b(\gamma_{a+1} + q^{-1}Z\gamma_{a-1}) = \gamma_a(\gamma_{b+1} + q^{-1}Z\gamma_{b-1})\quad \text{for}\quad a,b \geqslant 0.$$

If $\gamma(A)$ is a finite series in A, then (v) implies that it has degree $\leqslant -1$. In fact, the degree is $\leqslant -2$ by checking the coefficient of AB. Next assume that $\gamma(A)$ is an infinite series. Then, from (v), there is a complex number N such that

(vi) $$\gamma_{n+1} + q^{-1}Z\gamma_{n-1} = N\gamma_n \quad \text{for} \quad n \geqslant 0.$$

The absence of A^aB^b terms with $a > 0$ and $b \leqslant -2$ from the left hand side of (4.3) implies $c \geqslant -2$. To determine $\gamma(A)$ we need to know γ_n for $n = -2, -1$ and 0. For this, choose any integer $a \geqslant 0$ such that $\gamma_a \neq 0$ and compare the coefficients of $A^{a+1}B^{-1}$, A^{a+1} and $A^{a+1}B$ in (4.3). Using $\gamma_{-3} = \gamma_{-4} = 0$ and (vi), we arrive at

$$q^{-1}\gamma_a = Z\gamma_{-2}\gamma_a, \qquad N\gamma_a = (ZN\gamma_{-2} - Z\gamma_{-1})\gamma_a,$$

and

$$\gamma_a = (-N\gamma_{-1} + q^{-1}Z\gamma_{-2} + \gamma_0)\gamma_a,$$

in other words,

$$\gamma_{-2} = q^{-1}Z^{-1}, \quad \gamma_{-1} = (q^{-1} - 1)NZ^{-1}$$

and

$$\gamma_0 = (1 - q^{-2}) + (q^{-1} - 1)N^2Z^{-1}$$

since $\gamma_a \neq 0$. Writing $N = M + q^{-1}ZM^{-1}$, we get

$$\gamma(A) = \sum_{n \geqslant -2} \gamma_n A^n = \frac{\Gamma(\mathrm{MA})}{\Gamma(Z^{-1}\mathrm{MA}^{-1})} = \gamma^F(\mu A)\gamma^F(\nu A),$$

where $\mu = Mq^{1/2}$ and $\mu\nu = \omega = Z$.

We conclude that either $\gamma(A) = \gamma^F(\mu A)\gamma^F(\nu A)$ for some characters μ, ν of $F^\times$ with $\mu\nu = \omega$, or $\gamma(A)$ is a monomial in A of degree $\leqslant -2$ by the same argument as in Case 1. This completes the proof of Proposition 2.

In the course of the proof, we also showed that $c = -1 - f(\omega)$ if $\gamma(A)$ is not a monomial. Combined with the monomial case, we have shown Proposition 1, that is, the order of $\gamma(A)$ at $A = 0$ is $\leqslant -1 - f(\omega)$.

Now we proceed to prove Theorem 3. We know from Theorem 2 that $\gamma(\chi) = \gamma^F(\mu\chi)\gamma^F(\nu\chi)$ satisfies (MF). It remains to show that these are the only inhomogeneous ones. Suppose that γ is a rational function on $\mathcal{A}(F^\times)$ satisfying (MF) for the character ω and γ is not a monomial on the component of $\alpha \in \mathcal{A}(F^\times)$. Then, by the proposition above, we have

(4.4) $$\gamma(\alpha A) = \gamma^F(\mu\alpha A)\gamma^F(\nu\alpha A),$$

where μ, $\nu \in \mathcal{A}(F^\times)$, $\mu\alpha = Z$ is unramified and $\mu\nu = \omega$. We want to prove

$$\gamma(\beta) = \gamma^F(\mu\beta)\gamma^F(\nu\beta) \quad \text{for} \quad \beta \in \mathcal{A}(F^\times).$$

The complement formula together with the functional equation satisfied by γ^F shows that the above equality holds for β in the component of $\alpha^{-1}\omega^{-1}$. So we may assume $\alpha\beta^{-1}$ and $\alpha\beta\omega$ both ramified. We may also assume that β is not a pole of γ. The identity (MF)$'$ at $(\alpha A, \beta)$ with m large gives

$$\Gamma(\beta\alpha^{-1})\gamma(\alpha A)A^{f(\beta\alpha^{-1})} + \Gamma(\alpha\beta^{-1})\gamma(\beta)A^{-f(\alpha^{-1}\beta)} + R_{m,\gamma}(\alpha A, \beta) \tag{4.5}$$
$$= \gamma(\alpha A)\gamma(\beta)\Gamma(\alpha^{-1}\beta^{-1}\omega^{-1})(\alpha\beta\omega)(-1)A^{f(\alpha\beta\omega)},$$

where $R_{m,\gamma}(\alpha A, \beta)$ is a finite series in A with degree $\leqslant -1$ by Lemma 2. As before, for A with small absolute value, write $\gamma(\alpha A) = \Sigma_{n \geqslant c(\alpha)} \gamma_n A^n$. Choosing an integer $n > c$ and $n > c - f(\alpha\beta\omega) + f(\beta\alpha^{-1})$ such that $\gamma_n \neq 0$ and comparing the coefficient of $A^{n+f(\alpha\beta\omega)}$ in (4.5), we find

$$\gamma(\beta) = (\alpha\beta\omega)(-1)\frac{\Gamma(\beta\alpha^{-1})}{\Gamma(\alpha^{-1}\beta^{-1}\omega^{-1})}\,\frac{\gamma_{n-f(\alpha\beta\omega)+f(\beta\alpha^{-1})}}{\gamma_n}.$$

If $\omega\alpha^2$ is unramified, then $f(\alpha\beta\omega) = f(\beta\alpha^{-1})$ and hence

$$\gamma(\beta) = (\alpha\beta\omega)(-1)\frac{\Gamma(\beta\alpha^{-1})}{\Gamma(\alpha^{-1}\beta^{-1}\omega^{-1})} = \gamma^F(\alpha^{-1}\beta)\gamma^F(\alpha\omega\beta) = \gamma^F(\mu\beta)\gamma^F(\nu\beta)$$

since $\mu = \alpha^{-1}Z$ and $\nu = \alpha\omega Z^{-1}$. If $\omega\alpha^2$ is ramified, then (4.4) gives

$$\gamma_{n+f(\alpha\beta\omega)-f(\beta\alpha^{-1})}/\gamma_n = (Zq^{-1/2})^{f(\alpha\beta\omega)-f(\beta\alpha^{-1})}$$

by our choice of n, and hence

$$\gamma(\beta) = (\alpha\beta\omega)(-1)\frac{\Gamma(\beta\alpha^{-1}Zq^{-1/2})}{\Gamma(\alpha^{-1}\beta^{-1}\omega^{-1}q^{-1/2}Z)}$$
$$= (\beta\nu)(-1)\frac{\gamma^F(\mu\beta)}{\gamma^F(\nu^{-1}\beta^{-1})} = \gamma^F(\mu\beta)\gamma^F(\nu\beta),$$

as desired. This proves Theorem 3.

5. **Deep twist**. The deep twist property gives an explicit form for $\gamma(\chi)$ satisfying (MF) at characters χ with sufficiently large conductor. As before, let γ be a rational function satisfying (MF) with the character ω. Denote by c the order at zero of γ on the identity component. Let α be a character of $F^\times$ with conductor at least $-c$. By Proposition 1, the inequality $f(\alpha) \geqslant -c$ implies $f(\alpha) > f(\omega)$. In particular, $f(\alpha\omega) = f(\alpha) \geqslant 2$. The multiplicative formula (MF)′ at (α, B) with m large says

$$[(\alpha\omega)(-1)\Gamma(\alpha^{-1}\omega^{-1})\gamma(\alpha) - \Gamma(\alpha)]B^{f(\alpha)}\gamma(B)$$
$$= \Gamma(\alpha^{-1})\gamma(\alpha)B^{-f(\alpha)} + R_{m,\gamma}(\alpha, B).$$

Because $f(\alpha) \geqslant -c$, the left hand side is holomorphic at $B = 0$, while the right hand side is a polynomial, without constant term, in B^{-1} by Lemma 2. Hence both sides are identically zero; this gives

$$\gamma(\alpha) = \frac{\Gamma(\alpha)(\alpha\omega)(-1)}{\Gamma(\alpha^{-1}\omega^{-1})} = \frac{\gamma^F(\alpha)(\alpha\omega)(-1)}{\gamma^F(\alpha^{-1}\omega^{-1})} = \gamma^F(\alpha)\gamma^F(\alpha\omega),$$

which proves Theorem 4.

https://doi.org/10.4153/CMB-1985-043-0 Published online by Cambridge University Press

COROLLARY 2. *In* (MF), *the character* ω *is determined by* γ.

PROOF. Let m be an integer $\geqslant 2f(\omega)$ and $\geqslant -c$ as in Theorem 4. Then for $\chi \in \mathcal{A}(F^\times)$ with conductor $f(\chi) = m$, we have $\gamma^F(\omega\chi) = \gamma(\chi)/\gamma^F(\chi)$. Since the function on $\mathcal{P}^{\lceil m/2 \rceil}$ given by $u \longmapsto \chi(1 + u)$ is an additive character, there is an element $t_\chi \in F^\times$ of order m, unique mod$^\times(1 + \mathcal{P}^{[m/2]})$, such that $\chi(1 + u) = \psi(t_\chi^{-1}u)$. Then we have $\gamma^F(\chi\omega) = \omega(t_\chi^{-1})\gamma^F(\chi)$ (cf [1], 1.4), in other words, $\omega(t_\chi) = \gamma^F(\chi)^2/\gamma(\chi)$. As $[m/2] \geqslant f(\omega)$, these t_χ's determine the values of ω on the elements in $F^\times$ of order m. Therefore ω is completely determined by γ at the characters of conductor m and $m + 1$.

COROLLARY 3. *With* γ *and* c *as in Proposition 1, the multiplicative formula* (MF) *holds for* (α, β) *provided* m *is larger than or equal to* $2f(\alpha)$, $2f(\beta)$, $2f(\omega)$ *and* $-c$.

PROOF. From Theorem 4, we have $\gamma(\chi) = \gamma^F(\chi)\gamma^F(\chi\omega)$ for $f(\chi) \geqslant m$. Then Corollary follows from Lemma 3 by observing that, for two integers m, n, with $n \geqslant m$,

$$\frac{Z^{-n+1}}{1 - Z} - \frac{Z^{-m+1}}{1 - Z} = Z^{-m} + \ldots + Z^{-n+1}.$$

6. **Cuspidal condition and singularities of γ.** In this section, we prove Theorem 5. Let γ be a rational function on $\mathcal{A}(F^\times)$ satisfying the multiplicative formula (MF). Assume first that γ also satisfies the cuspidal condition for the same ω. If γ is not homogeneous, then by Proposition 3.2.4 of [2], γ has a simple pole at $\chi = \mu q$ for some character μ with $\omega = \mu^{-2}$. Then Theorem 3 implies that γ has the form

$$\gamma(\chi) = \gamma^F(\chi\mu^{-1}q^{-1/2})\gamma^F(\chi\mu^{-1}q^{1/2}) = \begin{cases} \gamma^F(\chi\mu^{-1})^2 & \text{if } \chi\mu^{-1} \text{ is ramified,} \\ -Z^{-2}\dfrac{qZ - 1}{q - Z} & \text{if } \chi\mu^{-1} = Z, \end{cases}$$

and consequently, μq is the only singularity of γ on $\mathcal{A}(F^\times)$.

Conversely, assume that γ has at most one simple pole at, say, $\chi = \mu q$. Then $\omega = \mu^{-2}$ if γ has a pole. Let r be a number satisfying $q|\omega|^{-1/2} > r > q^{1/2}|\omega|^{-1/2}$ and let r' be a number satisfying $r > r' > q|\omega|^{-1}r^{-1}$, where $|\omega|$ is the real number given by $|\omega|^{\mathrm{ord}\, t} = |\omega(t)|$ for all $t \in F^\times$. On each connected component $\chi\mathbb{C}^\times$ of $\mathcal{A}(F^\times)$, choose two circles C'_χ and C_χ with center origin and radius r', r, respectively. In case γ has a pole at μq, the condition $q|\omega|^{-1/2} = q|\mu| > r$ implies that this pole is outside C_μ, hence we may assume that the curve C_m in (MF) is the union of the C_χ's with $f(\chi) \leqslant m$. Since each circle C'_χ lies inside C_χ and, for each point α on C'_χ, the point $\alpha^{-1}\omega^{-1}q$ lies inside the circle $C_{\chi^{-1}\omega^{-1}}$, the multiplicative formula (MF) at $(\alpha, \alpha^{-1}\omega^{-1}q)$ holds for m sufficiently large. Let C'_m be the union of the curves C'_χ with $f(\chi) \leqslant m$.

Now we proceed to prove the cuspidal condition with the same ω as in (MF). Observe that if the condition holds for $(\nu, \tau) \in F^\times \times F$, then it holds for $(u^2\nu, u\tau)$ with $u \in F^\times$ by changing the variable $t \longmapsto ut$ in the integral. Thus it suffices to prove it for $\tau = 1 + \nu$, and $\nu \neq 1$. Our proof is similar to that of Lemma 3.4.4 of [2] in which we computed the case $\nu = 1$. Let m be a positive integer $\geqslant \max(2f(\omega), -c, 1 +$

https://doi.org/10.4153/CMB-1985-043-0 Published online by Cambridge University Press

$2\mathrm{ord}(\nu - 1))$, where c is as in Theorem 4. In particular, the deep twist formula is valid for χ in $\mathcal{A}(F^\times)$ with $f(\chi) \geqslant m$, and hence for a character α with $f(\alpha) \leqslant m$, the multiplicative formula at $(\alpha, \alpha^{-1}\omega^{-1}q)$ holds when integrated over C_{2m}.

It follows from our choice of C'_m that $\gamma(\alpha)$ and $\gamma(\alpha^{-1}\omega^{-1}q)$ are holomorphic at $\alpha \in C'_m$; since $\Gamma(\chi)$ has a zero at $\chi = q^{-1}$, the function $\Gamma(q^{-1})\gamma(\alpha)\gamma(\alpha^{-1}\omega^{-1}q)$ is equal to zero on C'_m. From (MF) we get

$$\oint_{C_{2m}} \Gamma(\alpha\chi^{-1})\Gamma(\alpha^{-1}\omega^{-1}q\chi^{-1})\gamma(\chi)\,d\chi = \omega(-1)(1 - q)^{-1}q^{-2m},$$

which yields

$$\oint_{C'_m} \left(\oint_{C_{2m}} \Gamma(\alpha\chi^{-1})\Gamma(\alpha^{-1}\omega^{-1}q\chi^{-1})\gamma(\chi)\,d\chi\right)\alpha(\nu)|\nu|\,d\alpha \tag{6.1}$$

$$= \omega(-1)(1 - q)^{-1}q^{-2m}\oint_{C'_m} \alpha(\nu)|\nu|\,d\alpha = 0$$

since $\nu \notin 1 + \mathcal{P}^m$. We split the integral on the left side as $\oint_{C'_m}\oint_{C_m} + \oint_{C'_m}\oint_{C_{2m}\setminus C_m}$ and compute each of them. As the integrand is continuous on $C'_m \times C_{2m}$, we may interchange the order of integration.

Our choice of C'_m and C_m allows us to express $\Gamma(\alpha\chi^{-1})$ and $\Gamma(\alpha^{-1}\omega^{-1}q\chi^{-1})$ in series:

$$\Gamma(\alpha\chi^{-1}) = \sum_{a \geqslant -m} \Gamma_a(\alpha\chi^{-1}), \quad \Gamma(\alpha^{-1}\omega^{-1}q\chi^{-1}) = \sum_{b \geqslant -m} \Gamma_b(\alpha^{-1}\omega^{-1}q\chi^{-1}),$$

where

$$\Gamma_a(\beta) = \int_{(a)} \beta(u)\psi(u)\,d^\times u,$$

and (a) denotes the set of elements in $F^\times$ of order a. Hence, for a character λ with $f(\lambda) \leqslant m$,

$$\oint_{C'_\lambda} \Gamma(\alpha\chi^{-1})\Gamma(\alpha^{-1}\omega^{-1}q\chi^{-1})\alpha(\nu)\,d\alpha$$

$$= \sum_{a \geqslant -m} \Gamma_{a - \mathrm{ord}\,\nu}(\lambda\chi^{-1})\Gamma_a(\lambda^{-1}\omega^{-1}q\chi^{-1})\lambda(\nu)$$

and the series converges absolutely. Summing over all components C'_λ of C'_m yields

$$\oint_{C'_m} \Gamma(\alpha\chi^{-1})\Gamma(\alpha^{-1}\omega^{-1}q\chi^{-1})\alpha(\nu)|\nu|\,d\alpha$$

$$= \int_{\nu^{-1}\mathcal{P}^{-m}} \left(\int_{\nu\mathcal{O}^\times} \sum_{f(\alpha)\leqslant m} (\alpha\chi^{-1})(t)\psi(t)(\alpha^{-1}\omega^{-1}q\chi^{-1})(tv)\right.$$

$$\left. \times\ \psi(tv)\alpha(\nu)|\nu|\,d^\times v\right) d^\times t$$

$$= \int_{\nu^{-1}\mathcal{P}^{-m}} (\omega^{-1}\chi^{-2})(t)|t|^{-1}(\omega^{-1}\chi^{-1})(\nu)\psi(t(1 + \nu))\,d^\times t.$$

Therefore

$$(6.2)\qquad \oint_{C_m}\left(\oint_{C'_m}\Gamma(\alpha\chi^{-1})\Gamma(\alpha^{-1}\omega^{-1}q\chi^{-1})\gamma(\chi)\alpha(\nu)|\nu|\,d\alpha\right)d\chi$$

$$= \omega(-\nu)^{-1}\oint_{C_m}\left(\int_{\nu^{-1}\mathcal{P}^{-m}}\gamma(\chi)(\omega^{-1}\chi^{-2})(t)\chi(\nu)^{-1}\psi(-t\tau)|t|^{-1}d^{\times}t\right)d\chi.$$

Because $\gamma(\chi)\chi(\nu)^{-1}$ is bounded and $|\omega^{-1}\chi^{-2}q| < 1$ on C_m, thus the integral in t converges absolutely. This allows us to interchange the order of integration in the above double integral, which becomes what appears in the cuspidal condition.

In view of (6.1) and (6.2), the cuspidal condition at $(\nu, 1 + \nu)$ will hold for m as chosen and $n \geqslant m + \operatorname{ord}\nu$ if we can show

$$\oint_{C'_m}\left(\oint_{C_{2m}\setminus C_m}\Gamma(\alpha\chi^{-1})\Gamma(\alpha^{-1}\omega^{-1}q\chi^{-1})\gamma(\chi)\alpha(\nu)\,d\chi\right)d\alpha = 0.$$

Recall that $\gamma(\chi) = \gamma^F(\chi)\gamma^F(\chi\omega)$ for $f(\chi) \geqslant m$, hence the above equation will result from

$$\oint_{C'_m}\left(\oint_{C_n\setminus C_{n-1}}\Gamma(\alpha\chi^{-1})\Gamma(\alpha^{-1}\omega^{-1}q\chi^{-1})\right.$$

$$\left.\times\ \Gamma(\chi)\Gamma(\chi\omega)\alpha(\nu)\,d\chi\right)d\alpha = 0 \quad\text{for}\quad m < n \leqslant 2m.$$

To evaluate this integral, express the four Γ-functions in their integral form and integrate against χ; it becomes

$$\oint_{C'_m}\int_{u\in(-n),\,v,\,r\in\mathcal{O}^{\times}}\alpha(v^{-1}\nu)\omega(r)\psi(u(1 + r)(1 + vr))\,d^{\times}u\,d^{\times}v\,d^{\times}r\,d\alpha.$$

If ν is not a unit, then integration against α gives zero and we are done. If ν is a unit, then integrating against α first and v next yields

$$(6.3)\qquad \int_{u\in(-n),\,r\in-1+\mathcal{P}^{n-m}}\omega(r)\psi(u(1 + r)(1 + r\nu))\,d^{\times}u\,d^{\times}r$$

$$= \int_{u\in(-n),\,w\in\mathcal{P}^{n-m}}\omega(-1 + w)\psi(uw(w + \nu^{-1} - 1))(1 - q^{-1})^{-1}\,d^{\times}u\,dw$$

under the change of variables $u \mapsto \nu^{-1}u$ and $r = 1 + w$. Using $\int_{(-n)}\psi(uv)\,d^{\times}u = 1 - q^{-1}, -q^{-1}, 0$ according as $\operatorname{ord} v \geqslant n$, $\operatorname{ord} v = n - 1$, $\operatorname{ord} v < n - 1$ respectively, we bring (6.3) to the form

$$(6.4)\qquad \int_{S'}\omega(-1 + w)\,dw - q^{-1}\int_{S}\omega(-1 + w)\,dw,$$

where

$$S = \{w \in \mathcal{P}^{n-m}\colon \operatorname{ord} w + \operatorname{ord}(w + \nu^{-1} - 1) \geqslant n - 1\}$$

and

$$S' = \{w \in \mathcal{P}^{n-m}\colon \text{ord } w + \text{ord } (w + \nu^{-1} - 1) \geq n\} \subset S.$$

Write e for ord $(\nu^{-1} - 1) = \text{ord } (\nu - 1)$. We have $m \geq 2e + 1$ and $m \geq 2f(\omega)$ by our choice.

CASE 1. $n - m > e$. Then ord $(w + \nu^{-1} - 1) = e$ for all $w \in \mathcal{P}^{n-m}$, and $S = \mathcal{P}^{n-1-e}$, $S' = \mathcal{P}^{n-e}$. From $n - 1 - e \geq m \geq 2f(\omega)$ we have $\omega(-1 + w) = \omega(-1)$ for $w \in S$, and, consequently, (6.4) is equal to zero.

CASE 2. $n - m \leq e$. Because $n + 1 \geq m + 2 > 2e$, an element $w \in S$ has order $\geq e$. Moreover, $n - 1 - e > e$, thus $w \in S$ has order $> e$ if and only if it is in $\mathcal{P}^{n-1-e}$, and it has order e if and only if it is in $1 - \nu^{-1} + \mathcal{P}^{n-1-e}$. This shows that $S = \mathcal{P}^{n-1-e} \cup (1 - \nu^{-1} + \mathcal{P}^{n-1-e})$. Similarly, $S' = \mathcal{P}^{n-e} \cup (1 - \nu^{-1} + \mathcal{P}^{n-e})$. From $n - 1 - e > n - 1 - m/2 \geq f(\omega)$ we know $\omega(-1 + w)$ is equal to $\omega(-1)$ for $w \in \mathcal{P}^{n-1-e}$ and $\omega(-\nu)^{-1}$ for $w \in 1 - \nu^{-1} + \mathcal{P}^{n-e}$. This proves that (6.4) vanishes. The proof of Theorem 5 is completed.

REFERENCES

1. P. Gérardin and P. Kutzko, *Facteurs locaux pour GL(2)*, Ann. Sc. Ec. Norm. Sup. **13** (1980), pp. 349–384.

2. P. Gérardin and W.-C. W. Li, *Fourier transforms of representations of quaternions* J. reine u. angewandte Math. (to appear).

3. R. Godement and H. Jacquet, *Zeta functions of simple algebras*, Lecture Notes in Mathematics 260, Springer-Verlag, Berlin-Heidelberg-New York, 1972.

4. H. Jacquet and R. P. Langlands, *Automorphic forms on GL(2)*, Lecture Notes in Mathematics 114, Springer-Verlag, Berlin-Heidelberg-New York, 1970.

5. W.-C. W. Li, *On the representations of GL(2). I ϵ-factors and n-closeness*, J. reine angew. Math. **313** (1980), pp. 27–42.

6. W.-C. W. Li, *Barnes' identities and representations of GL(2). II Nonarchimedean local case*, J. reine angew. Math. **345** (1983), pp. 69–92.

U.E.R. DE MATHÉMATIQUE ET
INFORMATIQUE
UNIVERSITÉ PARIS VII
2, PLACE JUSSIEU
75251 PARIS CEDEX 5
FRANCE
AND
DEPARTMENT OF MATHEMATICS
PENNSYLVANIA STATE UNIVERSITY
UNIVERSITY PARK, PA 16802
U.S.A.

DEPARTMENT OF MATHEMATICS
PENNSYLVANIA STATE UNIVERSITY
UNIVERSITY PARK, PA 16802
U.S.A.