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Martingale central limit theorems without uniform asymptotic negligibility

R.J. Adler and D.J. Scott

Central limit theorems are obtained for martingale arrays without the requirement of uniform asymptotic negligibility. The results obtained generalise the sufficiency part of Zolotarev's extension of the classical Lindeberg-Feller central limit theorem [V.M. Zolotarev, *Theor. Probability Appl.* 12 (1967), 608-618] and also the main martingale central limit theorem (not functional central limit theorem however) of D.L. McLeish [*Ann. Probability* 2 (1974), 620-628.

1. Introduction

In his 1967 paper [7] Zolotarev established two forms of a central limit theorem for sums of independent random variables, which did not require the summands to be uniformly asymptotically negligible as does the classical Lindeberg-Feller central limit theorem (see Loève [4], p. 280). The idea behind Zolotarev's investigation is quite simple; the summands are divided into "big" ones (those for which the uniform asymptotic negligibility condition doesn't hold) and "small" ones (those for which it does). If the big summands approach normality, and the small ones obey the Lindeberg condition, then the central limit theorem will hold, and moreover these requirements are necessary. In this paper we extend the sufficiency part of Zolotarev's results to the martingale case. The results obtained

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extend the central limit theorem of McLeish ([5], Theorem (2.3)) which constitutes the most general central limit theorem for martingales so far obtained, containing the results of Brown [1], Dvoretzky [2], and Scott [6]. The results are not functional central limit theorems however. It is clear that when uniform asymptotic negligibility conditions are not imposed it is not in general possible to obtain functional central limit theorems.

2. Notation and results

Let (Ω, A, P) be a probability space and for each $n \geq 1$ let $\{S_k(n), F_k(n); 1 \leq k \leq k_n\}$ be a martingale sequence defined on (Ω, A, P). Put $S_0(n) = X_0(n) = 0$ almost surely, $S_k(n) = \sum_{j=1}^{k} X_j(n)$ and assume $ES_k^2(n) = s_k^2(n) < \infty$ for $1 \leq k \leq k_n$ and all $n \geq 1$. We define

$$\sigma_j^2(n) = EX_j^2(n) ,$$

$$\tilde{\sigma}_j^2(n) = E\left\{X_j^2(n) \mid F_{j-1}(n)\right\} ,$$

$$\Phi(x) = \int_{-\infty}^{x} e^{-x^2/2} dx ,$$

$$\Phi_j^{(n)}(x) = \Phi\left(x/\sigma_j(n)\right) ,$$

and

$$\Delta_j^{(n)}(x) = P\left\{X_j(n) \leq x \mid F_{j-1}(n)\right\} - \Phi_j^{(n)}(x) .$$

THEOREM. *Let* γ_n *be a bounded sequence of positive real numbers. If*

$$\text{(1)} \qquad \gamma_n^{-1}\alpha_n\left(\log \alpha_n^{-1}\right)^{\frac{1}{2}} \to 0 \quad as \quad n \to \infty ,$$

where

$$\text{(2)} \qquad \alpha_n = \sup_{j \in U_n} \sup_x E\left|\Delta_j^{(n)}(x)\right|$$

and U_n *is the set of values of the index* j *such that* $\sigma_j^2(n) < \gamma_n$ *, and* *the following conditions hold*

$$\text{(A)}\quad\begin{cases} (3)\quad \sum_{j\in U_n} \sigma_j^2(n) \le 1\,, \\ (4)\quad \max_{j\in U_n} |X_j(n)| \xrightarrow{P} 0\,, \\ (5)\quad \max_{j\in U_n} |X_j(n)| \text{ is uniformly bounded in } L_2 \text{ norm}, \\ (6)\quad \sum_{j\in U_n} X_j^2(n) + \sum_{j\in \bar{U}_n} \sigma_j^2(n) \xrightarrow{P} 1\,, \end{cases}$$

then

$$P\left[S_{k_n}(n) \le x\right] \to \Phi(x) \quad \text{as} \quad n \to \infty\,, \quad \forall x\,.$$

Here "$\xrightarrow{P}$" denotes "converges in probability to".

We can obtain the result of the theorem under slightly more restrictive conditions, by means of the following lemma.

LEMMA. *Let* $s_{k_n}^2(n) = 1$. *Then the conditions* (A) *are simplified by the equivalent sets of conditions* (B), (C), *and* (D). *Under* $\gamma_n \to 0$, (B), (C), *and* (D) *are also equivalent to* (E), *and under* (1) *and* $\gamma_n \to 0$, *to* (F) *also.*

$$(7)\quad \text{(B)}\quad\begin{cases} \sum_{j\in U_n} \left[X_j^2(n)-\sigma_j^2(n)\right] \xrightarrow{P} 0\,, \\ \sum_{j\in U_n} X_j^2(n) I\left(|X_j(n)| \ge \varepsilon\right) \xrightarrow{P} 0\,, \quad \varepsilon > 0\,, \end{cases}$$

$$\text{(C)}\quad\begin{cases} (8)\quad \sum_{j\in U_n} \left[\tilde{\sigma}_j^2(n)-\sigma_j^2(n)\right] \xrightarrow{P} 0\,, \\ (9)\quad \sum_{j\in U_n} E\left\{X_j^2(n) I\left(|X_j(n)| \ge \varepsilon\right) \mid F_{j-1}(n)\right\} \xrightarrow{P} 0\,, \quad \forall\varepsilon > 0\,, \end{cases}$$

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$$(10)\quad \text{(D)}\quad \begin{cases} \sum_{j\in U_n} \left\{X_j^2(n)-\sigma_j^2(n)\right\} \xrightarrow{P} 0 , \\ \sup_{j\in U_n} X_j^2(n) \xrightarrow{P} 0 , \end{cases}$$

$$(11)\quad \text{(E)}\quad \begin{cases} \sum_{j\in U_n} \left\{\tilde{\sigma}_j^2(n)-\sigma_j^2(n)\right\} \xrightarrow{P} 0 , \\ \sum_{j\in U_n} \int_{|x|\ge\varepsilon} x^2 d\Delta_j^{(n)}(x) \xrightarrow{P} 0 , \quad \forall\varepsilon > 0 , \end{cases}$$

$$(12)\quad \text{(F)}\quad \begin{cases} \sum_{j\in U_n} \left\{\tilde{\sigma}_j^2(n)-\sigma_j^2(n)\right\} \xrightarrow{P} 0 , \\ \sum_{j=1}^{k_n} \int_{|x|<\varepsilon} x^2 d\Delta_j^{(n)}(x) \xrightarrow{P} 0 , \quad \forall\varepsilon > 0 . \end{cases}$$

We use $I(\cdot)$ to denote the indicator function.

When the martingale differences $\{X_j(n)\}$ are actually independent we can observe that the above results contain the sufficiency part of Zolotarev's results by choosing $\gamma_n = \beta_n^{\frac{1}{2}}$, where $\beta_n = \sup_{1\le j\le k_n} L\left\{P\left(X_j(n) \le x\right), \Phi_j^{(n)}(x)\right\}$ and L is the Levy metric. Noting Lemma 2 of [7] it is not difficult to show (1), (C), and $\gamma_n \to 0$ are equivalent to the conditions of the theorem of [7] and (1), (F), and $\gamma_n \to 0$ are equivalent to the conditions of the theorem (second version) of [7].

If the martingale difference array $\{X_j(n)\}$ satisfies the conditions of Theorem (2.3) of [5] then clearly

$$\sup_n \sup_{1\le j\le k_n} \sigma_j^2(n) < M < \infty ,$$

and taking $\gamma_n = 2M$, $\{X_j(n)\}$ also satisfies the conditions of our theorem. Thus the results above extend McLeish's Theorem (2.3) also.

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3. Proof of the theorem

We must show that

$$E \exp\left(itS_{k_n}(n)\right) \to e^{-t^2/2} \tag{13}$$

for each real t. For every $j \in \overline{U}_n$ we let $Y_j(n)$ be distributed as $N\left(0, \sigma_j^2(n)\right)$ independently of each other and of the σ-field generated by $\bigcup_{j=1}^{k_n} F_j(n)$ and for $j \in U_n$ simply set $Y_j(n) = X_j(n)$. Then letting

$$R_k(n) = \sum_{j=1}^{k} Y_j(n) + \sum_{j=k+1}^{k_n} X_j(n) , \quad 1 \le k \le k_n ,$$

we will show first that

$$\left|E \exp\left(itR_{k_n}(n)\right) - E \exp\left(itS_{k_n}(n)\right)\right| \to 0 . \tag{14}$$

Now, using the convention that $\sum_{j=k}^{l} a_j = 0$ for $k > l$,

$$\begin{aligned}
(15)\quad & \left|E \exp\left(itR_{k_n}(n)\right) - E \exp\left(itS_{k_n}(n)\right)\right| \\
&= \left|\sum_{k=0}^{k_n} E\left[\exp\left(\sum_{j=k+1}^{k_n} itY_j(n) + \sum_{j=1}^{k-1} itX_j(n)\right)\left(e^{itY_k(n)} - e^{itX_k(n)}\right)\right]\right| \\
&= \left|\sum_{k=0}^{k_n} E\left[\exp\left(\sum_{j=k+1}^{k_n} itY_j(n)\right)\right] E\left[\left(\exp \sum_{j=1}^{k-1} itX_j(n)\right)\left(e^{itY_k(n)} - e^{itX_k(n)}\right)\right]\right| \\
&\le \sum_{k=0}^{k_n} E\left|E\left\{\left(\exp \sum_{j=1}^{k-1} itX_j(n)\right)\left(e^{itY_k(n)} - e^{itX_k(n)}\right) \,\middle|\, F_{k-1}(n)\right\}\right| \\
&\le \sum_{k=0}^{k_n} E\left|E\left\{e^{itY_k(n)} - e^{itX_k(n)} \,\middle|\, F_{k-1}(n)\right\}\right| \\
&= \sum_{k \in U_n} E\left|E\left\{e^{itY_k(n)} - e^{itX_k(n)} \,\middle|\, F_{k-1}(n)\right\}\right| .
\end{aligned}$$

The sum in (15) can be treated in the following way. Define a sequence of

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numbers A_n by $A_n = \sqrt{2 \log \alpha_n^{-1}}$. By Feller [3] (page 175) we have

$$\begin{aligned}\Phi(-A_n) &= 1 - \Phi(A_n) \\ &< A_n^{-1} e^{-A_n^2/2} \\ &= \alpha_n \left(2 \log \alpha_n^{-1}\right)^{-\frac{1}{2}} .\end{aligned}$$

Since for $j \in \overline{U}_n$, $\sigma_j(n) \le 1$, it follows that

$$\begin{aligned}\Phi\left(-A_n/\sigma_j(n)\right) &= 1 - \Phi\left(A_n/\sigma_j(n)\right) \\ &< \alpha_n \left(2 \log \alpha_n^{-1}\right)^{-\frac{1}{2}} .\end{aligned} \tag{16}$$

We have thus

$$\begin{aligned}&E\left|E\left\{e^{itX_k(n)} - e^{itY_k(n)} \mid F_{k-1}(n)\right\}\right| \\ &\qquad = E\left|\int e^{itx} d\Delta_k^{(n)}(x)\right| \\ &\qquad \le E\left|\int_{-\infty}^{-A_n} e^{itx} d\Delta_k^{(n)}(x)\right| + E\left|\int_{-A_n}^{A_n} e^{itx} d\Delta_k^{(n)}(x)\right| + E\left|\int_{A_n}^{\infty} e^{itx} d\Delta_k^{(n)}(x)\right| \\ &\qquad = I_1 + I_2 + I_3 .\end{aligned}$$

Treating these forms separately

$$\begin{aligned}I_1 \le E\int_{-\infty}^{-A_n}\left|d\Delta_k^{(n)}(x)\right| &\le E\left(P\{X_k(n) \le -A_n \mid F_{k-1}(n)\} + \Phi\left(-A_n/\sigma_k(n)\right)\right) \\ &\le 2\Phi\left(-A_n/\sigma_k(n)\right) + \alpha_n \\ &\le 2\alpha_n\left(2 \log \alpha_n^{-1}\right)^{-\frac{1}{2}} + \alpha_n\end{aligned} \tag{17}$$

on using (2) and (16). Furthermore

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$$(18) \qquad I_2 \le E\left\{\left|\int_{-A_n}^{A_n} ite^{itx}\Delta_k^{(n)}(x)dx\right| + \left|\left[e^{itx}\Delta_k^{(n)}(x)\right]_{-A_n}^{A_n}\right|\right\}$$

$$\le tE\int_{-A_n}^{A_n}\left|\Delta_k^{(n)}(x)\right|dx + 2\alpha_n$$

$$\le 2tA_n\alpha_n + 2\alpha_n$$

$$= 2\alpha_n\left(1 + t\sqrt{2\log\alpha_n^{-1}}\right).$$

But $k \in \overline{U}_n$ entails $\sigma_k^2(n) \ge \gamma_n$ and since $\sum_{k\in\overline{U}_n} \sigma_k^2(n) \le 1$ there are at most γ_n^{-1} indices in $\overline{U}_n$. Combining this with (18), (17), and a similar bound for I_3, we obtain

$$\sum_{k\in\overline{U}_n} E\left|E\left\{e^{itX_k(n)} - e^{itY_k(n)} \;\middle|\; F_{k-1}(n)\right\}\right|$$

$$\le \gamma_n^{-1}\left\{4\alpha_n + 2t\alpha_n\sqrt{2\log\alpha_n^{-1}} + 4\alpha_n\left(2\log\alpha_n^{-1}\right)^{-\frac{1}{2}}\right\}.$$

Under (1) this goes to zero as $n \to \infty$ and thus (14) is proved.

We have

$$(19) \qquad E\exp\left(itR_{k_n}(n)\right) = \left[E\exp\left(\sum_{k\in\overline{U}_n} itY_k(n)\right)\right]\left[E\exp\left(\sum_{k\in U_n} itX_k(n)\right)\right]$$

$$= \exp\left(-t^2/2\sum_{k\in\overline{U}_n}\sigma_k^2(n)\right)\left[E\exp\left(\sum_{k\in U_n} itX_k(n)\right)\right],$$

and because of (14) it is clearly sufficient to prove the right hand side of (19) converges to $\exp\left(-t^2/2\right)$ for every real t, and in fact we need only show that for any subsequence $\{n'\}$ there exists a further subsequence $\{n''\}$ along which the convergence holds. Thus we may assume without loss of generality that

$$(20) \qquad \sum_{k\in\overline{U}_n}\sigma_k^2(n) \to L \quad \text{as} \quad n \to \infty$$

for some $0 \le L \le 1$, and thus that

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(21) $$\sum_{k \in U_n} X_k^2(n) \xrightarrow{P} 1 - L \quad \text{as} \quad n \to \infty .$$

Using this assumption we show that the small martingale differences constitute a martingale difference array which satisfies the conditions (a), (b), and (c) of Theorem (2.3) of [5] (with the slight modification that the convergence in (c) is to $1 - L$, not 1), so that

$$E \exp\left(\sum_{k \in U_n} itX_k(n)\right) \to \exp\left(-t^2/2\ [1-L]\right)$$

and thus the result follows from (14), (19), and (20).

For any n, let

$$\{X_j(n);\ j \in U_n\} = \left\{X_{j_l}(n);\ l = 1, 2, \ldots, m_n\right\} ,$$

where $j_{l_1} < j_{l_2}$ if $l_1 < l_2$, and put

$$Z_l(n) = X_{j_l}(n) .$$

(Strictly j_l should be indexed by n also, but this has been omitted for the sake of simplicity.) Then for each $n \geq 1$, we let

$$T_0(n) = Z_0(n) = 0 \text{ almost surely,}$$

$$T_m(n) = \sum_{l=1}^{m} Z_l(n) ,$$

$$G_m(n) = F_{j_{m+1}-1}(n) ,$$

and

$$t_m^2(n) = ET_m^2(n) < \infty , \quad m = 1, 2, \ldots, m_n .$$

Since

$$E\{X_l(n) \mid G_{l-1}(n)\} = E\left\{X_{j_l}(n) \mid F_{j_l-1}(n)\right\} = 0 \text{ almost surely,}$$

$\{T_m(n), G_m(n);\ m = 1, 2, \ldots, m_n\}$ is a martingale sequence for each

$n \geq 1$. Moreover we have from (A),

(22) $\max_{1 \leq l \leq m_n} |Z_l(n)| = \max_{j \in U_n} |X_j(n)|$ is uniformly bounded in L_2 norm,

(23) $\max_{1 \leq l \leq m_n} |Z_l(n)| = \max_{j \in U_n} |X_j(n)| \xrightarrow{P} 0$,

and from (21),

$$\sum_{l=1}^{m_n} Z_l^2(n) = \sum_{j \in U_n} X_j^2(n) \xrightarrow{P} 1 - L . \tag{24}$$

Thus $\{Z_l(n), G_m(n); m = 1, 2, \ldots, m_n\}$ is a martingale difference array which satisfies the conditions of Theorem (2.3) of [5] and the proof is complete.

4. Proof of lemma

We give an indication only of how to prove the results since the method in most cases is similar.

Suppose for instance conditions (B) hold. Then there exists for any subsequence $\{n'\}$ a further subsequence $\{n''\}$ such that

$$\sum_{j \in U_n} \sigma_j^2(n'') \to L \quad \text{as} \quad n'' \to \infty$$

for some $L \in [0, 1]$. Then by consideration of $\{Z_l(n), G_l(n), l = 1, \ldots, m_n\}$ for each $n \in \{n''\}$ we may use the method of [6] to show that the conditions (A) hold along the subsequence $\{n''\}$. But this implies conditions (A) hold as $n \to \infty$. Similarly by subsequencing and using the methods of Scott [6] we may show (B), (C), and (D) are equivalent under $s_k^2(n) = 1$.

To show that (B), (C), and (D) are equivalent to (E) under $s_{k_n}^2(n) = 1$ and $\gamma_n \to 0$, note that conditions (E) and (C) are equivalent since

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$$(25)\quad \sum_{j\in U_n} E\left\{X_j^2(n)I\left(|X_j(n)| \geq \varepsilon\right)|F_{j-1}(n)\right\}$$

$$= \sum_{j\in U_n}\int_{|x|\geq\varepsilon} x^2 d\Phi\left(x/\sigma_j(n)\right) + \sum_{j\in U_n}\int_{|x|\geq\varepsilon} x^2 d\Delta_j^{(n)}(x)$$

where the first term on the right hand side of (25) is bounded by

$$\sum_{j\in U_n}\sigma_j^2(n)\int_{|x|\geq\varepsilon/\sigma_j(n)} x^2 d\Phi(x) \geq \sup_{j\in U_n}\int_{|x|\geq\varepsilon/\sigma_j(n)} x^2 d\Phi(x)$$

$$= \int_{|x|\geq\varepsilon/\left(\sup_{j\in U_n}\sigma_j(n)\right)} x^2 d\Phi(x) ,$$

and $\sup_{j\in U_n}\sigma_j(n) \leq \gamma_n \to 0$ as $n \to \infty$.

Finally we show conditions (F) and (E) are equivalent under $s_{k_n}^2(n) = 1$, $\gamma_n \to 0$ and (1). Note (25) and observe that

$$(26)\quad \sum_{j\in U_n}\int_{|x|\geq\varepsilon} x^2 d\Delta_j^{(n)}(x)$$

$$= \sum_{j\in U_n}\left(\tilde{\sigma}_j^2(n)-\sigma_j^2(n)\right) - \sum_{j\in U_n}\int_{|x|<\varepsilon} x^2 d\Delta_j^{(n)}(x)$$

$$= \sum_{j\in U_n}\left(\tilde{\sigma}_j^2(n)-\sigma_j^2(n)\right) + \sum_{j\in \overline{U}_n}\int_{|x|<\varepsilon} x^2 d\Delta_j^{(n)}(x) - \sum_{j=1}^{k_n}\int_{|x|<\varepsilon} x^2 d\Delta_j^{(n)}(x)$$

and the result will follow provided we can show the second term in (26) converges to zero. Recalling that the number of indices in $\overline{U}_n$ does not exceed γ_n^{-1} we have

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$$E\left(\sum_{j\in U_n}\int_{|x|<\varepsilon} x^2 d\Delta_j^{(n)}(x)\right)$$

$$\le \gamma_n^{-1}E\left(\sup_{j\in U_n}\int_{|x|<\varepsilon} x^2 d\Delta_j^{(n)}(x)\right)$$

$$\le \gamma_n^{-1}E\left(\sup_{j\in U_n}\left\{\left|x^2\Delta_j^{(n)}(x)\right|_{-\varepsilon}^{\varepsilon} - 2\int_{|x|<\varepsilon}' x\Delta_j^{(n)}(x)dx\right\}\right)$$

$$\le \gamma_n^{-1}\left(2\varepsilon^2\alpha_n+4\varepsilon^2\alpha_n\right)$$

$$\to 0 \text{ as } n \to \infty .$$

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School of Mathematics,
University of New South Wales,
Kensington,
New South Wales;

Department of Probability and Statistics,
University of Sheffield,
Sheffield,
England.

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