SEMI-VALUATIONS AND GROUPS OF DIVISIBILITY

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Introduction. Associated with any integral domain R there is a partially ordered group A, called the group of divisibility of R. When R is a valuation ring, A is merely the value group; and in this case, ideal-theoretic properties of R are easily derived from corresponding properties of A, and conversely. Even in the general case, though, it has proved useful on occasion to phrase a ring-theoretic problem in terms of the ordered group A, first solve the problem there, and then pull back the solution if possible to R. Lorenzen (**15**) originally applied this technique to solve a problem of Krull, and Nakayama (**16**) used it to produce a counterexample to another question of Krull. More recently, Heinzer (**7**; **8**) has used the method to construct other interesting examples of rings.

Thus, one of the advantages of the process is its ability to produce interesting examples of domains, since partially ordered groups abound, whereas integral domains are not so easy to come by. The main theorem in the pull-back process from the group A to the ring R is due to Jaffard (**9**, p. 64, Theorem 1; or **12**, p. 78, Theorem 3) and asserts that any lattice-ordered group is the group of divisibility of a domain. Krull (**13**, p. 164) first established the theorem in the case that A is totally ordered, and it seems in (**16**) that Nakayama was certainly aware of some version of the theorem. (For the interested reader who finds difficulty in reading Nakayama's counterexample (**16**), I have written a simple version, based on Nakayama's ideas, which appears in (**5**, Appendix 4).)

Jaffard (**11**) has also given an example of a filtered ordered group which is *not* a group of divisibility, but in between these two extremes of a filtered group and a lattice-ordered group, nothing seems to have been known. In this paper we slightly close the gap by giving procedures for constructing a class of groups, not necessarily lattice, which *are* groups of divisibility, and on the other hand, a class of groups which are *not* groups of divisibility. The first class is sufficiently large to provide negative answers to a couple of questions raised by Jaffard in (**12**); and since this can be done directly, we begin with the counterexample, in § 2.

Most of our results are merely generalizations of classical theorems of valuation theory. Thus, § 3 is devoted to showing that the counterexample of § 2 comes from an appropriate treatment of the composite of two semi-valuations. Certain initial assumptions are made in § 3, and in § 4 we show that

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these assumptions are inherent in the nature of things. In § 5 we construct our class of groups which cannot be groups of divisibility. Section 6 is devoted to looking at the situation from a slightly different angle, namely from the viewpoint of extensions of semi-valuations. We show there that both Krull's construction of the Kronecker function ring and Jaffard's theorem can be considered as special cases of the same process. (This was originally given passing mention in (**17**, pp. 329–330).)

Finally, I wish to mention that the paper has benefited from a number of conversations which I had with William Heinzer during its preparation.

1. Definitions and immediate consequences. To begin, we review a few of the definitions, most of which can be found in the works of Jaffard, especially (**12**), or also in (**19**) or (**1**).

(a) *Ordered groups.* By an *ordered group* we mean a commutative group with a partial ordering. The ordered group A is called *filtered* if for any $a_1, a_2 \in A$, there exists $a \in A$ such that $a \leqq a_1, a_2$. A^+ denotes the elements greater than or equal to zero of the ordered group A. An *ordered subgroup* B of A is an ordered group contained in A such that $B^+ = A^+ \cap B$. We use the letter J to denote the additive group of integers with the usual order. If $D = \pi D_u$ is a direct product of ordered groups D_u, D is called the *ordered direct product* if $D^+ = \{d \in D | \, d_u \geqq 0 \text{ for all } u\}$; and one defines the *ordered direct sum* similarly.

Finally, if $a_0, a_1, \ldots, a_n$ are elements of an ordered group A, we define the expression $a_0 \geqq \inf_A\{a_1, \ldots, a_n\}$ by

$a_0 \geqq \inf_A\{a_1, \ldots, a_n\}$ if and only if $a_0 \geqq a$ for all $a \in A$ such that $a \leqq a_1, \ldots, a_n$.

(b) *Exact sequences.* A homomorphism α of an ordered group A into an ordered group B is called an *order homomorphism* (*homomorphisme croissant* (**12**, p. 10)) if $\alpha(A^+) \subset B^+$. α is an *order isomorphism* if α is a group isomorphism and $\alpha(A^+) = B^+$.

A short exact sequence of ordered groups

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0 \tag{1.1}$$

is called *order exact* if $\alpha(A^+) = \alpha(A) \cap B^+$ and $\beta(B^+) = C^+$. In particular, α and β are then order homomorphisms. The exact sequence (1.1) is called *lexicographically exact* if $B^+ = \{b \in B | \beta(b) > 0 \text{ or } b \in \alpha(A^+)\}$. A lexicographically exact sequence is also order exact. Sometimes the notation $(\alpha, \beta)\colon A \to B \to C$ will also be used for the short exact sequence (1.1). The group B is called an *ordered extension* of A by C or a *lexicographic extension* of A by C, depending on whether (1.1) is order exact or lexicographically exact.

A particularly important case of the above is the *lexicographic direct sum.* If A and C are ordered groups, we order the direct sum $A \oplus C$ by defining

$(A \oplus C)^+ = \{(a, c) \mid c > 0 \text{ or } c = 0 \text{ and } a \geqq 0\}$. (Note that we are using the reverse of the usual lexicographic ordering (**12**, p. 5). We do this since it seems better suited to our diagrams.) Thus, if l and π are the usual injection and projection maps, we have a lexicographically exact sequence

$$0 \to A \xrightarrow{l} A \oplus C \xrightarrow{\pi} C \to 0. \tag{1.2}$$

We shall always use the symbol $A \oplus C$ to denote the lexicographic direct sum.

We say that the ordered exact sequence (1.1) *splits* (*splits lexicographically*) if there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 \to & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \to 0 \\ & {\scriptstyle i_1}\downarrow & & {\scriptstyle i_2}\downarrow & & {\scriptstyle i_3}\downarrow & \\ 0 \to & A & \xrightarrow{l} & A \oplus C & \xrightarrow{\pi} & C & \to 0 \end{array}$$

where i_1 and i_3 are the identity maps, and i_2 is an isomorphism (order isomorphism). If (1.1) is lexicographically exact and splits, then it splits lexicographically.

Remarks. (1) (**10**, pp. 204–205). If (1.1) is lexicographically exact and $A, C \neq 0$, then (i) B is filtered if and only if C is filtered, and (ii) B is lattice if and only if A is lattice and C is totally ordered.

(2) If (1.1) is order exact, then B totally ordered implies that A and C are totally ordered and the sequence is lexicographically exact.

The proof is easy and is therefore omitted. Bourbaki (**2**(a), p. 164, Lemma 2) proved (2) in the case where the sequence splits. However, there exists an ordered exact sequence of totally ordered groups which does not split (the example can be found in (**3**; **18**, p. 25; **19**, p. 57)); thus, (2) is a legitimate generalization of the Bourbaki lemma.

(c) *Semi-valuations.* A domain will always mean a commutative ring with identity and without divisors of zero. If R is a domain, we use R^* to denote the multiplicative semigroup of non-zero elements of R. $U(R)$ is the multiplicative group of units of R.

A *semi-valuation* of a field K is a map w of K^* into an additive ordered group A such that for all $x, y \in K^*$,

$$\begin{aligned} &\text{(i)} \ \ w(xy) = w(x) + w(y), \\ &\text{(ii)} \ \ w(x + y) \geqq \inf_{w(K^*)}\{w(x), w(y)\}, \\ &\text{(iii)} \ \ w(-1) = 0. \end{aligned} \tag{1.3}$$

$w(K^*)$ is called the *semi-value group* of w.

Some authors take w to be a map defined on K by specifying that $w(0) = +\infty$. In this case, (ii) and (iii) can be replaced by the single axiom

$$\text{(ii)}' \ w(x - y) \geqq \inf_{w(K)}\{w(x), w(y)\}.$$

(This is the axiom originally used by Zelinsky (**21**, p. 1148).) Two semi-

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valuations w, w' of K having respective semi-value groups A, A' are called *equivalent* if there exists an order isomorphism ϕ from A to A' such that $\phi w = w'$.

If R is a subring of a field K, a pre-order is defined on K^* by taking $(K^*)^+ = R^*$; and then the natural map of K^* onto the associated ordered group $K^*/U(R)$ written additively is a semi-valuation. Whenever we consider the group $K^*/U(R)$, we shall assume that it has this order.

Conversely, if w is any semi-valuation of K, then

$$R_w = \{x \in K^* | \, w(x) \geqq 0\} \cup \{0\}$$

is a subring of K, called the *semi-valuation ring of* w. w is then equivalent to the natural semi-valuation map of K^* onto $K^*/U(R_w)$. Thus, there is a one-to-one correspondence between equivalence classes of semi-valuations of K and subrings of K.

If A is the semi-value group of a semi-valuation w of K, A is called *a group of divisibility* whenever A is filtered. This occurs if and only if the semi-valuation ring R_w has K as quotient field (**12**, p. 8). If R is a domain with K as quotient field, the ordered group $K^*/U(R)$ is called *the group of divisibility of* R.

A semi-valuation w of K will be called an *additive semi-valuation* if

(1.4) $\quad w(x) < w(y)$ implies $w(x + y) = w(x)$ whenever $x + y \in K^*$.

It is easily seen that w is additive if and only if its semi-valuation ring R_w is quasi-local.

(d) *V-homomorphisms*. If B and C are ordered groups and β is a homomorphism of B into C, then β is called a *V-homomorphism* if

(1.5) for any $b_0, b_1, \ldots, b_n \in B$,
$$b_0 \geqq \inf_B\{b_1, \ldots, b_n\} \text{ implies that } \beta(b_0) \geqq \inf_C\{\beta(b_1), \ldots, \beta(b_n)\}.$$

(The above notation is to be interpreted as explained in § 1 (a).) A V-homomorphism is then, in particular, an order homomorphism. (The notion of V-homomorphism subsumes a couple of other definitions of Jaffard: If C is totally ordered, Jaffard (**12**, p. 46) calls a V-homomorphism a V-valuation (whence our name); while if B and C are both lattice, our V-homomorphisms are the *coréticule* homomorphisms (**12**, p. 13) of Jaffard. We hope to say more about these notions in a future paper.)

A *V-isomorphism* is an isomorphism such that it and its inverse are V-homomorphisms. A *V-embedding* of B in C is a V-homomorphism which is one-to-one. A subgroup B of C is a *V-subgroup* if the identity map is a V-homomorphism. We next collect some immediate properties of V-homomorphisms.

Properties. (1) Let v be a semi-valuation of K with semi-value group B. If β is a V-homomorphism of B into an ordered group C, then βv is also a semi-valuation of K.

(2) If α: $A \to B$ and β: $B \to C$ are V-homomorphisms, then $\beta\alpha$ is also.

(3) If (α, β): $A \to B \to C$ is lexicographically exact, then α is a V-homomorphism.

(4) If B and C are lattice-ordered groups, then the homomorphism β is a V-homomorphism if and only if $\beta(\inf_B\{b_1, \ldots, b_n\}) = \inf_C\{\beta(b_1), \ldots, \beta(b_n)\}$.

The importance of the notion of V-homomorphism is mainly due to property (1). The converse to (1) is false, and counterexamples are easy to construct; in particular, Example 4.4 works. Furthermore, (1) is no longer true if β is merely assumed to be an order homomorphism, even when C is totally ordered (thus contradicting (**2**(b), p. 82, exercise 5b)). An example which shows this runs as follows: Let Q be the field of rationals. Any $x \in Q^*$ can be uniquely written as $x = \pm\pi p_i^{n_i}$, $i \in I$, where $\{p_i\}$ is the set of all prime integers. Let B be the ordered direct sum of I copies of Z, and let v be the semi-valuation of Q defined by: $v(x)$ is that function $F \in B$ such that $F(i) = n_i$, $i \in I$. Now let β be the order homomorphism of B onto Z defined by $\beta(F) = \sum F(i)$. βv is not a semi-valuation since $\beta v(4) = 2$, $\beta v(9) = 2$, but $\beta v(4 + 9) = 1$.

Finally, we remark that in property (3), β need not be a V-homomorphism. We give necessary and sufficient conditions in Theorem 4.2 for such a β to be a V-homomorphism.

2. The counterexample to Jaffard's questions. Let A be an ordered group. $a, b \in A$ are called *disjoint* if $0 = \inf\{a, b\}$. They are called *relatively prime* if $0 \leqq a, b$ and whenever $0 \leqq c \leqq a, b$, then $c = 0$ (**12**, p. 6). Jaffard (**12**, pp. 80–81) asks the following questions. Does there exist a group of divisibility possessing two elements which are relatively prime but not disjoint? Is an ordered group in which every element is a difference of two relatively prime elements necessarily lattice? The answer to both questions is "no". To show this, it suffices to construct an ordered group B with the following properties:

(i) B is a group of divisibility,

(ii) there exist two relatively prime elements of B which are not disjoint,

(iii) any element of B is a difference of two relatively prime elements.†

To begin, let k_0 be a field and X, Y be indeterminates over k_0; let $k = k_0(X)$, $K = k_0(X, Y)$; let w be the Y-adic valuation of K/k; and let C (the additive group of integers) be the value group of w. Let A be the ordered group k^*/k_0^*, and let u be the associated semi-valuation of k. Let B be the lexicographic direct sum $A \oplus C$. Define a semi-valuation v of $k_0[X, Y]$ by

$$v(p_i Y^i + \ldots + p_{i+n} Y^{i+n}) = (u(p_i), i),$$

†The referee has made (May, 1968) the astute observation that any Noetherian domain R which is not a UFD has a group of divisibility which satisfies (i), (ii), and (iii); such an R satisfies (ii) since there exist $a, b, a', b' \in R$ such that a and b are distinct irreducible elements and $aa' = bb'$ but $a'/b \notin R$. For (iii), if $\xi \in K^*$, choose $a, b \in R$ such that $\xi = a/b$ and such that the ideal (a, b) is maximal with respect to this choice.

whenever $p_{i+j} \in k_0[X]$ and $p_i \neq 0$; and extend v to K^*. Then v has semi-value group B, and since C is filtered, B is also. Thus, B is a group of divisibility.

Note that $A^+ = \{0\}$; thus, $(a, 0) \geqq (0, 0)$ if and only if $a = 0$, for any $a \in A$. Therefore, if $a \neq 0 \in A$, then $(a, 1)$ and $(-a, 1)$ are relatively prime. However, $(a, 0) \leqq (a, 1)$, $(-a, 1)$ but $(a, 0) \nleqq (0, 0)$; therefore, $(a, 1)$, $(-a, 1)$ are not disjoint.

Finally, let us check that any element (a, i) of B is a difference of two relatively prime elements. If $i > 0$, then $(a, i) = (a, i) - (0, 0)$; and similarly, if $i < 0$, then $(a, i) = (0, 0) - (-(a, i))$. If $i = 0$ and $a \neq 0$, then $(a, i) = (a, 1) - (0, 1)$, where $(a, 1)$ and $(0, 1)$ are relatively prime. Thus, B has the required properties.

Observe that the semi-valuation ring of v is $k_0 + m_w$, where m_w is the maximal ideal of the ring of w. Further facts about this ring can be found in (**6**, § 5 or **4**, Example 4.5).

3. Composite semi-valuations. Throughout this section we fix the following notation. Let w be a semi-valuation of the field K, and *assume that the semi-valuation ring R_w is quasi-local* with maximal ideal m_w and residue field $k = R_w/m_w$. Let h be the canonical homomorphism of R_w onto k. Now let u be a semi-valuation of k with semi-valuation ring R_u; and let v be the semi-valuation of K (determined up to equivalence) having semi-valuation ring $R_v = h^{-1}(R_u)$. v is said to be composite with w and u (the word "composite" seems to be bad terminology, but we adhere to it because of its general use in valuation theory, e.g. (**20**, p. 43)).

Let, furthermore, A_u, B_v, and C_w denote the respective semi-value groups of u, v, and w; and let U_u, U_v, and U_w be the respective multiplicative groups of units $U(R_u)$, $U(R_v)$, and $U(R_w)$.

LEMMA 3.1. *$U_v + m_w \subset U_v$ and $U_v = h^{-1}(U_u)$.*

Proof. Since $m_w \subset R_v$, it is sufficient to show for the first assertion that $1/(t + y) \in U_v + m_w$ whenever $y \in m_w$, $t \in U_v$. However, $1/(t + y) = 1/t - y/t(t + y) \in U_v + m_w$.

As for the second assertion, $U_v \subset h^{-1}(U_u)$ is clear. Conversely, $x \in h^{-1}(U_u)$ implies that there exists $x' \in R_w$ such that $xx' \in 1 + m_w \subset U_v$. Then $h(x') = 1/h(x) \in U_u$, thus $x' \in h^{-1}(U_u) \subset R_v$. Therefore, $x \in U_v$.

THEOREM 3.2. *There exist homomorphisms α, β which complete the commutative diagram* (3.1) *below and which make the bottom row lexicographically exact.*

(3.1)
$$\begin{array}{ccccccccc}
 & & U_w & \xrightarrow{i} & K^* & & & & \\
 & & \downarrow{\scriptstyle uh'} & & \downarrow{\scriptstyle v} & \searrow{\scriptstyle w} & & & \\
0 & \rightarrow & A_u & \overset{\alpha}{\dashrightarrow} & B_v & \overset{\beta}{\dashrightarrow} & C_w & \rightarrow & 0
\end{array}$$

where i is the identity and h' is the restriction of h to U_w.

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Proof. $U_w = \ker w \supset \ker v = U_v$, since $R_v \subset R_w$; thus, the epimorphism β is defined in the obvious way. Similarly, $\ker(uh') = h^{-1}(U_u) = U_v$, by Lemma 3.1; and $\ker(vi) = U_w \cap U_v = U_v$. Therefore, α can also be defined in the obvious way, and the result is a monomorphism.

It remains to show that the bottom row is lexicographically ordered. Let $b \in B_v^+$, and choose $x \in K^*$ such that $v(x) = b$. It follows that $w(x) = \beta(b)$. Then either $w(x) > 0$ or $x \in U_w \cap R_v$. In the latter case, $h'(x) \in R_u$, and hence $uh'(x) \in A_u^+$ and $b = \alpha uh'(x) \in \alpha(A_u^+)$.

Conversely, suppose that for $b \in B_v$ either $\beta(b) > 0$ or $\beta(b) = 0$ and $b \in \alpha(A_u^+)$. In the first case, there exists $x \in K^*$ such that $w(x) = \beta(b) > 0$; and then $x \in m_w \subset R_v$; thus, $b = v(x) \in B_v^+$. In the second case, since uh' is onto, there exists $x \in U_w$ such that $\alpha uh'(x) = b$. Therefore, $uh'(x) \in A_u^+$; thus $h'(x) \in R_u$. Since $h^{-1}(R_u) = R_v$, this implies that $x \in R_v$; and thus $v(x) = b \in B_v^+$.

Since the short exact sequence (α, β): $A_u \to B_v \to C_w$ always splits when C_w is free (as a J-module), we conclude the following result.

COROLLARY 3.3. *Let C_w be the group of divisibility of a quasi-local domain R_w having residue field k, and let A_u be the semi-value group of a semi-valuation of k. If C_w is free, then $A_u \oplus C_w$ is also a group of divisibility.*

By (**2**(b), p. 32, Theorem 1), the domains having group of divisibility order-isomorphic to the ordered direct sum of copies of J are exactly the unique factorization domains (UFD)s. Thus, Corollary 3.3 is applicable whenever R_w is a quasi-local UFD. It is crucial that R_w be quasi-local. For example, the ordered direct sum C of a finite number, greater than or equal to 2, of copies of J can only be the group of divisibility of a finite intersection of greater than or equal to 2 discrete rank 1 valuation rings, and hence never produces a quasi-local R_w. It will follow from Theorem 4.1 and Corollary 4.4 that for such a group C, $A \oplus C$ is *never* a group of divisibility when A is not 0.

An example to which Corollary 3.3 does apply is, for instance, $R_w = k[X, Y]_{(X, Y)}$. Then, if A_w is the group of divisibility of R_w and A_u is any semi-value group of a semi-valuation u of k, then $A_u \oplus A_w$ is also a group of divisibility.

Let us investigate some further conditions under which the bottom row of (3.1) splits.

PROPOSITION 3.4. *The bottom row of* (3.1) *splits if and only if there exists a set $M = \{x_c \in K^*\}$, $c \in C_w$, such that $w(x_c) = c$ and $(x_c \cdot x_d)/x_{c+d} \in U_v$.*

Proof. Sufficiency. $\{v(x_c)\}$, $x_c \in M$, forms a subgroup of B_v isomorphic to C_w, and B_v is the direct sum of $\alpha(A_u)$ and this subgroup.

Necessity. If the sequence splits, then there exists a subgroup S of B_v such that β maps S isomorphically onto C_w. Let b_c, $c \in C_w$, denote that element of S such that $\beta(b_c) = c$. Then $b_c + b_d = b_{c+d}$. Choose $x_c \in K^*$ such that $v(x_c) = b_c$. Then $\{x_c\}$, $c \in C_w$, has the required properties.

Note that since the bottom row of (3.1) is lexicographically exact, when it splits, it splits lexicographically. Furthermore, in case the set M is a multiplicative system, then $x_c x_d = x_{c+d}$; thus, the second condition of Proposition 3.4 is then trivially satisfied.

Application. If C_w is assumed totally ordered and K is the quotient field of the group algebra $\mathscr{A}_k(C_w)$ of C_w over the field k, then a classical result of Krull (**13**, p. 164) asserts that there exists a valuation w of K having value group C_w and residue field k and such that $w(x_c) = c$ for each generator x_c of the algebra. The $\{x_c\}$, $c \in C_w$, thus trivially satisfy the conditions of Proposition 3.4. Therefore we have the following result.

COROLLARY 3.5. *If A_u is any semi-value group and C_w is any totally ordered group, then $B = A_u \oplus C_w$ is a group of divisibility.*

In view of Corollaries 3.3 and 3.5 and the remarks following Corollary 3.3, one might reasonably conjecture that if A is a group of divisibility and C is, say, an ordered direct sum of an *infinite* number of copies of J, then $A \oplus C$ is also a group of divisibility. We shall devote the remainder of this section to showing why this conjecture is false.

PROPOSITION 3.6. *Let R be a quasi-local domain with residue field k. If R is not a valuation ring, then R contains at least* card(k) *elements which are not associates.*

Proof. Since R is not a valuation ring, there exist non-zero $x, y \in R$ such that $x/y, y/x \notin R$. Let a and b be elements of R such that $h(a) \neq h(b)$, where h is the canonical homomorphism of R onto k.

Claim. $x + ay$ and $x + by$ are not associates. For, suppose that $x + ay = u(x + by)$ for some $u \in U(R)$. Then $(1 - u)x = (ub - a)y$. Since x/y, $y/x \notin R$, $1 - u$ and $ub - a$ must both be non-units of R. Therefore, $1 - h(u) = 0$ and $h(u)h(b) - h(a) = 0$. However, this implies that $h(b) = h(a)$, a contradiction to our choice of a, b. Thus, if S is a set of representatives in R for the elements of k, then $\{x + ay \mid a \in S\}$ is a set of non-associated elements of R having the same cardinality as S, and hence the same cardinality as k.

COROLLARY 3.7. *Let K be a field containing the quasi-local domain R, and let k be the residue field of R. If R is not a valuation ring, then* $\text{card}(K^*/U(R)) \geqq \text{card}(k)$.

By applying Corollary 3.7 along with Theorems 4.1 and 4.2 of the next section, one can now conclude the following.

COROLLARY 3.8. *If A is filtered and not* 0 *and C is not totally ordered, and if $A \oplus C$ is a semi-value group, then* $\text{card}(C) \geqq \text{card}(A)$.

Thus, in particular, $A \oplus C$ is *not* a group of divisibility whenever (i) A is filtered and not 0, (ii) C is not totally ordered, and (iii) $\mathrm{card}(C) < \mathrm{card}(A)$. In § 5 we shall construct another class of groups which are not groups of divisibility.

4. The converse situation and V-homomorphisms. Given a lexicographically exact sequence (α, β): $A_u \to B_v \to C_w$ and a semi-valuation v with semi-value group B_v, we investigate here the corresponding ring-theoretic situation and how nearly it approximates the assumptions of § 3; in particular, we show that under rather mild restrictions the situation is indeed that originally hypothesized in § 3.

THEOREM 4.1. *Suppose that* (α, β): $A_u \to B_v \to C_w$ *is lexicographically exact and* v *is a semi-valuation of* K *with semi-value group* B_v *and ring* R_v. *If* $w = \beta v$ *is also a semi-valuation, and if* $A_u \neq 0$, *then* (i) *the semi-valuation ring* R_w *of* w *is quasi-local* (*with a maximal ideal* m_w), (ii) $m_w \subset R_v \subset R_w$, *and* (iii) *there exists a semi-valuation* u *of the residue field* k *of* R_w *having semi-value group* A_u *and for which the commutative diagram* (3.1) *is valid.*

Proof. (i) We must see that $x_1, x_2, x_1 + x_2 \in K^*$ *and* $w(x_i) > 0$, $i = 1, 2$, implies that $w(x_1 + x_2) > 0$. Since w is a semi-valuation, certainly

$$w(x_1 + x_2) \geqq 0.$$

If $w(x_1 + x_2) = 0$, then $v(x_1 + x_2) \in \alpha(A_u)$. Since $A_u \neq 0$, there exists $a \nleqq 0$ in A_u; thus, $a' = \alpha(a) + v(x_1 + x_2) \in \alpha(A_u)$. However, $v(x_i) \geqq a'$, since $\beta(v(x_i) - a') = \beta v(x_i) = w(x_i) > 0$. Since v is a semi-valuation, we therefore have $v(x_1 + x_2) \geqq a'$, which implies that $0 \geqq \alpha(a)$, a contradiction to $a \nleqq 0$. Thus, $w(x_1 + x_2) > 0$.

(ii) is immediate from the lexicographic ordering.

(iii) Let h denote the canonical homomorphism of R_w onto k, and let h' be as in (3.1), the restriction of h to U_w. Define the homomorphism $\{uh'\}$ of (3.1) to be the map $\alpha^{-1}vi$. Since v is a semi-valuation, so also is $\{uh'\}$ (here we mean semi-valuation in the generalized sense of a map which is defined on a multiplicative *subgroup* K^* of a field and which satisfies the axioms (1.3)). Now define the homomorphism u of k^* onto A_u by the equation $\{uh'\} = uh'$. Such a u is well-defined since $\ker h' = 1 + m_w \subset U_v = \ker\{uh'\}$, by Lemma 3.1. Finally, u is a semi-valuation since h' preserves addition and $\{uh'\}$ is a semi-valuation.

To truly treat the converse situation to § 3, we should not assume in Theorem 4.1 that w is a semi-valuation. Thus, it is important to find sufficient conditions involving only the semi-valuation v and the groups A_u, B_v, C_w in order for w to be a semi-valuation. Of course, a sufficient condition is that β be a V-homomorphism. In Theorem 4.2 we completely characterize, in terms of the groups involved, the instances when β is a V-homomorphism. However, we show in Example 4.4 that w may well be a semi-valuation without β being a V-homomorphism; thus, Theorem 4.2 does not completely solve the problem.

THEOREM 4.2. *Suppose that* $(\alpha, \beta): A \to B \to C$ *is lexicographically exact. If* A *is filtered, then* β *is a* V*-homomorphism. If* A *is not filtered, then* β *is a* V*-homomorphism if and only if* C *satisfies:*

(4.1) *for* $c_1', c_2' \in C$, $c_2' > c'$ *for all* $c' < c_1'$ *implies that* $c_2' \geqq c_1'$.

Proof. Let $b, b_1, \ldots, b_n$ be elements of B such that $b \geqq \inf_B\{b_1, \ldots, b_n\}$ (where this expression is to be interpreted as explained in § 1 (a)). To see that β is a V-homomorphism one must check that $\beta(b) \geqq \inf_C\{\beta(b_1), \ldots, \beta(b_n)\}$, i.e. that $\beta(b) \geqq d'$ for any $d' \in C$ such that $d' \leqq \beta(b_1), \ldots, \beta(b_n)$. Let d be an element of B such that $\beta(d) = d'$; then $\beta(b_i - d) \geqq 0$.

Consider first the case that A is filtered. If $\beta(b_i - d) > 0$, then $b_i \geqq d$; while if $\beta(b_i - d) = 0$, then $b_i - d \in \alpha(A)$. In either case, $d \leqq b_i - a_i$ for some $a_i \in \alpha(A)$. Since A is filtered, there exists $a \in \alpha(A)$ such that $a \leqq a_1, \ldots, a_n$. Therefore, $d \leqq b_i - a$, and hence $d + a \leqq b_1, \ldots, b_n$. Then $b \geqq \inf_B\{b_1, \ldots, b_n\}$ implies that $b \geqq d + a$. Therefore, $\beta(b) \geqq \beta(d) = d'$, which proves the first assertion.

Now consider the case that A is not filtered and C satisfies (4.1). Suppose that c' is an element of C such that $c' < d'$. For any $c \in B$ such that $\beta(c) = c'$, $\beta(b_i) \geqq d' > c'$ implies $b_i \geqq c$. Therefore $b \geqq c$. Since this is true for any pre-image of c', we also conclude that $b \geqq c + a$ for any $a \in \alpha(A)$. If $b = c$, then $0 \geqq a$ for any $a \in \alpha(A)$, which would imply that $A = 0$, contrary to our assumption that A is not filtered. Thus, $b > c$ for any $c \in B$ such that $\beta(c) = c'$. This implies that $\beta(b) > \beta(c) = c'$. Therefore by (4.1), $\beta(b) \geqq d'$.

For the final assertion of the theorem, we assume that A is not filtered and that β is a V-homomorphism. Let c_1' and c_2' be elements of C such that $c_2' > c'$ for all $c' < c_1'$. Choose a pre-image c_1 of c_1' in B. Since A is not filtered, we can find $a_1, a_2 \in \alpha(A)$ such that there does not exist $a \in \alpha(A)$ with $a \leqq a_1, a_2$. Suppose that $f \leqq c_1 + a_1$, $c_1 + a_2$. Then $\beta(f) \leqq c_1'$. If $\beta(f) = c_1'$, then $f = c_1 + a$ for some $a \in \alpha(A)$; and then $a \leqq a_1, a_2$, a contradiction. Thus, $\beta(f) < c_1'$. Therefore, by our initial assumption, $c_2' > \beta(f)$. Hence, for any pre-image $c_2 \in B$ of c_2', $c_2 \geqq f$. By the choice of f and the assumption that β is a V-homomorphism, we conclude that $c_2' \geqq \inf\{\beta(c_1 + a_1), \beta(c_1 + a_2)\} = c_1'$.

COROLLARY 4.3. *If* $(\alpha, \beta): A \to B \to C$ *is lexicographically exact and* C *is lattice-ordered, then* β *is a* V*-homomorphism.*

Proof. Any lattice-ordered group satisfies (4.1).

It is possible for the maps v, w of the commutative diagram (3.1) (with lexicographically exact bottom row) to be semi-valuations without β being a V-homomorphism. In view of Theorem 4.2, to show this we need only construct an example for which A_u is not filtered and C_w does not satisfy (4.1).

Example 4.4. Let k_1 be a field and let Z, X, Y be indeterminates over k_1. Set $k_0 = k_1(Z)$, and construct, as in § 2, a semi-valuation w of $K = k_0(X, Y)$ having semi-value group $C_w = D \oplus J$, where D is the semi-value group

$k_0(X)^*/k_0^*$. As remarked in § 2, the ring of w has the form $k_0 + m$, where m is the maximal ideal of the Y-adic valuation of K over $k_0(X)$; thus, in particular, this ring is quasi-local with residue field k_0. Therefore, the results of § 3 apply. Let $A_u = k_0^*/k_1^*$ and let u be the corresponding semi-valuation of k_0. Let v be the composite of w and u. Then the commutative diagram (3.1) is valid, with lexicographically exact bottom row. Certainly, A_u is not filtered. Moreover, C_w does not satisfy (4.1) since if $d \neq 0 \in D$, then $(d, 0) > c$ for all $c \in D \oplus J$ such that $c < (0, 0)$, but $(d, 0) \ngeqq (0, 0)$.

Note that this example even splits, i.e. the semi-value group B_v is order-isomorphic to $A_u \oplus (D \oplus J)$. For, let

$$M = \{\xi \in K^* | \xi = (f(X)/g(X))Y^i, f(X), g(X) \in k_0[X], f(0) = g(0) = 1\}.$$

By definition of w, $w(\xi) = (d, i)$, where d is the residue class of $f(X)/g(X)$ in $k_0(X)^*/k_0^*$. It is then easy to check that w gives a one-to-one correspondence between the elements of M and the elements of $C_w = D \oplus J$. Moreover, M is a multiplicative system, hence, it satisfies the requirements of Proposition 3.4.

5. Groups which are not groups of divisibility. In (**11**), Jaffard has given an example of a filtered group which is not a group of divisibility. We shall show in this section how to construct a large class of such groups. In particular, Theorem 5.3 will provide additional groups (to those given at the end of § 3) of the form $A \oplus C$, where A is not 0 and C is lattice, which are not groups of divisibility.

LEMMA 5.1. *Suppose that* (α, β): $A \to B \to C$ *is lexicographically exact and* v *is a map of* K^* *onto* B, *and let* $w = \beta v$. *If* $A \neq 0$ *and if there exist* $x, y \in K^*$ *such that* $w(x + y) < w(x), w(y)$, *then* v *is not a semi-valuation.*

Proof. $w(x + y) < w(x)$, $w(y)$ implies $v(x + y) + a \leqq v(x)$, $v(y)$ for all $a \in \alpha(A)$. Therefore, if v is a semi-valuation, we must have $v(x + y) \geqq v(x + y) + a$, for all $a \in \alpha(A)$. However, this implies that $A = 0$, a contradiction.

Discussion. Let C be an ordered group and let l be an embedding of C in an ordered direct product $D = \pi D_u$ of filtered groups. Let p_u be the projection of D on D_u, and let $l_u = p_u l$. Then $l = \pi l_u$. Since the p_u are always V-homomorphisms, it follows that l is a V-homomorphism if and only if all l_u are. Moreover, if w is a map of K^* onto C and l is a V-homomorphism, then w is a semi-valuation if and only if each $u = l_u w$ is a semi-valuation. If, moreover, w is a semi-valuation and R_w and R_u are the semi-valuation rings of w and u, respectively, then $R_w = \bigcap R_u$.

If C is a lattice-ordered group, Lorenzen has proved that there always exists such a V-embedding l of C in a product of totally ordered groups (**12**, p. 37, Theorem 2). Also, by Jaffard's theorem, when C is lattice there exists a semi-valuation w of a field K having semi-value group C; and it thus follows that in this case the resulting semi-valuations u are actually valuations. We shall present a new proof of Jaffard's theorem in § 6 from this point of view.

LEMMA 5.2. *Suppose that C is a lattice-ordered group and l is a V-embedding of C in the ordered direct product $D = \pi D_u$ of totally ordered groups, and suppose that w is a semi-valuation of a field K with semi-value group C. If C satisfies:*

(5.1) *there exist $c_1, c_2 \in C$ such that $c_1 \not\geq c_2$ and $c_2 \not\geq c_1$ and such that $l_u(c_1) \neq l_u(c_2)$ for all u,*

then there exist $x_1, x_2 \in K^*$ such that $w(x_1 + x_2) < w(x_1), w(x_2)$.

Proof. Choose $x_1, x_2 \in K^*$ such that $w(x_i) = c_i$. As in the Discussion, let $u = l_u w$. Since u is a *valuation* and $u(x_1) \neq u(x_2)$, $u(x_1 + x_2) = \inf\{u(x_1), u(x_2)\}$. Then, since $lw = \pi u$, $w(x_1 + x_2) = \inf\{w(x_1), w(x_2)\}$ also. However, $\inf\{w(x_1), w(x_2)\} < w(x_1), w(x_2)$ by our hypothesis that c_1 and c_2 are unrelated.

Note that there exist lattice-ordered groups satisfying (5.1); for example, any ordered direct product of at least two copies of J.

THEOREM 5.3. *Let C be a lattice-ordered group which satisfies* (5.1), *and suppose that (α, β): $A \to B \to C$ is lexicographically exact and A is not* 0. *Then B is not a group of divisibility.*

Proof. Suppose that B is a group of divisibility, and let v be a semi-valuation of a field K having semi-value group B. By Corollary 4.3, β is a V-homomorphism. Therefore, $w = \beta v$ is a semi-valuation. By Lemma 5.2, there exist $x_1, x_2 \in K^*$ such that $w(x_1 + x_2) < w(x_1), w(x_2)$; thus, by Lemma 5.1, v is not a semi-valuation.

COROLLARY 5.4. *Any ordered group A is an ordered subgroup of an ordered group B which is not a group of divisibility.*

Proof. Let C be a lattice-ordered group which satisfies (5.1), and take $B = A \oplus C$.

We conclude this section by mentioning an unpublished result of R. L. Pendleton; namely, Pendleton has classified the filtered orders on J and has shown that the only ones which produce groups of divisibility are the two obtained by taking as positive elements either J^+ or $-J^+$.

6. Extensions of semi-valuations. The construction of § 3 may also be regarded as a result on extensions of semi-valuations. For example, Corollary 3.5 may be rephrased:

Let u be a semi-valuation of a field k with semi-value group A_u. Let K be the quotient field of $\mathscr{A}_k(C)$, where C is a totally ordered group. Then u extends to a semi-valuation v of K having semi-value group $B_v = A_u \oplus C$.

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Under the additional assumption that u is additive (see (1.4)), the following theorem includes both this result and a classical theorem on extensions of valuations to simple transcendental field extensions (**2**(a), p. 160, Lemma 1).

THEOREM 6.1. *Let u be an additive semi-valuation of k with semi-value group A_u, let A_u and C be ordered subgroups of a larger ordered group, and let K be the quotient field of $\mathscr{A}_k(C)$. If C satisfies:*

(6.1) $\qquad c \neq 0 \in C$ *implies* $c + a$ *is related to* 0 *for all* $a \in A_u$,

then u extends to a semi-valuation v of K, the semi-valuation v being defined by

$$v(s_1X^{c_1} + \ldots + s_nX^{c_n}) = \inf\{u(s_i) + c_i\}, \tag{6.2}$$

where $c_i \neq c_j$ for $i \neq j$, and $s_1, \ldots, s_n \neq 0$.

Note that (6.1) assures that $u(s_1) + c_1, \ldots, u(s_n) + c_n$ are totally ordered, and hence that $\inf\{u(s_i) + c_i\}$ exists. The theorem itself is merely intended as a passing remark, and since its proof is very similar to (**2**(a), p. 160, Lemma 1), it will be omitted.

We have already remarked that a semi-valuation u is additive if and only if its ring R_u is quasi-local. In the case of a lexicographic extension, we also have the following result.

LEMMA 6.2. *Suppose that v is a semi-valuation of K with semi-value group B_v and (α, β): $A \to B_v \to C$ is lexicographically exact. If $w = \beta v$ is an additive semi-valuation, then for $x, y \in K^*$, $w(x) < w(y)$ implies $v(x + y) = v(x)$.*

Proof. $w(x) < w(y)$ implies $y/x \in m_w \subset R_v$; where m_w is the maximal ideal of the quasi-local ring R_w. By Lemma 3.1, $1 + y/x \in U_v$. Therefore, $v(1 + y/x) = 0$, hence $v(x + y) = v(x)$.

Note that the above lemma does *not* assert that v is additive when w is. For it is possible, for example, to extend a non-additive semi-valuation u to a semi-valuation v of the quotient field K of $\mathscr{A}_k(C_w)$, where C_w is totally ordered; and the resulting v is then certainly not additive.

If in Theorem 6.1 the semi-value group of v is a lexicographic extension of a totally ordered group C, then (6.2) is in many instances the only possible way of extending u. For, we have the following result.

PROPOSITION 6.3. *Let $K = k(x)$ be a simple extension of a field k, and suppose that v is a semi-valuation of K with semi-value group B_v. If moreover (α, β): $A \to B_v \to C$ is lexicographically exact and $w = \beta v$ is a valuation such that $nw(x) \in w(k^*)$, $n \in Z$, implies $n = 0$, then*

$$v(s_0 + s_1x + \ldots + s_nx^n) = \inf\{v(s_i) + iv(x)\},$$

where $s_0, \ldots, s_n \in k^*$.

Proof. $w(s_ix^i) \neq w(s_jx^j)$ for $j > i$; for if equality holds, then

$$(j - i)w(x) \in w(k^*),$$

a contradiction. Therefore, by Lemma 6.2, $v(s_0 + s_1x + \ldots + s_nx^n) = \inf\{v(s_ix^i)\}$.

We return now to the situation described in the Discussion in order to give, among other things, a new proof of Jaffard's theorem. Let $D = \pi D_u$ be an ordered direct product of ordered groups, let B be an ordered subgroup of D, let C be an arbitrary non-empty subset of D, and let p_u denote the projection map of B into D_u. Let $(X) = \{X_c\}$, $c \in C$, be indeterminates over a field k, and let $K = k(X)$. We henceforth *assume that B is the semi-value group of a semi-valuation v of k and that the maps $u = p_uv$ are also semi-valuations of k.* (One way of obtaining this situation is to start with a domain R_v which is an intersection of domains R_u, all contained in a field k. Then take u to be the natural semi-valuation map of k^* onto $k^*/U(R_u) = D_u$, and let $v = \pi u$. It follows that v is a semi-valuation of k with semi-valuation ring R_v and semi-value group $B \subset \pi D_u$, and $u = p_uv$.)

If each u extends to a semi-valuation u' of K such that $u'(X_c) = c_u$, then $v' = \pi u'$ is an extended semi-valuation of v to K having semi-value group $B' \supset B + C$ (here, $B + C = \{b + c | \, b \in B, c \in C\}$; it need not be a group). Moreover, the semi-valuation rings satisfy $R_v = \bigcap R_u$, $R_{v'} = \bigcap R_{u'}$, and $R_{v'} \cap k = R_v$.

It is important to know what the group B' looks like. Under the following hypotheses, we can describe B' in some interesting cases:

(i) the D_u are all totally ordered,

(ii) u extends to u' by defining $u'(X_c) = c_u$, and

(6.3) $u'(f(X)) = \inf\{u'(M_i)\}$, where M_i are the distinct monomials occurring in $f(X) \in k[X]$.

We assume in the following that the hypotheses (6.3) are in effect; see (**2**(a), p. 160) for the existence of such u'. Then D is a lattice-ordered group; thus, if $(\ )_l$ denotes the smallest lattice V-subgroup of D containing ( ), we have:

$$B + C \subset B' \subset (B + C)_l.$$

Applications. (1) Consider the case where $C = 0$. The resulting ring $R_{v'}$ is called the *Kronecker function ring* of R_v with respect to the set of valuations $\{u\}$ (**14**, pp. 558–561). It is easily established that the Kronecker function ring is Bezoutian, that is, that every finitely generated ideal is principal; see (**14**, p. 559) or Theorem 6.6. From this fact and the equality $R_{v'} = \bigcap R_{u'}$, it follows that the semi-value group B' is lattice-ordered and is a V-subgroup of D. Thus, $B \subset B' \subset B_l$ implies $B' = B_l$.

(2) Consider the case where $B = 0$ and C is a lattice-ordered group. Any such C can be V-embedded in a product of totally ordered groups $D = \pi D_u$ (**12**, p. 37); thus, $C = C_l$, and hence $C \subset B' \subset C_l$ implies $C = B'$. Thus, we have proved the following theorem.

THEOREM (Jaffard). *Any lattice-ordered group is a group of divisibility.*

One can establish the following general fact about B'.

THEOREM 6.4. *Assume that* (6.3) *is valid. If* $B = B_l$, *then* $B' = (B')_l$.

Proof. It is sufficient to see that $b_1', b_2' \in B'$ implies $b_1' \wedge b_2' \in B'$, where $\wedge$ denotes infimum in D. Choose $x_i \in K^*$ such that $v'(x_i) = b_i'$. $x_i = y_i/z$, y_i, $z \in k[X]$. Then $v'(x_1) \wedge v'(x_2) = v'(y_1) \wedge v'(y_2) - v'(z)$. Thus, it is sufficient to see that $v'(y_1) \wedge v'(y_2) \in B'$. Write $y_i = \sum s_{ij}P_j$, where $s_{ij} \in k$ and the P_j are distinct power products in the (X), and where possibly some $s_{ij} = 0$. Then by the definition of v',

$$v'(y_i) = \bigwedge_j [v(s_{ij}) + v'(P_j)]$$

(where we omit those j for which $s_{ij} = 0$). Therefore,

$$\begin{aligned} v'(y_1) \wedge v'(y_2) &= \bigwedge_j [v(s_{1j}) + v'(P_j)] \wedge \bigwedge_j [v(s_{2j}) + v'(P_j)] \\ &= \bigwedge_j [[v(s_{1j}) \wedge v(s_{2j})] + v'(P_j)]. \end{aligned}$$

Since B is a lattice V-subgroup of D (i.e., since $B = B_l$), $v(s_{1j}) \wedge v(s_{2j}) \in B$. Therefore, there exist $c_j \in k$ such that $v(c_j) = v(s_{1j}) \wedge v(s_{2j})$. Then $v'(\sum c_jP_j) = v'(y_1) \wedge v'(y_2) \in B'$.

COROLLARY 6.5. $B = B_l$ *implies* $B' = (B + C)_l$.

Finally, we shall show that in case $0 \in C$, the ring-theoretic situation is similar to that of the Kronecker function ring.

THEOREM 6.6. *Assume that* (6.3) *is valid. If* $0 \in C$, *then* $B' = (B')_l$ *and* $R_{v'}$ *is Bezoutian.*

Proof. Let $\xi, \eta \in k(X)$, where $\xi = f(X)/h(X)$, $\eta = g(X)/h(X)$, $f, g, h \in k[X]$. Choose $m >$ deg X_0 in $f(X)$, and let $\gamma = \xi + X_0^m\eta$. Then

$$v'(\gamma) = (v'(f) \wedge v'(g)) - v'(h) = v'(\xi) \wedge v'(\eta).$$

Therefore, $B' = (B')_l$. Moreover, then $\gamma R_{v'} = (\xi, \eta)R_{v'}$; thus $R_{v'}$ is Bezoutian.

The above proof actually shows that when $0 \in C$, the ring $R_{v'}$ is Bezoutian with the further property that for any $\xi, \eta \in R_{v'}$ there exists $r \in R_{v'}$ such that $(\xi, \eta) = (\xi + r\eta)$. Such rings are exactly the Bezoutian rings with 1 in the stable range of (**4**); see, in particular (**4**, Proposition 5.1).

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