EMBEDDING CIRCLE-LIKE CONTINUA IN THE PLANE

R. H. BING

A finite sequence of open sets $L_1, L_2, \ldots, L_n$ is called a linear chain if each L_i intersects only the L's adjacent to it in the sequence. The finite sequence is a circular chain if we also insist that the first and last links intersect each other. The 1-skeleton of the covering is an arc for a linear chain and a simple closed curve for a linear chain.

A compact metric continuum X is called snake-like if for each $\epsilon > 0$, X can be covered by a linear chain of mesh less than ϵ. Likewise, X is called circle-like if for each $\epsilon > 0$, X can be irreducibly covered with a circular chain of mesh less than ϵ. This definition is more restrictive than that given in (**3**, p. 210) for there a pseudo-arc is not circle-like but here it is. The present usage is in keeping with definitions of Burgess.

When a person thinks of a linear chain in the plane, he is likely to think of a finite sequence of open disks. However, the definition of a chain does not insist that the links be open disks or even that they be connected. Hence, a person who thinks only of chains with connected links is considering only a special sort of embedding as pointed out by Example 3 of (**1**). Theorems 10 and 11 show that in the category sense, most plane continua can be covered by chains whose links are small open disks. Theorems 5, 6, and 7 show that even for chainable continua which cannot be covered by chains with small connected links, it is the embedding that is special rather than the topology of the embedded continua.

1. Circling. We use $C(L_1, L_n, \ldots, L_2)$ to denote the chain C with links $L_1, L_2, \ldots, L_n$. In dealing with circular chains, we use L_{i-1} as the link preceding L_i and in case $i = 1$, we interpret L_{i-1} to mean L_n. If $i = n$, the link L_{i+1} following L_n is L_1. To facilitate this convention, we understand L_0 to be another name for L_n.

Number of times C_{i+1} circles C_i. Suppose the circular chain C_{i+1} $(A_1, A_2, \ldots, A_n)$ is a refinement of the circular chain $C_i(B_1, B_2, \ldots, B_m)$. Let $f'(A_k)$ be the subscript of one of the elements of C_i containing A_k. In some cases there are two choices for $f'(A_k)$ but a definite one of them is selected. We note that $f'(A_k), f'(A_{k+1})$ are adjacent or one is m and the other is 1. Also, $f'(A_1), f'(A_n)$ are adjacent in this sense.

Let f be a map of the integers $(0, 1, 2, \ldots, n)$ into the integers defined as follows.

Received December 12, 1960. Work on this paper was supported by contract NSF–G11665.

https://doi.org/10.4153/CJM-1962-009-3 Published online by Cambridge University Press

$$f(0) = f'(A_n),$$

$$f(i+1) = \begin{cases} f(i) - 1 \text{ if } f'(A_{i+1}) \text{ precedes } f'(A_i), \\ f(i) \text{ if } f'(A_{i+1}) = f'(A_i), \\ f(i) + 1 \text{ if } f'(A_{i+1}) \text{ follows } f'(A_i). \end{cases}$$

Thus it is true that $f(i) \equiv f'(A_i) \bmod m$ and $f(0) = f'(A_n) \equiv f(n) \bmod m$. Recall that we are using the convention that $f'(A_{i+1})$ precedes $f'(A_i)$ if $f'(A_{i+1}) = m$ and $f'(A_i) = 1$ while $f'(A_{i+1})$ follows $f'(A_i)$ if $f'(A_{i+1}) = 1$ and $f'(A_i) = m$—so in a sense we sometimes treat m as 0. We note that $f'(A_i)$ and $f(i)$ may differ by a multiple of m as may $f(0)$ and $f(n)$.

The number of times that C_{i+1} circles C_i is defined to be

$$|f(n) - f(0)|/m.$$

We note that this number is invariant under taking different elements of C_{i+1}, C_i as the first element or in ordering the elements in a counter fashion. Nor does it matter which of the two choices for $f'(A_k)$ we made when we had a choice.

THEOREM 1. *If C_2 circles C_1 x times and C_3 circles C_2 y times, then C_3 circles C_1 xy times.*

Proof. Suppose $(A_1, A_2, \ldots, A_m)$, $(B_1, B_2, \ldots, B_n)$, and $(D_1, D_2, \ldots, D_r)$ are the links of C_3, C_2, C_1 respectively and f, g, h are the functions determining how many times C_3 circles C_2, C_2 circles C_1, and C_3 circles C_1. We suppose that the first links of C_2 and C_1 are selected so that $f(0) = g(0) = 0$ and that the choice of $h'(A_i)$ is such that

$$h'(A_i) = g'(B_{f'(A_i)}).$$

We note that g is only defined on the integers between 0 and n inclusive. For convenience we suppose $g(n) \geqslant g(0)$. Suppose that g is extended to all the integers so that

$$g(i + kn) = kxr + g(i).$$

It follows by induction that

$$h(i) = g(f(i)).$$

Hence

$$|h(m) - h(0)| = |g(f(m)) - 0| = g(\pm yn)| = |\pm xyr| = xyr.$$

THEOREM 2. *Suppose X is a compact continuum in a locally connected metric space and ϵ is a positive number such that X cannot be covered by any ϵ-chain. If C_1, C_2 are two circular ϵ chains covering X such that C_2 is a refinement of C_1, then there is a chain C_3 circling C_1 a positive number of times such that each link of C_3 is a component of a link of C_2.*

Note that we do not assert that C_3 covers X.

Proof. Let $D_1, D_2, \ldots, D_m$ be a finite collection of components of links of C_2 such that the collection covers X. Suppose the D's are ordered so that D_{i+1} intersects $\sum_{j=1}^{i} D_j$.

Let $f'(D_k)$ be a subscript of a link of C_1 containing D_k and f be a map of the integers $(1, 2, \ldots, m)$ into the integers defined as follows:

$$f(1) = f'(D_1).$$

After $f(1), f(2), \ldots, f(i-1)$ have been defined, pick a $j < i$ such that $D_i \cdot D_j \neq 0$ and define

$$f(i) = \begin{Bmatrix} f(j) - 1 \text{ if } f'(D_i) \text{ precedes } f'(D_j), \\ f(j) \text{ if } f'(D_i) = f'(D_j), \\ f(j) + 1 \text{ if } f'(D_i) \text{ follows } f'(D_j) \end{Bmatrix}.$$

Case 1. If f has the property that $D_i \cdot D_k \neq 0$ implies that $|f(i) - f(k)| \leqslant 1$, X can be covered by a linear ϵ-chain each of whose links is the sum of D's. The link containing D_j is the sum of all D_i's such that $f(j) = f(i)$. In this case the definition of $f(i)$ was independent of the D_j picked that intersects D_i.

Case 2. Suppose there are two elements D_i, D_j such that $D_i \cdot D_j \neq 0$ but $|f(i) - f(j)| > 1$. Let n be a minimal integer such that there is an ordered subcollection $(E_1, E_2, \ldots, E_n)$ of the D's such that E_{i+1} intersects $\sum_{j=1}^{i} E_j$ and there is a g defined on the E's in a fashion similar to that in which f was defined on the D's that is not independent of the $j < i$ selected such that $E_i \cdot E_j \neq 0$. We shall show that a reordering of the E's gives a circular chain which circles in C_1. A reordering is needed since E_3 may intersect E_1 instead of E_2.

The reordering is accomplished as follows. Let $F_1 = E_1$, $F_2 = E_2$, F_3 be one of the remaining E's intersecting F_2, F_4 one of the remaining E's intersecting $F_3, \ldots$. If we come to a place where we cannot select another F, we have used up our E's or else our collection of E's was not minimal with respect to there being no independent g. Also, F_n (the last F) intersects an F_r such that for g defined on the F's, $|g(n) - g(r)| > 1$. Furthermore, $r = 1$ or we could delete $F_1, F_2, \ldots, F_{r-1}$ from our minimal collection. If F_n intersects any F_j $(j \neq 1, n-1, n)$, then $F_{j+1}, \ldots, F_{n-1}$ could have been deleted from our minimal collection. Hence F_n intersects only F_1 and F_{n-1}. A similar argument shows that each F intersects only the F's adjacent to it in $F_1, F_2, \ldots, F_n, F_1$.

THEOREM 3. *Suppose ϵ is a positive number, X is a circle-like continuum in the plane that cannot be covered by a linear ϵ-chain, and C_1 is a circular ϵ-chain in the plane covering X. Then there is a positive number n such that no chain C_2 covering X circles C_1 more than n times.*

Proof. Let T be a triangulation of the plane such that each closed 2-simplex of T that intersects X lies in a link of $C_1(L_1, L_2, \ldots, L_m)$. For each link L_i of C_1, let S_i be the collection of closed 2-simplexes of T that intersect X and lie in L_i. Let K be the collection of 1-simplexes common to an element of S_1 and an element of $S_2 - S_1$. The number n promised by the theorem is the number of elements in K.

Assume some chain C_2 covers X and circles C_1 more than n times. We suppose that the links of C_2 are "cut down to size" so that each point of a link of C_2 lies in a 2-simplex of T that intersects X. It follows from Theorem 2 that there is a chain C_3 with connected links which circles C_2 a positive number of times and from Theorem 1 that such a C_3 would circle C_1 more than n times. We are not interested in whether or not C_3 covers X.

Let $(D_1, D_2, \ldots, D_r)$ be the links of C_3 and f be a map of the integers $(0, 1, 2, \ldots, r)$ into the integers showing that C_3 circles C_1 more than n times. Then

D_r lies in the $f(0)$th link of C_1,
D_i lies in the $(f(i) \bmod m)$th link of C_1,
$|f(i) - f(i+1)| \leqslant 1$, and
$|f(r) - f(0)| = n'm$ where n' is an integer larger than n.

Let $A_1A_2, A_2A_3, \ldots, A_rA_1$ be a collection of polygonal arcs such that

$$A_iA_{i+1} \subset D_i, \qquad A_rA_1 \subset D_r,$$

$A_1A_2 + A_2A_3 + \ldots + A_rA_1$ is a simple closed curve J, and
J does not contain an endpoint of an element of K.

We suppose that no vertex of J lies on an element of K and no straight line segment in J intersects two elements of K.

Let $P_1, P_2, \ldots, P_s$ be the vertices of J ordered in a natural fashion on J. We suppose that each A_i is a vertex of J, that $P_1 = A_1$, and the first P's are vertices of A_1A_2. Let g be a map of the integers $(0, 1, 2, \ldots, s)$ into the integers such that

$g(j) = f(i)$ if P_j is a point of $(A_iA_{i+1} - A_{i+1})$,
$g(j) = f(r)$ if P_j is a point of $(A_rA_1 - A_1)$,
$g(0) \equiv g(s) \bmod m$ and is adjacent to $g(1)$.

Then

$$|g(s) - g(0)| = |f(r) - f(0)| = n'm.$$

We now adjust g on certain integers. If P_j lies in an element of S_1, then it lies in D_1 but perhaps also in either D_2 or D_r. We wish to treat it as though it were only in D_1. If P_j is in an element of $S_2 - S_1$, we treat it as though it did not lie in D_1. So as not to disturb the adjusted g on 0 and s, we suppose that P_1 lies in an element of S_1. We define the adjusted g as follows.

$g'(j) = g(j)$ unless P_j lies in an element of $S_1 + S_2$,
$g'(j) = f(1)$ if P_j lies in an element of S_1,
$g'(j) = f(2)$ if P_j lies in an element of $S_2 - S_1$.

We note that

https://doi.org/10.4153/CJM-1962-009-3 Published online by Cambridge University Press

$$|g'(i+1)| - g'(i)| \leqslant |,$$
$$|g'(s) - g'(0)| = n'm.$$

Let i be an integer such that one of $g'(i)$, $g'(i+1)$ is 1 mod m and the other is 2 mod m. The corresponding segment P_iP_{i+1} of J crosses an element of K. The crossing is from an element of S_1 to an element of $S_2 - S_1$ if $g'(i+1) > g'(i)$ and from an element of $S_2 - S_1$ to an element of S_1 if $g'(i+1) < g'(i)$. There must be n' more crossings one way than the other since $|g'(s) - g'(0)| = n'm$. We show that this is impossible since K has only n elements and on no one of these elements can the number of crossings one way exceed by more than 1 the number of crossings in the other direction.

Suppose K_i is an element of K and these crossings are ordered on K_i rather than on J. Since as one moves along K_i he alternately moves in and out of the disk bounded by J as he passes these crossings, J crosses K_i from different directions for two adjacent crossings (adjacent on K_i). Hence, the number of crossings of K_i by J in one direction does not exceed by more than 1 the number of crossings in the other direction and it is false that $n' > n$. The contradiction arose from the false assumption that there is a chain C_2 in X which circles C_1 more than n times.

Question. Theorem 3 gives no clue as to the minimum size of n. Suppose C_1 has i links and no link of C_1 has more than j components. Perhaps the minimum n is a function of i and j. An early conjecture that $n = j$ turned out to be false on considering a circle J in the plane covered by a circular chain of twelve links $D_1, D_2, \ldots, D_{12}$ such that each D_i is the interior of a round disk. One notes that J circles the chain consisting of $(D_1 + D_5 + D_9, D_2 + D_6 + D_{10}, D_3 + D_7 + D_{11}, D_4 + D_8 + D_{12})$ three times and each link has three components. However, it is possible to join D_1 to D_5 by a path in the exterior of J, D_6 to D_{10} by a path in the exterior of J, D_3 to D_7 by a path on the interior of J and D_8 to D_{12} by a path on the interior of J so as to get a chain of four links ($D_1 + D_5 + D_9$ + path from D_5 to D_9), ($D_2 + D_6 + D_{10}$ + path from D_6 to D_{10}), ($D_3 + D_7 + D_{11}$ + path from D_3 to D_7), ($D_4 + D_8 + D_{12}$ + path from D_8 to D_{12}) each of which has only two components and J circles the chain three times. Here we spoke of a curve rather than a chain circling but no confusion should result.

There are several proofs of the result that ordinary solenoids cannot be embedded in the plane (the circle is a degenerate solenoid which can be embedded). One of these follows from Bing's result **(3)** that the simple closed curve is the only homogeneous bounded plane continuum that contains an arc. Another follows from the results that for a decomposition of E^3 whose only non-degenerate element is an ordinary solenoid, the decomposition space is not simply connected but for a decomposition of E^3 whose only non-degenerate element is a plane continuum, the decomposition space is simply connected **(0)**. However, Theorem 3 gives a more straightforward proof of this result so we state it as a corollary.

https://doi.org/10.4153/CJM-1962-009-3 Published online by Cambridge University Press

COROLLARY. *The circle is the only solenoid which can be embedded in the plane.*

2. Plane embeddings. Theorem 3 with its corollary showed that certain circle-like continua cannot be embedded in the plane. In this section we show what sorts of circle-like continua can be embedded in the plane.

THEOREM 4. *Suppose X is a circle-like continuum and $C_1, C_2, \ldots,$ is a sequence of circular chains covering X such that mesh C_i approaches 0 as i increases without limit and C_{i+1} circles C_i exactly once. Then there is a homeomorphism h of X into the plane such that for each positive number ϵ, $h(X)$ can be covered by a circular chain each of whose links is the interior of a round disk of diameter less than ϵ.*

Proof. By dropping out certain elements from the chains $C_1, C_2, \ldots,$ one can get a sequence whose meshes converge to 0 very fast. Hence, we suppose with no loss of generality that C_{i+1} has such small mesh that each link L of C_{i+1} is contained in a link L' of C_i such that the distance from L to the boundary of L' is more than twice the mesh of C_{i+1}.

Suppose C_i has n_i links. Let f_i be a map of the integers $(0, 1, 2, \ldots, n_{i+1})$ into the integers such that

the n_{i+1}th link of C_{i+1} lies in the $f_i(0)$th link of C_i,
the jth link of C_{i+1} lies in the $(f_i(j) \bmod n_i)$th link of C_i,
$|f_i(j) - f_i(j+1)| \leqslant 1$,
$|f_i(n_{i+1}) - f_i(0)| = n_i$,
one of $f_i(j-1), f_i(j+1)$ is equal to $f_i(j)$.

That it is possible to place the last restriction on f_i follows from the fact that each link of C_{i+1} was required to lie way inside a link of C_i.

Let J_1 be a polygonal simple closed curve in the plane which is the sum of n_1 arcs

$$A_1A_2, A_2A_3, \ldots, A_{n_1}A_1$$

such that each of these arcs is of length less than 1 and if two of the arcs intersect each other, the intersection is an endpoint of each. A first approximation to h takes X onto J_1 by taking the part of X in the ith link of C_1 onto the arc A_iA_{i+1}.

Let $F_1(L_1, L_2, \ldots, L_{n_1})$ be a circular chain of mesh less than 1 covering J_1 such that L_i covers A_iA_{i+1}, L_{n_1} covers $A_{n_1}A_1$, and the sum of the links of F_1 is the sum of the elements of a circular chain each of whose elements is the interior of a round disk of diameter less than 1. Note that as we build the F_i's in the plane to match the C_i's, we shall not insist that the links of the F_i's be circular disks but rather that the sum of these links be the sum of the links of a circular chain each of whose links is the interior of a round disk.

We want to build a chain F_2 of mesh less than $\frac{1}{2}$ such that F_2 has the same number of links as C_2, the jth link of F_2 lies in the $(f_1(j) \bmod n_1)$th link of F_1, and the sum of links of F_2 is the sum of the elements of a circular chain each of whose elements is the interior of a round disk of diameter less than $\frac{1}{2}$. To facilitate the construction of such an F_2 we build a simple closed curve J_2 very close to J_1 that follows J_1 around in the same fashion that C_2 follows C_1. There is a problem in showing that there is such a J_2 since one might suppose that if P is a path near J_1, that follows J_1 around in a particular way, then P would have to cross itself if its two ends were to be joined. The following two paragraphs show that as long as C_2 circles C_1 exactly once, P need not cross itself and there is a J_2.

If g is a map of the interval $(0, 2\pi)$ into the positive reals such that $g(2\pi) - g(0) = 2\pi$ and ϵ is a small positive number, then

$$h(1, \theta) = (1 - \epsilon(g(\theta) - \theta), g(\theta))$$

is a homeomorphism of the unit circle onto a simple closed curve where the equation is given in polar co-ordinates since if $g(\theta_1) - \theta_1 = g(\theta_2) - \theta_2$ and $g(\theta_1) = g(\theta_2)$, then $\theta_1 = \theta_2$.

Let J_2 be a polygonal simple closed curve very close to J_1 which is the sum of the arcs $B_1B_2, B_2B_3, \ldots, B_{n_2}B_1$ each of which is of length less than $\frac{1}{2}$ such that $B_iB_{i+1} \subset L_j$ if $f_1(i) \equiv j \bmod n_1$. In getting the B_1B_{i+1}'s to be of diameter less than $\frac{1}{2}$ we make use of the conditions that the A_jA_{j+1}'s have length less than 1 and one of $f_1(k-1), f_1(k+1)$ is equal to $f_1(k)$. To see that J_2 need not cross itself we regard J_2 as a circle (rather than a polygon) with A_i the point $(1, 2\pi i/n_2)$ in polar co-ordinates. Let g be defined on the values $2\pi j/n_2$ $(j = 0, 1, \ldots, n_2)$ by $g(2\pi j/n_2) = 2\pi f_1(j)n_1$. Then extend g linearly to all of $[0, 2\pi]$. Suppose $f_1(i-1) \neq f_1(i) = f_1(i+1) = \ldots = f_1 (i+k) \neq f_1(i+k+1)$. Then $B_iB_{i+1} + B_{i+2}B_{i+3} + \ldots + B_{i+k}B_{i+k+1}$ is the graph of $h(1, [2\pi a/n_2, 2\pi b/n_2])$ where h is the homeomorphism given in the preceding paragraph, a is i or $i-1$ according as $f_1(i-1)$ is less than or greater than $f_1(i)$ and b is $i+k+1$ or $i+k$ according as $f_1(i+k+1)$ is greater than or less than $f_1(i+k)$.

Let F_2 be a circular chain covering J_2 such that the ith link of F_2 contains B_iB_{i+1}, the closure of the ith link of F_2 lies in the $(f_1(i) \bmod n_1)$th link of F_1, each link of F_2 is of diameter less than $\frac{1}{2}$, and the sum of the elements of F_2 is the sum of the links of a circular chain each of whose links is a round disk of diameter less than $\frac{1}{2}$.

We continue in this fashion to define chains $F_3, F_4, \ldots$ such that

F_i is mesh less than 2^{1-i},
F_i has n_1 links,
the closure of the jth links of F_{i+1} lies in the $(f_i(j) \bmod n_i)$th link of F_i,
the sum of the links of F_i is the sum of the links of a circular chain each of whose links is the interior of a round disk of diameter less than 2^{1-i}.

The homeomorphism h is defined as follows. For each point p of X let $R_i(p)$ denote the sum of the closures of all links of F_i whose subscripts differ mod n_i from the subscript of a link of C_i containing p by no more than 1. Then $R_i(p)$ is the sum of the closures of three or four links of F_i according as p belongs to one or two links of C_i. Then $R_1(p)$, $R_2(p)$, . . . , is a decreasing sequence of closed sets converging to the point $h(p)$. The map h defined may be shown to be a homeomorphism.

Had we wanted to insist that the links of the circular chain be interiors of square disks, interiors of rectangular disks, interiors of triangular disks, or even annuli instead of interiors of round disks, this would have been no problem since the methods used in the proof of Theorem 4 could have been used to get a continuum so embedded that it could have been covered with circular chains whose links are of the required sort. In Theorem 4 of **(1)** we embedded a snake-like continuum in the plane so that it could be covered by linear chains whose links were the interiors of rectangles. The methods of Theorem 4 of the present paper can be used to give the following variation of that result.

THEOREM 5. *Each snake-like continuum can be embedded in the plane in such a way that for each* $\epsilon > 0$ *there is a linear chain covering the embedded snake-like continuum such that each link of the chain is the interior of a round disk of diameter less than* ϵ.

To construct such an embedding of a snake-like continuum X one would consider a sequence of linear chains $C_1, C_2, \ldots$, covering X and with meshes converging to 0 so fast that each collection of five consecutive links of C_{i+1} lies in a link of C_i.

Suppose C_i has n_i links. Let $A_1A_2 \ldots A_{n_1}A_{n_1+1}$ be a polygonal arc in the plane such that each A_iA_{i+1} is a segment of length less than 1. Let

$$F_1(L_1, L_2, \ldots, L_{n_1})$$

be a linear chain of mesh less than 1 covering $A_1A_2 \ldots A_{n_1+1}$ such that L_i contains A_iA_{i+1} and the sum of the links of F_1 is the sum of the elements of a linear chain each of whose elements is the interior of a round disk of diameter less than 1.

Let $B_1B_2 \ldots B_{n_2+1}$ be a polygonal arc very close to $A_1A_2 \ldots A_{n_1+1}$ such that each B_iB_{i+1} is a segment of length less than $\frac{1}{2}$ and lies in a link L_j of F_1 such that the ith link of C_2 lies in the jth link of C_1. Let F_2 be a linear chain of mesh less than $\frac{1}{2}$ such that this ith link of F_2 contains B_iB_{i+1} and its closure lies in the jth link of F_1 while the sum of the links of F_2 is the sum of the links of a linear chain each of whose elements is the interior or a round disk of diameter less than $\frac{1}{2}$.

Continuing as in the proof of Theorem 4 we get chains $F_3, F_4, \ldots$. Let X' be the intersection of the sum of the links of F_1, the sum of the links of $F_2, \ldots$. Then X' is the homeomorphic image of X that can for each positive number ϵ be covered by a linear chain each of whose links is the interior of a round disk of diameter less than ϵ.

The following result may also be proved in a similar fashion.

THEOREM 6. *If X is an indecomposable snake-like continuum, then there is a homeomorphism h of X into the plane such that for each positive number ϵ there is a linear chain C irreducibly covering $h(X)$ such that each link of C is the interior of a round disk of diameter less than ϵ and the sum of the first and last disks of C lies in a round disk of diameter less than ϵ such that this disk does not intersect any other links of C other than the second and the next to the last.*

It follows as a corollary of the above result that each indecomposable snake-like continuum is also circle-lke. In fact, the class of snake-like continua that are also circle-like is exactly the class which are also indecomposable.

The following theorem which follows as a consequence of Theorems 3, 4, and 6 answers a question raised by Burgess in **(5)**.

THEOREM 7. *Each circle-like continuum which can be embedded in the plane can be embedded in such a way that for each positive number ϵ, the embedded continuum can be irreducibly covered by a circular chain each of whose links is a round disk of diameter less than ϵ.*

THEOREM 8. *Each circle-like continuum can be embedded in the cartesian product of a triod and an arc.*

Proof. The proof of this theorem is similar to the proof of Theorem 4. The only difference is that if we have a sequence of circular chains defining the circle-like continuum, we cannot suppose that the chains do not circle each other more than once. We use a fin sticking up out of one of the links as a cross-over place.

Let $C_1, C_2, \ldots$, be a sequence of circular chains covering the circle-like continuum X where each link L of C_{i+1} is contained in a link L' of C_i such that the mesh of C_{i+1} is less than the distance from L to the boundary of L'. We suppose that C_i has n_i links.

Let D be a disk in the $z = 0$ plane and E be a disk in the $x = 0$ plane such that $D \cdot E$ is an arc spanning D and $D + E$ is homeomorphic to the cartesian product of an arc and a triod.

Let J_1 be a polygonal simple closed curve in D which is the sum of n_1 arcs $A_1A_2, A_2A_3, \ldots, A_{n_1}A_1$ such that each of these arcs is of diameter less than 1, A_1A_2 crosses the straight line segment $D \cdot E$, and if two of the arcs intersect each other, the intersection is an endpoint of each. A first approximation of a homeomorphism taking X into $D + E$ takes X onto J_1 by taking the part of X in the ith link of C_1 onto the arc A_iA_{i+1}.

Let $F_1(L_1, L_2, \ldots, L_{n_1})$ be a circular chain of mesh less than 1 covering J_1 such that the L_1 is an open subset of $D + E$ covering A_1A_2. We can throw away part of certain links of F_1. However, we keep all of L_1. Let $L_i' = L_i \cdot D$ if $i > 1$ and $L_1' = L_1$. It may be convenient to think of L_1' as a disk with a fin, L_1' $(i > 1)$ as a disk, and $\sum L_i'$ as an open annular ring with a fin.

https://doi.org/10.4153/CJM-1962-009-3 Published online by Cambridge University Press

We build F_2 in $\sum L_i'$ as was done in Theorem 4, using $L_1' \cdot E$ as a cross-over place in case C_2 circles C_1 more than once. Each link of F_2 except one is then reduced to a disk and this link has a fin that is used as a cross-over for building F_3 in case C_3 circles C_2 more than once. We continue to build circular chains $F_3, F_4, \ldots$, as done in the proof of Theorem 4. The limiting set of the sum of the links of F_1, the sum of the links of $F_2, \ldots$, is a circle-like continuum in $D + E$ homeomorphic to X.

Since separating the plane into a certain number of pieces is a topological property for closed sets and the continuum $h(X)$ described in Theorem 4 separates the plane into exactly two pieces, but no snake-like continuum separates the plane, we have the following result which shows that part of the hypothesis of a result announced in **(6)** is not needed.

THEOREM 9. *For each circle-like plane continuum X which is not snake-like, the complement of X in the plane has exactly two components and X is the boundary of each.*

3. Categories of continua in the plane. The space of compact continua of a metric space S is the metric space $C(S)$ whose points are the compact continua of S, where the distance between two elements C_1, C_2 of $C(S)$ is the Hausdorff distance between them in S, namely,

$$\text{Least upper bound } \rho(x, C_i)(x \in C_1 + C_2, i = 1, 2).$$

The metric space $C(E^n)$ is a complete metric space.

A subset of a complete metric space is of the second category if the subset contains a dense G_δ (inner limiting set) subset of the space. The complement of a set of the second category is of the first category.

We say that most compact continua of E^n have a certain property if in the metric space $C(E^n)$, the set of elements with the property is associated with a subset of $C(E^n)$ of the second category. It is shown in **(2)** that most compact continua in E^n $(n \geqslant 2)$ are pseudo-arcs. We now show that most of these can be covered by chains each of whose links is the interior of a small round disk.

THEOREM 10. *Most bounded continua in the plane are pseudo-arcs which for each $\epsilon > 0$ can be covered with a linear chain each of whose links is the interior of a round disk of diameter less than ϵ.*

Proof. The proof is the same as that given in **(2)**. Let F_i be the collection of all compact continua C in E^2 such that C cannot be covered by a linear chain each of whose links is the interior of a disk of diameter less than $1/i$. If $C_1, C_2, \ldots$, is a sequence of elements of F_i converging to a compact continuum C_0, C_0 is an element of F_i because if a chain of a certain sort covers C_0, it also covers some C_j. Hence, F_i is closed in $C(E^2)$.

The collection of all simple polygonal arcs is dense in $C(E^2)$. To get a polygonal arc within ϵ of a bounded continuum X one could let U be a neighbourhood of X such that each point of U is within ϵ of X, let $p_1, p_2, \ldots, p_n$ be a finite set of points of X which is ϵ dense in X, and take any polygonal arc in U that contains the p's. Since each of these broken lines can be covered by a chain each of whose links is the interior of a small circular disk, $C(E^2)-\sum F_i$ is a dense G_δ subset of $C(E^2)$.

As pointed out in **(2)**, the set of all compact continua in E^2 which are not hereditarily indecomposable is the sum of a countable number of nowhere dense closed sets. Hence, the set of elements of $C(E^2)$ which are not pesudo-arcs that can be covered by chains whose links are the interiors of small round disks is of the first category.

The following theorem can be proved in a similar fashion.

THEOREM 11. *Suppose U is the interior of a circle in E^2 and W is the collection of all compact continua C in E^2 such that U lies in a bounded component of $E^2 - C$. Then most elements of W are circle-like continua C such that*
each proper subcontinuum of C is a pseudo-arc and
for each $\epsilon > 0$, C can be covered by a circular chain each of whose links is the interior of a round disk of diameter less than ϵ.

Questions. If X_1, X_2 are two non-degenerate hereditarily indecomposable linearly chainable compact continua, then X_1, X_2 are pseudo-arcs (and hence are topologically equivalent) and each is homogeneous. Suppose Y_1, Y_2 are hereditarily indecomposable circularly chainable compact planar continua neither of which is linearly chainable. Are they topologically equivalent? Is each homogeneous? See the question raised about Example 2 in **(2)**.

4. Circle-like continua in 2-manifolds. We show in this section that each circle-like continuum that can be embedded in a 2-manifold can be embedded in the plane. Of course it cannot be concluded that an open subset of the 2-manifold can be embedded along with the circle-like continuum as can be seen by considering the centre simple closed curve in a Moebius band. However, we have the following result.

THEOREM 12. *If a circle-like continuum X lies in a 2-manifold M, either it lies in an open subset of M topologically equivalent to a subset of the plane or it lies in an open subset of M topologically equivalent to a subset of a Moebius band.*

Proof. Let T be a triangulation of M such that no vertex of T lies on X. We suppose that M is metrized so that each 1-simplex of T is a unit segment, each 2-simplex of T is an equilateral triangle, and the distance between two points is the length of some simple polygonal arc joining the two points. (We assume M to be connected.)

Let 2ϵ be a positive number such that no vertex of T is within 2ϵ of X. About each vertext v of T remove from M the closed disk whose centre is v

https://doi.org/10.4153/CJM-1962-009-3 Published online by Cambridge University Press

and whose radius is ϵ. Let M' be the remainder of M after all of these disks are removed.

Let $C(L_1, L_2, \ldots, L_n)$ be a circular chain of mesh less than $\epsilon/2$ irreducibly covering X. We are supposing that M is the total space. Hence, the links of C are open subsets of M'. If the sum of two adjacent links intersects two 2-simplexes of T, the two 2-simplexes share a common edge.

The open subset of M promised by the Theorem is $\sum L_i$, the sum of the links of C. In order to embed $\sum L_i$ in a set of the proper sort, we consider a covering space of M' into which $\sum L_i$ can be lifted.

Covering spaces for M'. Suppose S is a unit equilateral triangular disk from which has been removed each point whose distance from a vertex is less than or equal to ϵ. We call S an ϵ-blunted triangle. We note that the intersection of M' and a 2-simplex of T is an ϵ-blunted triangle. An ϵ-blunted triangle is a 2-manifold-with-boundary with three boundary components.

Suppose N is a 2-manifold with boundary which is the sum of a locally finite collection of ϵ-blunted triangles such that if two of these ϵ-blunted triangles are adjacent, their intersection is a common boundary component of each. If to each boundary component of N is added an ϵ-blunted triangle, there results a new 2-manifold-with-boundary N' with twice as many boundary components as N such that Int N' is homeomorphic with Int N. We suppose that no two of the added ϵ-blunted triangles are adjacent to each other. We say that N grew to N' in one step. If in turn we permit N' to grow a step, the resulting set to grow a step, . . . , the result after a countably infinite number of steps is a 2-manifold without boundary which is homeomorphic with Int N.

Suppose N is a 2-manifold-with-boundary with a decomposition into ϵ-blunted triangles and g is a map of N into M' such that

g is a local homeomorphism and
g is an isometry in taking each ϵ-blunted triangle in the decomposition of N onto the intersection of M' and a 2-simplex of T.

If N grows in one step to N', there is a unique extension of g so that the new g has the required properties for g and takes N' into M'. In fact, if N^∞ is the result of N after a countably infinite number of steps of growth, N^∞ is a 2-manifold without boundary homeomorphic with Int N and g can be uniquely extended to take N^∞ onto M' so that g is a local homeomorphism and g is an isometry in taking each ϵ-blunted triangle in the decomposition of N^∞ into M'.

The group G. We now consider a certain group G that will be useful in building an open set to contain $\sum L_i$. Suppose the 2-simplexes of T are ordered $D_1, D_2, \ldots$, and the 1-simplexes of T are ordered $E_1, E_2, \ldots$. Consider the free group G with generators $g_1, g_2, \ldots$. If an arc crosses E_i in going from D_j to D_{j+k}, the crossing generates the element g_i of G. If an arc crosses E_i in going from D_{j+k} to D_j, the crossing corresponds to g_i^{-1}. If a path from D_j

https://doi.org/10.4153/CJM-1962-009-3 Published online by Cambridge University Press

goes to E_i and then back into D_j without crossing into D_{j+k}, the element $g_i g_i^{-1}$ is generated. If a path from D_{j+k} goes to E_i and then back into D_{j+k} without crossing into D_j, the element $g_i^{-1} g_i$ is generated. We note that the group G is not the fundamental group of M' since it contains elements not generated by any loop. However, certain loops in M' do correspond to elements in the group G.

Let G' be the collection of words that can be made with letters $g_1, g_1^{-1}, g_2, g_2^{-1}, \ldots$. We use words of G' to represent loops rather than for purposes of multiplication. A word of G' is a finite ordered set whose elements are in $g_1, g_1^{-1}, g_2, g_2^{-1}, g_3, \ldots$, and an element of G is an equivalence class of words of G'. Two words are different as elements of G' if they have different spellings even though they may be in the same equivalence class corresponding to an element of G.

If a link L_i of C lies in a 2-simplex of T, let a_i be the centre of this 2-simplex. If L_i intersects two 2-simplexes of T, let a_i be the centre of the 1-simplex common to these two 2-simplexes. Consider the closed curve (perhaps singular) consisting of the segments from a_1 to a_2, a_2 to $a_3, \ldots,$ and a_n to a_1. It generates a word W of G' as it crosses or touches the various E_i's.

Suppose $W_1, W_2, \ldots, W_m$ is a sequence of words of G' such that:

$W = W_1$.

W_{i+1} is obtained from W_i by cancelling two letters in W_i such that one of these letters is the inverse of the other and the two letters are either adjacent or they are at opposite ends of W_i.

W_m cannot be further reduced.

The word W_1 corresponds to the closed curve $a_1 a_2 \ldots a_n a_1$ and W_{i+1} corresponds to a closed curve (perhaps singular) in M' which is homotopic to the preceding in M'. We consider two cases.

Case 1. *W_m has no letters.* We suppose that W_{m-1} is $g_1 g_1^{-1}$ and that E_1 is the common edge of D_1 and D_2. Let N^1 be a 2-manifold with boundary which is the sum of two ϵ-blunted triangles joined along a common boundary component of each and g be a homeomorphism of N^1 onto $M' \cdot (D_1 + D_2)$ which is an isometry in taking each ϵ-blunted triangle of N^1 onto the intersection of M' and one of D_1, D_2. Since g is a homeomorphism, there is a closed curve in N^1 that is taken by g onto the closed curve associated with W_{m-1}.

The closed curve associated with W_{m-2} is the same as the closed curve associated with W_{m-1} except that an arc in the first closed curve which crosses an E_i and then crosses back (or merely touches and then backs away) is replaced by an arc which does not touch E_i. Hence, if N^2 is the result of N^1 after a one-step growth, there is a closed curve in N^2 that maps under g onto the closed curve associated with W_{m-2}. By continuing this procedure, one finds that if N^{m-1} is the result of N^1 after $m - 2$ steps of growth, there is a closed curve $b_1 b_2 \ldots b_n b_1$ in N^{m-1} that maps by the extended g onto $a_1 a_2 \ldots a_n a_1$.

https://doi.org/10.4153/CJM-1962-009-3 Published online by Cambridge University Press

The homeomorphism h of $\sum L_i$ into Int N^{m-1} is described as follows. If a_1 is the centre of D_j, $h = g^{-1}$ in taking L_i into the sum of the two ϵ-blunted triangles of N^{m-1} having b_i on their common edge. We note that h takes $\sum L_i$ homeomorphically into Int N^{m-1} and Int N^{m-1} is topologically equivalent to the plane since Int N^1 is.

Case 2. $W_m = x_1x_2 \ldots x_r$. Let $E(x_i)$ be the 1-simplex of T associated with x_i or x_i^{-1}. Since there is a closed curve in M' associated with W_m, $E(x_i)$ and $E(x_{i+1})$ are different edges of the same 2-simplex of T as are $E(x_1)$ and $E(x_r)$.

Let $A_1, A_2, \ldots, A_r$ be r ϵ-blunted triangles joined in a fashion to be described. Let h be an isometry of A_i onto the intersection of M' and the 2-simplex of T having $E(x_i)$ and $E(x_{i+1})$ as edges (h takes A_r onto the intersection of M' and the 2-simplex of T having $E(x_1)$ and $E(x_r)$ as edges). Suppose that A_i is joined to A_{i+1} along an edge of each so that h is the same on this edge. Also, A_1 and A_r are joined in a similar fashion. We note that A_{i+2} and A_i are not joined to A_{i+1} along the same edge since $E(x_i)$, $E(x_{i+1})$ are different edges of the same simplex of T. We note that if N is the sum of these A_i's, Int N is either the interior of an annulus ring or the interior of a Moebius band.

By the same argument as that used in Case 1, it follows that $\sum L_i$ can be embedded in the result after N is grown through several steps. Hence, either $\sum L_i$ can be embedded in an annulus ring (and hence in the plane) or it can be embedded in a Moebius band.

The methods used in the proof of Theorem 12 give the following result.

THEOREM 13. *Each tree-like continuum on a 2-manifold M lies on an open subset of M which can be embedded in the plane.*

Since each neighbourhood of a planar tree-like continuum contains a disk containing the continuum, Thorem 13 actually implies that some disk in M contains the tree-like continuum. This was proved in Lemma 2 of **(7)** by other methods.

THEOREM 14. *Each circle-like continuum that can be embedded in a 2-manifold can be embedded in the plane.*

Proof. We suppose that $C_1, C_2, \ldots,$ is a sequence of circular chains covering X such that mesh C_i approaches 0 as i increases without limit and C_{i+1} is a refinement of C_i. Since a circle-like continuum X can be embedded in the plane if it is snake-like, we suppose that each C_{i+1} circles C_i a positive number of times. Since it follows from Theorem 4 that X can be embedded in the plane if with only a finite number of exceptions C_{i+1} circles C_i exactly once, we do not consider this case. In view of Theorem 1 we suppose with no loss of generality that each C_{i+1} circles C_i more than twice. We finish the proof of Theorem 14 by showing that this case cannot occur.

Suppose $C_1(L_1, L_2, \ldots, L_n)$ is the chain described in the proof of Theorem 12 such that either $\sum L_i$ can be embedded in the plane or $\sum L_i$ can be embedded in a Moebius band so that C_1 circles the band exactly once. If C_1 circles a Moebius band exactly once, then C_2 circles the band more than twice and it follows from Theorem 2 that there is a chain with connected links in C_2 that circles C_1 more than twice. Hence, there is a simple closed curve in the Moebius band that circles the band more than twice. But no simple closed curve circles a Moebius band more than twice.

COROLLARY. *The circle is the only solenoid that can be embedded in a 2-manifold.*

THEOREM 15. *If X is a circle-like continuum on an orientable 2-manifold M, then X lies on an annulus in M.*

Proof. We showed in Theorem 12 that X lies in an open subset U of M such that U can either be embedded in the plane or U can be embedded in a Moebius band.

Suppose U can be embedded in the plane and M' is a connected 2-manifold-with-boundary in U such that Int M' contains X. Then M' is a punctured disk. Since $E^2 - X$ has one or two components according as X is snake-like or not, it is possible to remove canals from M' so that X lies in the remaining punctured disk which has at most two boundary components. If the remainder has two boundary components, it is an annulus. If it has only one boundary component, it is a disk. An annulus in the disk contains X.

Finally we consider the case where no open subset of M which contains X can be embedded in the plane. We show that this case cannot occur. It follows from Theorem 13 that if it did, X would not be snake-like.

Let $C_1, C_2, \ldots,$ be a decreasing sequence of circular chains covering X such that each C_{i+1} circles in C_i exactly once, each link of each C_i is an open subset of M, and the sum of the links of C_1 can be embedded in a Moebius band as described in Theorem 12 with C_1 circling this Moebius band exactly once.

Suppose that C_i has n_i links and each link of C_{i+1} lies in a component of a link of C_i. Let f_i $(i = 1, 2)$ be a map of $(0, 1, 2, \ldots, n_{i+1})$ into the integers such that the $f_i(j)$th link of C_i is the first link of C_i with a component that contains the jth link of C_{i+1} and $f_i(0) = f_i(n_{i+1})$. Since C_{i+1} circles C_i once,

$$\sum_{j=0}^{ni+1} (f_i(j+1) - f_i(j)) = \pm n_i$$

where $(n_i - 1)$, $(1 - n_i)$ are considered as -1, 1 in computing this sum. Since C_3 circles C_1 once, we also have that

$$\sum_{j=0}^{n_3} (f_1f_2(j+1) - f_1f_2(j)) = \pm n_1.$$

Let U_j be the component of the $f_2(j)$th link of C_2 containing the jth link of C_3. Suppose U_r and U_s intersect with $r < s$. Then either

$$f_1f_2(r) - f_1f_2(s) + \sum_{j=r}^{s-1} (f_1f_2(j+1) - f_1f_2(j))$$

or

https://doi.org/10.4153/CJM-1962-009-3 Published online by Cambridge University Press

$$f_1f_2(s) - f_1f_2(r) + \sum_{j=0}^{r-1} (f_1f_2(j+1) - f_1f_2(j)) + \sum_{j=s}^{n_3} (f_1f_2(j+1) - f_1f_2(j))$$

is an odd multiple of n_1.

By a continuation of the elimination of the U's, one can obtain a circular chain of the U's that circles C_1 an odd number of times. Then a simple closed curve J in the sum of these U's circles the Moebius band an odd number of times. This odd number cannot be more than one since no simple closed curve in a Moebius band circles the Moebius band more than twice. Any open subset of the Moebius band that contains J also contains a Moebius band. Hence, the sum of the links of C_1 contains a Moebius band which is contrary to the hypothesis that M is an orientable manifold.

COROLLARY. *If X is a circle-like continuum in an orientable connected 2-manifold M, then $M - X$ has at most two components.*

REFERENCES

1. R. H. Bing, *Snake-like continua*, Duke Math. J., *18* (1951), 653–663.
2. ——— *Concerning hereditarily indecomposable continua*, Pacific J. Math., *1* (1951), 43–51.
3. ——— *A simple closed curve is the only homogeneous bounded plane continuum that contains an arc*, Can. J. Math., *12* (1960), 209–230.
4. ——— *Conditions under which monotone decompositions of E^3 are simply connected*, Abstract 325, Bull. Amer. Math. Soc., *63* (1957), 143.
5. C. E. Burgess, *Homogeneous continua which are almost chainable*, Can. J. Math., *13* (1961), 519–528.
6. ——— *Homogeneous continua which are circularly chainable and contain an arc*, Abstract 545-16, Amer. Math. Soc. Notices, *5* (1958), 229.
7. R. H. Roberts and N. E. Steenrod, *Monotone transformations of two-dimensional manifolds* Ann. Math., *39* (1938), 851-863.

University of Wisconsin

https://doi.org/10.4153/CJM-1962-009-3 Published online by Cambridge University Press