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ON A THEOREM OF A. A. GOL'DBERG

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Abstract It is shown that a condition on the order of a meromorphic function in a result of A. A. Gol'dberg cannot be relaxed.

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1. Introduction

Suppose that f is a meromorphic function in the plane and that

$$\liminf_{r\to\infty} \frac{N(r,0)+N(r,\infty)}{(\log r)^2} \leqslant \sigma, \tag{1.1}$$

for some positive number σ, where

$$N(r,a) = \int_0^\infty \frac{n(t,a)}{t}\,\mathrm{d}t,$$

$n(t,a)$ being the number of a-points of f in $\{z : |z| \leqslant t\}$. (We assume that f has neither a zero nor a pole at the origin, which ensures the existence of $N(r,0)$ and $N(r,\infty)$; as will be apparent, no loss of generality is entailed in doing so.) In response to a conjecture of Barry [**1**, p. 485], Gol'dberg [**2**] showed that if in addition f has order 0, then

$$\limsup_{r\to\infty} \frac{m(r)}{M(r)} \geqslant C(\sigma), \tag{1.2}$$

where

$$C(\sigma) = \left(\prod_{j=1}^{\infty} \frac{1-\mathrm{e}^{-(2k-1)/(4\sigma)}}{1+\mathrm{e}^{-(2k-1)/(4\sigma)}}\right)^2 \tag{1.3}$$

and

$$m(r) = \min_{|z|=r} |f(z)|, \qquad M(r) = \max_{|z|=r} |f(z)|,$$

the minimum and maximum moduli of f. The constant $C(\sigma)$ is best possible, as Barry showed [**1**, p. 484]. Gol'dberg commented [**2**, p. 434]:

> It is likely that [in this theorem] it is possible to replace the requirement that f has order 0 by the weaker restriction that f be a function of genus 0. However, we have not been able to prove this.

The intention here is to show that order 0 cannot be replaced by genus 0 in the hypotheses of the theorem.

2. An example

Given a number ρ, with $0 < \rho < 1$, and a positive number σ, we will construct an entire function $f(z)$ of order ρ for which

$$\liminf_{r\to\infty} \frac{N(r)}{(\log r)^2} \leqslant \sigma \tag{2.1}$$

(for brevity $N(r)$ is used here and in what follows instead of $N(r,0)$), while

$$\limsup_{r\to\infty} \frac{m(r)}{M(r)} < C(\sigma). \tag{2.2}$$

Let m_j, $j = 1, 2, \dots$, be an increasing sequence of positive integers which is sparse in the sense that

$$\sum_{l=1}^{j} \mathrm{e}^{\rho m_l/(2\sigma')} = o(m_{j+1}) \quad (j \to \infty), \tag{2.3}$$

where

$$\sigma' = \sigma/(1-\rho^2), \tag{2.4}$$

and let $R_j = \mathrm{e}^{m_j/(2\sigma')}$. Let K_j be the largest positive integer such that $R_j \mathrm{e}^{K_j/(2\sigma')} \leqslant R_{j+1}^{1-\rho}$. In view of (2.3) and the definition of R_j,

$$\sum_{l=1}^{j} R_l^{\rho} = o(\log R_{j+1}); \tag{2.5}$$

also,

$$\begin{aligned} K_j &= [(1-\rho)m_{j+1} - m_j] = (1-\rho+o(1))m_{j+1} \\ &= 2\sigma'(1-\rho+o(1))\log R_{j+1}, \end{aligned} \tag{2.6}$$

where $[\cdot]$ denotes the integral part. Finally, given a positive integer p, set

$$\alpha = \mathrm{e}^{p/(2\sigma)} \tag{2.7}$$

and define $f(z)$ to be the entire function formed from its zeros using the simplest Weierstrassian factors, with the zeros specified as follows: for each positive integer j, $[\alpha R_j^{\rho}]$

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zeros are placed at $-R_j$, and, again for each j, simple zeros are placed at $-R_j e^{k/(2\sigma')}$, $k = 1, 2, \dots, K_j$. The zeros of f thus occur in blocks corresponding to the intervals $(-R_{j+1}, -R_j]$, with a concentration of zeros at $-R_j$, a regular distribution of simple zeros from $-R_j$ to about $-R_{j+1}^{1-\rho}$, and the remaining part of the interval free of zeros.

The counting function, $n(r)$, of the zeros of f satisfies, for each j,

$$n(r) = n(R_j) + [2\sigma' \log(r/R_j)], \quad R_j \leqslant r \leqslant R_j e^{K_j/(2\sigma')}, \tag{2.8}$$

$$n(r) = n(R_j) + K_j, \quad R_j e^{K_j/(2\sigma')} \leqslant r < R_{j+1}, \tag{2.9}$$

$$n(R_{j+1}) = n(R_j) + K_j + \alpha R_{j+1}^{\rho}. \tag{2.10}$$

Let us first check that f has order ρ. From (2.10), $n(R_{j+1}) \geqslant \alpha R_{j+1}^{\rho}$, so, from Jensen's Theorem, f has order at least ρ. On the other hand, from (2.8) and (2.9), for $R_j \leqslant r < R_{j+1}$,

$$\begin{aligned} n(r) &= \sum_{l=1}^{j} \alpha R_j^{\rho} + O(\log r) \\ &= \alpha R_j^{\rho} + \sum_{l=1}^{j-1} \alpha R_j^{\rho} + O(\log r) \\ &= \alpha R_j^{\rho} + O(\log r), \end{aligned} \tag{2.11}$$

in view of (2.5), and so f has order at most ρ. Also, from (2.11) and (2.5), $n(R_j) = o(\log R_{j+1})$, and therefore, using (2.8), (2.9) and (2.6),

$$\begin{aligned} N(R_{j+1}) &= \left\{ \int_0^{R_j} + \int_{R_j}^{R_{j+1}^{1-\rho}} + \int_{R_{j+1}^{1-\rho}}^{R_{j+1}} \right\} \frac{n(t)}{t}\, dt \\ &\leqslant n(R_j) \log R_j + \int_{R_j}^{R_{j+1}^{1-\rho}} (n(R_j) + 2\sigma' \log(t/R_j)) \frac{dt}{t} + \rho(n(R_j) + K_j) \log R_{j+1} \\ &= (1-\rho) n(R_j) \log R_{j+1} + \sigma' (\log(R_{j+1}^{1-\rho}/R_j))^2 + \rho(n(R_j) + K_j) \log R_{j+1} \\ &= \sigma'(1-\rho)^2 (\log R_{j+1})^2 + 2\sigma' \rho(1-\rho+o(1))(\log R_{j+1})^2 + o(\log R_{j+1})^2 \\ &= \sigma'(1-\rho^2+o(1))(\log R_{j+1})^2 = (\sigma + o(1))(\log R_{j+1})^2, \end{aligned} \tag{2.12}$$

from (2.4), so that (2.1) is satisfied.

It remains to verify (2.2), and to do so we consider three subcases for r in any of the intervals $[R_j, R_{j+1})$:

(i) $R_j \leqslant r \leqslant \beta R_j$;

(ii) $\beta^{-1} R_{j+1}^{1-\rho} \leqslant r < R_{j+1}$;

(iii) $\beta R_j \leqslant r \leqslant \beta^{-1} R_{j+1}^{1-\rho}$.

Here

$$\beta = \mathrm{e}^{p/(2\sigma')}, \tag{2.13}$$

p being the positive integer introduced earlier (cf. (2.7)).

(i) $R_j \leqslant r \leqslant \beta R_j$:

$$\begin{aligned}\frac{m(r)}{M(r)} &\leqslant \left|\frac{1-r/R_j}{1+r/R_j}\right|^{\alpha R_j^{\rho}} \\ &\leqslant \left(\frac{\beta-1}{\beta+1}\right)^{\alpha R_j^{\rho}} = o(1),\end{aligned} \tag{2.14}$$

as $j \to \infty$.

(ii) $\beta^{-1}R_{j+1}^{1-\rho} \leqslant r < R_{j+1}$:

$$\begin{aligned}\frac{m(r)}{M(r)} &\leqslant \left(1-\frac{r}{R_{j+1}}\right)^{\alpha R_{j+1}^{\rho}} \\ &\leqslant \left(1-\frac{\beta^{-1}}{R_{j+1}^{\rho}}\right)^{\alpha R_{j+1}^{\rho}} \\ &= (1+o(1))\mathrm{e}^{-\alpha\beta^{-1}} \\ &= (1+o(1))\mathrm{e}^{-\mathrm{e}^{p(\sigma'-\sigma)/(2\sigma\sigma')}},\end{aligned} \tag{2.15}$$

from (2.7) and (2.13).

(iii) $\beta R_j \leqslant r \leqslant \beta^{-1}R_{j+1}^{1-\rho}$ (i.e. $\mathrm{e}^{(m_j+p)/(2\sigma')} \leqslant r \leqslant \mathrm{e}^{((1-\rho)m_{j+1}-p)/(2\sigma')}$):

$$\begin{aligned}\frac{m(r)}{M(r)} &\leqslant \prod_{k=1}^{K_j}\left|\frac{1-(r/R_j)\mathrm{e}^{-k/(2\sigma')}}{1+(r/R_j)\mathrm{e}^{-k/(2\sigma')}}\right| \\ &= \prod_{k=1}^{K_j}\left|\frac{1-r\mathrm{e}^{-(k+m_j)/(2\sigma')}}{1+r\mathrm{e}^{-(k+m_j)/(2\sigma')}}\right| \\ &= \varPi_1^{-1}\varPi_2^{-1}\prod_{k=1}^{\infty}\left|\frac{1-r\mathrm{e}^{-k/(2\sigma')}}{1+r\mathrm{e}^{-k/(2\sigma')}}\right|,\end{aligned} \tag{2.16}$$

where

$$\varPi_1 = \prod_{k=m_j+K_j+1}^{\infty}\left|\frac{1-r\mathrm{e}^{-k/(2\sigma')}}{1+r\mathrm{e}^{-k/(2\sigma')}}\right|, \qquad \varPi_2 = \prod_{k=1}^{m_j}\left|\frac{1-r\mathrm{e}^{-k/(2\sigma')}}{1+r\mathrm{e}^{-k/(2\sigma')}}\right|.$$

Since $r \leqslant \mathrm{e}^{((1-\rho)m_{j+1}-p)/(2\sigma')}$ and, from (2.6), $m_j + K_j + 1 \geqslant (1-\rho)m_{j+1}$,

$$\varPi_1 \geqslant \prod_{k=p}^{\infty}\left(\frac{1-\mathrm{e}^{-k/(2\sigma')}}{1+\mathrm{e}^{-k/(2\sigma')}}\right) = \pi(p),$$

say, where $\pi(p) \to 1$ as $p \to \infty$. Similarly, but using $r \geqslant \mathrm{e}^{(m_j+p)/(2\sigma')}$,

$$\Pi_2 \geqslant \pi(p),$$

and we conclude that

$$\frac{m(r)}{M(r)} \leqslant \pi(p)^{-2} \prod_{k=1}^{\infty} \left| \frac{1 - r\mathrm{e}^{-k/(2\sigma')}}{1 + r\mathrm{e}^{-k/(2\sigma')}} \right|.$$

Barry [**1**, p. 484] has shown that

$$\limsup_{r\to\infty} \prod_{k=1}^{\infty} \left| \frac{1 - r\mathrm{e}^{-k/(2\sigma')}}{1 + r\mathrm{e}^{-k/(2\sigma')}} \right| = C(\sigma'),$$

and therefore, for all large j, for $\beta R_j \leqslant r \leqslant \beta^{-1} R_{j+1}^{1-\rho}$,

$$\frac{m(r)}{M(r)} \leqslant (1 + o(1))\pi(p)^{-2} C(\sigma').$$

Combining the results from (i), (ii) and (iii), we obtain

$$\limsup_{r\to\infty} \frac{m(r)}{M(r)} \leqslant \max\{\mathrm{e}^{-\mathrm{e}^{p(\sigma'-\sigma)/(2\sigma\sigma')}}, \pi(p)^{-2} C(\sigma')\}.$$

Since p may be arbitrarily large,

$$\limsup_{r\to\infty} \frac{m(r)}{M(r)} \leqslant C(\sigma'),$$

which establishes (2.2), $C(\sigma')$ being less than $C(\sigma)$ since $\sigma' > \sigma$.

References

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