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THE C^*–ALGEBRAS OF SOME INVERSE SEMIGROUPS

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We discuss the structure of some inverse semigroups and the associated C^*-algebras. In particular, we study the bicyclic semigroup and the free monogenic inverse semigroup, following earlier work of Conway, Duncan and Paterson. We then associate to each zero-one matrix A an inverse semigroup $\mathcal{C}_A$, and show that the C^*-algebra $\mathcal{O}_A$ of Cuntz and Krieger is closely related to the semigroup algebra $C^*(\mathcal{C}_A)$.

An inverse semigroup is a semigroup S in which each element x has a unique "inverse" x^* satisfying $xx^*x = x$ and $x^*xx^* = x^*$; it turns out that the map $x \to x^*$ is then an involution on S ([**12**], V.1.4). Through this involution, the semigroup algebra $\mathbb{C}S$ becomes a $*$-algebra, and has an enveloping C^*-algebra, called the C^*-algebra of the semigroup (see below). Here we shall discuss several specific inverse semigroups, investigate the structure of their C^*-algebras, and study the connections with other well-known C^*-algebras.

We shall take the point of view suggested by Duncan and Paterson [**9**]. They observed that inverse semigroups are precisely those semigroups which can be realised as $*$-semigroups of partial isometries on a Hilbert space, in which the "inverse" of an element is the adjoint of the corresponding linear operator ([**9**], 1.1). (For this reason, we shall call x^* the adjoint of x.) Using this approach and the structure theory of partial isometries, Conway, Duncan and Paterson gave an elegant classification of the monogenic (that is, singly generated) inverse semigroups ([**3**, Section 1]; Preston has also given a purely algebraic version [**16**]).

If S is an inverse semigroup, the $*$-algebraic structure on $\mathbb{C}S = \operatorname{sp}\{\delta_x : x \in S\}$ is defined by $\delta_x\delta_y = \delta_{xy}$ and $(\delta_x)^* = \delta_{x^*}$; the $*$-representations $\pi : \mathbb{C}S \to B(H)$ are then the linear extensions of the $*$-representations of S as partial isometries on the Hilbert space H. (From now on, all our representations will be assumed $*$-preserving.) Since every partial isometry has norm 0 or 1, for any representation π of $\mathbb{C}S$ and $f \in \mathbb{C}S$, we have

$$\|\pi(f)\| = \left\|\pi\left(\sum_{x\in S} f(x)\delta_x\right)\right\| \leqslant \sum_{x\in S} |f(x)|\,\|\pi(\delta_x)\| \leqslant \sum_{x\in S} |f(x)|;$$

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thus

$$\|f\| = \sup\{\|\pi(f)\| : \pi\colon \mathbb{C}\mathcal{S} \to B(H) \text{ is a representation}\}$$

is a well-defined C^*-seminorm on $\mathbb{C}\mathcal{S}$ — in fact, it is a C^*-norm, because the regular representation is known to be faithful [**20**]. The completion $C^*(\mathcal{S})$ of $\mathbb{C}\mathcal{S}$ in this norm is the *C^*-algebra of the semigroup $\mathcal{S}$*: note that its representations are by definition in one-to-one correspondence with those of $\mathcal{S}$. (Other equivalent definitions are given in [**9**, p.44].)

The structure of the C^*-algebra $C^*(\mathcal{S})$ has already been investigated for various semigroups and classes of semigroups ([**14**, **9**, **3**]). In particular, Conway, Duncan and Paterson have described the C^*-algebras of the monogenic inverse semigroups ([**3**], Sections 2, 3), and for the bicyclic semigroup and the free monogenic inverse semigroup these turned out to be algebras which had often appeared in other contexts. We shall start by giving an alternative analysis of these two examples; our results go a little further than those of [**3**], and some of them also appear to be slightly different. In any case, we hope our systematic approach will shed some light on the results of [**3**], and also motivate our subsequent analysis of other semigroup algebras.

In Section 2 we introduce an apparently new class of semigroups which we call the *Cuntz–Krieger semigroups.* Their definition was inspired by work of Cuntz and Krieger, in which they study the C^*-algebras generated by families of partial isometries whose initial and range projections satisfy relations governed by a zero-one matrix A (see [**7**]). We modify their relations slightly, replacing them with ones which only involve multiplication, and let $\mathcal{C}_A$ be the semigroup defined by this new set of relations. We prove that $\mathcal{C}_A$ is an inverse semigroup, by constructing a faithful representation of $\mathcal{C}_A$ as a *-semigroup of partial isometries. The C^*-algebra $C^*(\mathcal{C}_A)$ is not quite the same as the algebra $\mathcal{O}_A$ of [**7**], but it is very closely related: each $C^*(\mathcal{C}_A)$ contains an ideal I isomorphic to a few copies of the compact operators, and the quotient $C^*(\mathcal{C}_A)/I$ is isomorphic to $\mathcal{O}_A$.

In the special case where A has all entries 1, the algebra $\mathcal{O}_A$ is the Cuntz algebra $\mathcal{O}_n$ [**4**], and the connection with an inverse semigroup O_n had been noticed by Renault [**17**], although he did not explicitly identify $\mathcal{O}_n$ with a quotient of $C^*(O_n)$. However, the generalisation to arbitrary A may also be of some interest: the semigroup $\mathcal{C}_A$ is a new ingredient in the interplay between the structure of $\mathcal{O}_A$ and that of the topological Markov chain associated to A (see [**7**, **6**]).

While we hope we have already provided good evidence that the C^*-algebras of inverse semigroups can be interesting, we can also point to other places in the theory of C^*-algebras where inverse semigroups may arise naturally. It seems likely, for example, that the algebras studied by Salas [**18**] will be associated to those of inverse semigroups: indeed, Robin Balean has recently verified that they are generated by *-semigroups of partial isometries, although it is not yet clear how many different inverse semigroups are involved or what the results of [**18**] say about their algebras. For a more speculative example, we observe that there are various constructions in the literature (for example [**13, 5**]) which look suspiciously like crossed products by actions of inverse semigroups.

1. Monogenic inverse semigroups

The first semigroup we shall consider is the *bicyclic semigroup* $\mathcal{C}$, which is the semigroup generated by two elements p, q subject to the relation $qp = 1$. One way to see that this is an inverse semigroup is as follows. First observe that every element of $\mathcal{C}$ can be written in the form $p^n q^m$ for some $n, m \geqslant 0$. Next, let S be the unilateral shift on l^2, and note that $p = S$, $q = S^*$ satisfy the relation $qp = 1$; thus there is a homomorphism ϕ of $\mathcal{C}$ onto the semigroup generated by S and S^*. But the set $\{S^n (S^*)^m\}$ is closed under the adjoint operation, so this is an inverse semigroup by [**9**, 1.1], and ϕ must be an isomorphism since the operators $S^n(S^*)^m$ are distinct.

If π is a representation of $\mathcal{C}$ on Hilbert space, then the operator $\pi(p)$ is an isometry and π is therefore equivalent to a representation in which $\pi(p) = U \oplus (S \otimes 1_\alpha)$, where U is unitary and $S \otimes 1_\alpha$ is a shift of multiplicity α, possibly infinite ([**10**], p.117). We may as well suppose $\alpha \geqslant 1$, since $\|\pi(f)\|$ is not decreased by adding another summand to π. Now an elegant argument of Coburn ([**2**], Section 3) shows that the map $S \to U \oplus (S \otimes 1_\alpha)$ induces an isometric isomorphism of $C^*(S)$ onto $C^*(U \oplus (S \otimes 1_\alpha))$: thus the representation π which sends p to S is isometric for the C^*-norm on $\mathbb{C}\mathcal{C}$, and $C^*(\mathcal{C}) \cong C^*(S)$. Coburn has also proved that $C^*(S) \subset B(l^2)$ contains the algebra $\mathcal{K}$ of compact operators and has $C^*(S)/\mathcal{K} \cong C(\mathbf{T})$; this result and this algebra are of fundamental importance in C^*-algebra theory and operator theory, where $C^*(S)$ appears as the C^*-algebra generated by the Toeplitz operators with continuous symbol (see, for example, [**8**, Chapter 7]).

Next we wish to consider the *free monogenic inverse semigroup* $\mathcal{F}$: monogenic means it is generated as a semigroup by one element u and its adjoint u^*, and free means that, whenever x is an element of an inverse semigroup S, there is a homomorphism $\phi: \mathcal{F} \to S$ with $\phi(u) = x$. Such a semigroup is obviously unique up to isomorphism, but, again, the requirement of uniqueness of adjoints means it is not at all obvious that there is one – indeed, Schein has shown that such an $\mathcal{F}$ cannot be realised in terms of generators and finitely many relations [**19**]. We shall sketch a proof of existence, similar in spirit to that given by Preston ([**16**], Sections 1, 2), and which is implicit in [**3**].

Lemma 1.1. *Let S be a monogenic inverse semigroup with generator u.*

(a) *Every element of S has the form $(u^*)^k u^l (u^*)^m$, where k, l, m are integers satisfying $l \geqslant 0$, $l \geqslant k \geqslant 0$ and $l \geqslant m \geqslant 0$.*

(b) *If the elements in (a) are all distinct, then $(\mathcal{S}, u)$ is a free monogenic inverse semigroup.*

Part (a) is proved in [16], Section 1; the proof is a tedious reduction, but is elementary in the sense that it uses only the commutativity of idempotents in $\mathcal{S}$ ([12], V.1.27). If x belongs to another inverse semigroup $\mathcal{T}$, and the given elements of $\mathcal{S}$ are distinct, we can define $\phi\colon \mathcal{S} \to \mathcal{T}$ by

$$\phi\Big((u^*)^k u^l (u^*)^m\Big) = (x^*)^k x^l (x^*)^m;$$

since the reductions involved in part (a) are valid in any inverse semigroup, the map defined this way is a homomorphism.

Now consider the *-subsemigroup of $B\big(l^2 \oplus l^2\big)$ generated by $S \oplus S^*$, where S is again the unilateral shift. Since any element of this semigroup has the form $S^n(S^*)^m \oplus S^k(S^*)^l$, this is a *-semigroup of partial isometries and hence an inverse semigroup. To see that it is a free monogenic inverse semigroup, we just need to check that the elements

$$(S^* \oplus S)^k (S \oplus S^*)^l (S^* \oplus S)^m = S^{l-k}(S^*)^m \oplus S^k (S^*)^{l-m}$$

are distinct, and this can be easily done by applying them to the sequence $\{1/n\} \in l^2$.

REMARK 1.2. We can deduce from this by taking free inverse products of $\mathcal{F}$ that there are free inverse semigroups with arbitrary sets of generators. Since the construction of free inverse products is straightforward (see [**12**, VII.4.5]), and the known constructions of free inverse semigroups are quite complicated (for example [**15**]), this may be a relatively efficient existence proof!

The structure of $C^*(\mathcal{F})$ is described in the following theorem of Conway, Duncan and Paterson ([3], Section 2). In it, J_n denotes the truncated shift on $\mathbb{C}^n$ defined with respect to the usual basis by $J_n(z_1, \dots, z_n) = (0, z_1, \dots, z_{n-1})$. The operator $\oplus_{n=2}^{\infty} J_n$ had previously been studied by Bunce and Deddens [1], and the treatment in [3] used their results. However, our proof is more direct, and the facts about $\oplus J_n$ proved in [1] will follow from it. Both proofs are based on the same theorem of Halmos and Wallen [**11**].

THEOREM 1.3. *If u is a generator for the free monogenic inverse semigroup $\mathcal{F}$, then $u \to J = \oplus_{n=2}^{\infty} J_n$ extends to an isomorphism of $C^*(\mathcal{F})$ onto $C^*(J)$.*

PROOF: If π is any representation of $\mathcal{F}$, then in particular $\pi(u^n)$ is a partial isometry for all n, and hence $\pi(u)$ is a power partial isometry in the sense of [**11**]. Thus by [**11**], $\pi(u)$ can be decomposed as a direct sum

$$\pi(u) = W \oplus (S \otimes 1_\alpha) \oplus (S^* \otimes 1_\beta) \oplus \big(\oplus_{n=2}^{\infty}(J_n \otimes 1_{k_n})\big),$$

where W is unitary, S is the unilateral shift, J_n is the truncated shift on $\mathbb{C}^n$, and α, β, k_n are multiplicities, possibly infinite. Conversely, the *-semigroup generated by any operator T of this form consists of partial isometries, hence is an inverse semigroup, and there is a representation π of $\mathcal{F}$ with $\pi(u) = T$. Thus it will be enough for us to prove that, if $k_n \geqslant 1$ for all n, then the map

$$T = W \oplus (S \otimes 1_\alpha) \oplus (S^* \otimes 1_\beta) \oplus (\oplus_{n=2}^{\infty}(J_n \otimes 1_{k_n})) \to J = \oplus_{n=2}^{\infty} J_n$$

extends to an isomorphism of $C^*(T)$ onto $C^*(J)$.

The elements of the form $p(T, T^*)$, where $p(X, Y)$ is a polynomial in two non-commuting variables, are dense in $C^*(T)$, and the map $p(T, T^*) \to p(J, J^*)$ is well-defined since it just picks off some direct summands. We shall show that $\|p(T, T^*)\| = \|p(J, J^*)\|$ for any such polynomial. The multiplicities do not affect the norms, and therefore

$$\|p(T, T^*)\| = \max\{\|p(W, W^*)\|, \|p(S, S^*)\|, \|p(S^*, S)\|, \|p(J, J^*)\|\}.$$

It is well-known that $\|p(W, W^*)\| \leqslant \|p(S, S^*)\|$ ([2], Section 3), and we claim also that $\|p(S, S^*)\| \leqslant \|p(J, J^*)\|$.

To see this, fix $\varepsilon > 0$ and choose $\xi = (\xi_1, \xi_2, \dots) \in l^2$ such that $\|\xi\| = 1$, $\xi_k = 0$ for large k (say for $k > N$) and $\|p(S, S^*)\xi\| > \|p(S, S^*)\| - \varepsilon$. Choose $n >$ deg $P + N$, set $\xi^n = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$, and observe that

$$p(S, S^*)(\xi) = (p(J_n, J_n^*)(\xi^n), 0, 0 \cdots);$$

this works because there are enough 0's at the end of ξ^n to ensure that no non-zero entry can get killed by J_n^r for $r \leqslant$ deg P. Since $\|\xi^n\| = 1$, this implies

$$\|p(J_n, J_n^*)\| \geqslant \|p(J_n, J_n^*)(\xi^n)\| = \|p(S, S^*)\xi\| > \|p(S, S^*)\| - \varepsilon,$$

and we therefore have

$$\|p(J, J^*)\| = \sup_n \|p(J_n, J_n^*)\| \geqslant \|p(S, S^*)\|,$$

as claimed.

Next we observe that the unitary $U_n \in U_n(\mathbb{C})$ defined by $U_n(\xi_1, \dots, \xi_n) = (\xi_n, \dots, \xi_1)$ satisfies $U_n J_n U_n^* = J_n^*$, and hence $U = \oplus U_n$ satisfies $UJU^* = J^*$. Thus if $q(X, Y) = p(Y, X)$, the previous claim implies

$$\|p(S^*, S)\| = \|q(S, S^*)\| \leqslant \|q(J, J^*)\| = \|p(J^*, J)\| = \|Up(J, J^*)U^*\| = \|p(J, J^*)\|.$$

We have now shown that $\|p(T, T^*)\| = \|p(J, J^*)\|$ for any polynomial p; thus the map $p(T, T^*) \to p(J, J^*)$ is isometric, and extends to an isomorphism of $C^*(T)$ onto $C^*(J)$. $\square$

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REMARK 1.4. It follows trivially from this theorem that the partial isometry $J = \oplus J_n$ must also generate the free inverse semigroup $\mathcal{F}$. In view of this, it is perhaps slightly surprising that $C^*(S \oplus S^*)$ and $C^*(J)$ are not isomorphic as C^*-algebras: $C^*(S \oplus S^*)$ is a proper quotient of $C^*(J)$, as was proved in [3] via an analysis of $C^*(S \oplus S^*)^\wedge$ and $C^*(\mathcal{F})^\wedge$ (see Proposition 1.5 below). In fact this distinction can be seen at the purely algebraic level: while the *-semigroup generated by $S \oplus S^*$ is isomorphic to $\mathcal{F}$, the representation of $\mathbb{C}\mathcal{F}$ which sends the generator U to $T = S \oplus S^* \in B(l^2 \oplus l^2)$ is not faithful. Indeed, the operator

$$\begin{aligned} &T^*T^2T^* - (T^*)^2T^3T^* - T^*T^3(T^*)^2 + (T^*)^2T^4(T^*)^2 \\ &\qquad = SS^* \oplus SS^* - SS^* \oplus S^2(S^*)^2 - S^2(S^*)^2 \oplus SS^* + S^2(S^*)^2 \oplus S^2(S^*)^2 \end{aligned}$$

vanishes. This appears to contradict the assertion made at the botom of p.20 of [3], and hence raises again the question of whether $\mathbb{C}\mathcal{F}$ has a unique C^*-norm (see [3, Corollary 2.15]). To settle this, we give a description of the spectrum of $C^*(\mathcal{F})$.

PROPOSITION 1.5. *The spectrum of $C^*(\mathcal{F})$ consists of the representations π_n (for $n \geqslant 2$), which send u to J_n; π_S, π_{S^*}, which send u to S, S^* respectively; and ρ_θ (for $0 \leqslant \theta < 2\pi$), which send u to $e^{i\theta}$. The topology is described by:*

(a) *$\{\pi_S, \pi_{S^*}\} \cup \{\rho_\theta : 0 \leqslant \theta < 2\pi\}$ is a closed subset C of $C^*(\mathcal{F})^\wedge$, in which $\overline{\pi_S} = \pi_S \cup \{\rho_\theta\}$, $\overline{\pi_{S^*}} = \pi_{S^*} \cup \{\rho_\theta\}$ and $\{\rho_\theta\}$ is topologically a circle; C is the spectrum of the quotient $C^*(S \oplus S^*)$ of $C^*(\mathcal{F})$;*

(b) *the subset $\{\pi_n : n \geqslant 2\}$ is discrete, and π_n converges to every point of C as $n \to \infty$.*

PROOF: As we saw in the proof of the preceding theorem, any representation of $\mathcal{F}$ sends u to a power partial isometry, and, conversely, any power partial isometry generates a representation of $\mathcal{F}$. It follows from the Halmos–Wallen Theorem that the only power partial isometries which act irreducibly are S, S^*, J_n and the scalars $\{e^{i\theta}\}$, so the first part is clear. Notice that $\pi_n(C^*(\mathcal{F})) = M_n(\mathbb{C})$, $\rho_\theta(C^*(\mathcal{F})) = \mathbb{C}$ and $\pi_S(C^*(\mathcal{F})) = \pi_{S^*}(C^*(\mathcal{F})) = C^*(S)$ contains $\mathcal{K}(l^2)$ (by Coburn's theorem), so $C^*(\mathcal{F})$ is type I and $C^*(\mathcal{F})^\wedge =$ Prim $C^*(\mathcal{F})$.

The quotient map $\pi_S : C^*(\mathcal{F}) \to C^*(S)$ embeds $C^*(S)^\wedge$ as a closed subset C_1 of $C^*(\mathcal{F})^\wedge$; Coburn's theorem identifies the representations in C_1 as π_S and $\{\rho_\theta\}$, topologised so that $e^{i\theta} \to \rho_\theta$ is a homeomorphism of $\mathbf{T}$ onto $\{\rho_\theta\}$, and the closure of π_S is the whole of C_1. Similarly, $\pi_{S^*} : C^*(\mathcal{F}) \to C^*(S^*)$ embeds $C^*(S^*)^\wedge = C^*(S)^\wedge$ as a closed subset $C_2 = \pi_{S^*} \cup \{\rho_\theta\}$, topologised the same way. (This is slightly more subtle than it looks: if ε_θ is the representation of $C^*(S)$ which sends S to $e^{i\theta}$, then $\varepsilon_\theta \circ \pi_S = \rho_\theta = \varepsilon_{-\theta} \circ \pi_{S^*}$.) Thus $C = C_1 \cup C_2$ has the topology described. To see that C is closed, we observe that the element

$$t = u^*u^2u^* - (u^*)^2u^3u^* - u^*u^3(u^*)^2 + (u^*)^2u^4(u^*)^2 \tag{1}$$

belongs to $\ker\pi_S \cap \ker\pi_{S^*} = I_C = \cup\{\ker\pi : \pi \in C\}$, but not to any $\ker\pi_n$ for $n \leqslant 2$; thus $I_C \not\subset \ker\pi_n$ for any n, and no π_n belongs to $\overline{C}$. The direct sum $\pi_S \oplus \pi_{S^*}$ has kernel I_C, and hence induces an isomorphism of the quotient A/I_C with spectrum C into $C^*(S) \oplus C^*(S^*)$; since $\pi_S \oplus \pi_{S^*}$ is the representation which sends u to $S \oplus S^*$, the image of $\pi_S \oplus \pi_{S^*}$ is just $C^*(S \oplus S^*)$ and we have identified C with $C^*(S \oplus S^*)^\wedge$. We have now proved (a).

We next show that each point π_n is closed in $C^*(\mathcal{F})^\wedge$. The element

$$(u^*)u - uu^* - (u^*)^{n-1}u^{n-1} + u^{n-1}(u^*)^{n-1}$$

of $\mathbf{C}\mathcal{F}$ belongs to $\ker\pi_n$, but not to $\ker\pi_m$ for $m \neq n$, to $\ker\pi_S$ or to $\ker\pi_{S^*}$, and u^n belongs to $\ker\pi_n$ but not to $\ker\rho_\theta$ for any θ; thus $\ker\pi_n$ is not contained in any other primitive ideal of $C^*(\mathcal{F})$, and $\{\pi_n\}$ is closed in $C^*(\mathcal{F})^\wedge =$ Prim $C^*(\mathcal{F})$. To see that each point π_n is also open, it is enough to show that π_n is not in the closure of $\{\pi_m : m > n\} \cup C$; by part (a), this is equivalent to showing $\bigcap_{m>n} \ker\pi_n$ is not contained in $\ker\pi_n$. But if $n = 2$, the element

$$v = \left(uu^* - u^2(u^*)^2\right)(uu^* - u^*u) \tag{2}$$

of $\mathbf{C}\mathcal{F}$ belongs to $\ker\pi_m$ for all $m > 2$, but not to $\ker\pi_n$; if $n > 2$ the element

$$w = \left(uu^* - u^2(u^*)^2\right)\left(u^*u - (u^*)^{n-1}u^{n-1}\right) \tag{3}$$

belongs to $\ker\pi_m$ for all $m > n$ but not to $\ker\pi_n$. We have now shown that $\{\pi_n\}$ is discrete. The final statement follows from the inequality $\|p(S \oplus S^*,\, S^* \oplus S)\| \leqslant \|p(J,\, J^*)\|$ established in the proof of Theorem 2.3, which implies that $\ker\pi_S \oplus \pi_{S^*} \supset \cap\ker\pi_n$. ∎

Corollary 1.6. *The semigroup algebra* $\mathbf{C}\mathcal{F}$ *has a unique* C^**-norm.*

Proof: If A is the completion of $\mathbf{C}\mathcal{F}$ in any C^*-norm, then A must be a quotient of $C^*(\mathcal{F})$, and hence its spectrum must be a closed subset D of $C^*(\mathcal{F})^\wedge$. There are three possibilities:

(a) D is a finite subset of $\{\pi_n\}$;
(b) $D = F \cup E$ for some finite set $F \subset \{\pi_n\}$ and E closed in C;
(c) D contains infinitely many π_n's and hence all of C.

We can discard (a), because for large n, u^n would be an element of $\mathbf{C}\mathcal{F}$ whose image in A vanished. Similarly, we can discard (b), because if $t \in \mathbf{C}\mathcal{F}$ is defined by (1) and n is large, tu^n would have zero image in A. Thus D must have the form (c). But if $n > 2$ and $\pi_n \notin D$, and w is given by (3), then the element $u^{n-1}w$ of $\mathbf{C}\mathcal{F}$ goes to zero in A; if π_2 is missing, the element v given by (2) goes to zero. Thus D must be all of $C^*(\mathcal{F})^\wedge$, and $A = C^*(\mathcal{F})$. ∎

REMARK 1.7. Proposition 1.5 includes the facts about $C^*(\oplus J_n)$ established in [**1**], p.268–270. Since the regular representation λ of $C\mathcal{F}$ is known to be faithful [**20**], it follows easily from the corollary that the C^*-algebra $C^*_r(\mathcal{F})$ generated by the regular representation is isomorphic to $C^*(\mathcal{F}) \cong C^*(\oplus J_n)$; this was proved in [**3**], Corollary 2.12 by computing $\lambda(u) = \oplus(J_n \otimes 1_n)$ and invoking the results of [**1**].

2. THE CUNTZ-KRIEGER SEMIGROUPS

DEFINITION 2.1: Let $A = (A(i,j))_{i,j=1}^n$ be an $n \times n$ matrix with each entry $A(i,j) = 0$ or 1. The *Cuntz-Krieger semigroup* $\mathcal{C}_A$ is the semigroup with 0 element generated by the set $\{s_i, t_i : 1 \leqslant i \leqslant n\}$ subject to the relations

(a) $t_i s_i t_i = t_i$, $s_i t_i s_i = s_i$;
(b) $t_j s_i = 0$ for $j \neq i$;
(c) $(t_i s_i)(s_j t_j) = A(i,j)(s_j t_j) = (s_j t_j)(t_i s_i)$;
(d) $(t_i s_i)(t_j s_j) = (t_j s_j)(t_i s_i)$;

where, for $x \in \mathcal{C}_A$, $A(i,j)x$ means x if $A(i,j) = 1$, or 0 if $A(i,j) = 0$.

REMARK 2.2. We should point out that, even when $A(i,j) = 1$ for all i,j, this is not the same as the Cuntz semigroup O_n studied in [**17**], p.141: there the relations (a) are replaced by the stronger condition $t_i s_i = 1$. However, O_n is naturally a quotient of this $\mathcal{C}_A$, and we shall see later what happens at the C^*-algebraic level.

We want to show that $\mathcal{C}_A$ is an inverse semigroup. Our proof will follow the same general principles as those of Section 2: we begin by reducing elements of $\mathcal{C}_A$ to a simple form, then show that every element x of $\mathcal{C}_A$ has an adjoint x^*, and finally construct a faithful representation of $\mathcal{C}_A$ by partial isometries. Since our relations are abstractions of those considered in [**7**], we can achieve the first step in this program by checking that the reductions in [**7**], Section 2, are still valid in our setting. For this we shall need some notation. We write $p_i = s_i t_i$, $q_i = t_i s_i$ and note that our relations imply that p_i, q_i are commuting projections. More generally, for any multi-index $\mu = (\mu_1, \dots, \mu_k)$ with $1 \leqslant \mu_i \leqslant n$, we set $|\mu| = k$ and

$$s_\mu = s_{\mu_1} s_{\mu_2} \cdots s_{\mu_k}, \quad t_\mu = t_{\mu_k} t_{\mu_{k-1}} \cdots t_{\mu_1}, \quad p_\mu = s_\mu t_\mu, \quad q_\mu = t_\mu s_\mu.$$

(Caution: we deliberately wrote μ backwards when defining t_μ.) In fact, p_μ and q_μ are projections, but this is not yet obvious – it will follow from Lemma 2.3.

LEMMA 2.3. (a) $s_i s_j \neq 0 \Rightarrow A(i,j) = 1$; $t_i t_j \neq 0 \Rightarrow A(j,i) = 1$.

(b) *If* $\mu = (\mu_1, \cdots, \mu_k)$ *is a multi-index such that* $s_\mu \neq 0$, *then* $A(\mu_j, \mu_{j+1}) = 1$ *for* $1 \leqslant j < k$, *and* $q_\mu = t_{\mu_k} s_{\mu_k}$.

(c) *Every element of* $\mathcal{C}_A$ *has the form* $s_\mu q_\lambda t_\nu$ *for some multi-indices* μ, λ, ν.

PROOF: We have

$$s_is_j = s_it_is_is_jt_js_j = s_i(A(i,j)s_jt_j)s_j = A(i,j)s_is_j,$$

which immediately gives the first part of (a); the second is similar. The first assertion in (b) follows from (a), and then

$$\begin{aligned} t_\mu s_\mu &= t_{\mu_k}\cdots t_{\mu_2}(t_{\mu_1}s_{\mu_1})(s_{\mu_2}t_{\mu_2})s_{\mu_2}s_{\mu_3}\cdots s_{\mu_k} \\ &= t_{\mu_k}\cdots t_{\mu_2}A(\mu_1,\mu_2)s_{\mu_2}t_{\mu_2}s_{\mu_2}\cdots s_{\mu_k} \\ &= t_{\mu_k}\cdots t_{\mu_2}s_{\mu_2}s_{\mu_3}\cdots s_{\mu_k}, \end{aligned}$$

since $A(\mu_1,\mu_2) = 1$. After k steps like this, we obtain $q_u = t_{\mu_k}s_{\mu_k}$. To see that (c) works, we just have to observe that

$$\begin{aligned} t_js_i &= \delta_{ij}q_i, \\ t_jq_i &= (t_js_jt_j)(t_is_i) = t_jA(i,j)s_jt_j = A(i,j)t_j, \\ q_is_j &= (t_is_i)(s_jt_js_j) = A(i,j)s_jt_js_j = A(i,j)s_j, \end{aligned}$$

and we can therefore reduce any word to one of the form $s_\mu q_\lambda t_\nu$. ∎

LEMMA 2.4. *For every element x of C_A there is an element y satisfying $yxy = y$ and $xyx = x$; indeed, if $x = s_\mu q_\lambda t_\nu$, we can take $y = s_\nu q_\lambda t_\mu$.*

PROOF: Use Lemma 2.3 to write $x = s_\mu q_\lambda t_\nu$, and take y as suggested. Then

$$\begin{aligned} yxy &= s_\nu q_\lambda (q_{\mu_k}) q_\lambda (q_{\nu_m}) q_\lambda t_\mu \quad \text{by Lemma 2.3(b)} \\ &= s_\nu q_{\nu_m} q_\lambda q_{\mu_k} t_\mu \quad \text{since the } q_i \text{ commute} \\ &= s_\nu q_\lambda t_\mu. \end{aligned}$$

Since the situation is completely symmetric, the same argument gives $xyx = x$. ∎

PROPOSITION 2.5. *For any zero-one matrix A, the Cuntz-Krieger semigroup C_A is an inverse semigroup in which $s_i^* = t_i$.*

PROOF: Take $H = \mathbb{C}^{n+1} \oplus l^2$, let P_i be the projection onto the subspace of l^2 spanned by $\{e_{i+kn} : k \in \mathbb{N}\}$, and let Q_i be defined by

$$Q_i(z,\xi) = \left(\widehat{z}_i, \sum_j A(i,j)P_j(\xi)\right),$$

where $\widehat{z}_i$ is z with the ith co-ordinate replaced by 0. Let S_i be a partial isometry with initial space $Q_i(H)$ and range space $P_i(H)$ – defined, say, by viewing $(z,\xi) \in Q_i(H)$ as a sequence

$$(z_1, z_2, \cdots, \widehat{z}_i, \cdots z_{n+1}, \xi_1, \xi_2, \cdots),$$

and writing this sequence in the slots labelled $i+kn$ for $k \in \mathbb{N}$. Since it is easy to check that the operators S_i, $T_i = S_i^*$ satisfy the relations of Definition 2.1, we can define a representation $\pi\colon \mathcal{C}_A \to B(H)$ by $\pi(s_i) = S_i$, $\pi(t_i) = S_i^*$. We shall show that π is faithful; since Lemma 2.4 shows that each $\pi(x)$ is a partial isometry and Lemma 2.3 that $\pi(\mathcal{C}_A)$ is a *-subsemigroup of $B(H)$, this will establish the result by ([9], 1.1).

We first show that $s_\mu q_\lambda t_\nu \neq 0$ implies $\pi(s_\mu q_\lambda t_\nu) \neq 0$. By Lemma 2.3(b), we have $A(\mu_j, \mu_{j+1}) = 1$ for all j; thus $P_{\mu_j}(H) \subset Q_{\mu_{j+1}}(H)$ for all j, and S_μ is a partial isometry with initial space $Q_{\mu_k}(H)$ – in particular, $S_\mu \neq 0$. Similarly, $\pi(t_\nu) = S_\nu^*$ is a non-zero partial isometry with range space $Q_{\nu_m}(H)$. Since $Q_{\nu_m}(H)$, $Q_\lambda(H)$ and $Q_{\mu_k}(H)$ all contain the vector $(e_{n+1}, 0) \in \mathbb{C}^{n+1} \oplus l^2$, this means $S_\mu Q_\lambda S_\nu^* = \pi(s_\mu q_\lambda t_\nu)$ is non-zero.

Now suppose that $S_\mu Q_\lambda S_\nu^* = S_{\mu'} Q_{\lambda'} S_{\nu'}^* \neq 0$. We have

$$\text{range } S_\mu Q_\lambda S_\nu^* \subset \text{ range } S_\mu = P_{\mu_1}(H), \ \text{ range } S_{\mu'} Q_{\lambda'} S_{\nu'}^* \subset P_{\mu_1'}(H)$$

and $P_i(H) \perp P_j(H)$ for $i \neq j$, so μ_1 must equal μ_1'. For any i, j we either have $P_j(H) \perp Q_i(H)$ or $P_j(H) \subset Q_i(H)$; since $S_{\mu_1} S_{\mu_2} \neq 0$, we must have $P_{\mu_2}(H) \subset Q_{\mu_1}(H)$ and $S_{\mu_2} = Q_{\mu_1} S_{\mu_2}$. Thus

$$S_{\mu_2} \cdots S_{\mu_k} Q_\lambda S_\nu^* = S_{\mu_1}^* (S_\mu Q_\lambda S_\nu^*) = S_{\mu_1}^* \left(S_{\mu'} Q_{\lambda'} S_{\nu'}^*\right) = S_{\mu_2'} \cdots S_{\mu_l'} Q_{\lambda'} S_{\nu'}^*.$$

Repeating this argument, and supposing without loss of generality that $|\mu'| \geqslant |\mu|$, we obtain $Q_\lambda S_\nu^* = S_{\mu''} Q_{\lambda'} S_{\nu'}^*$ for some shorter multi-index μ''. Applying the same reasoning to $S_\nu Q_\lambda = S_{\nu'} Q_{\lambda'} S_{\mu''}^*$ yields either (a) $Q_\lambda = S_{\nu''} Q_{\lambda'} S_{\mu''}^*$, or (b) $S_{\nu''} Q_\lambda = Q_{\lambda'} = Q_{\lambda'} S_{\mu''}^*$.

Suppose (a) occurs. Then we must have ν'' trivial, for otherwise

$$0 \neq Q_\lambda(H) = S_{\nu''} Q_{\lambda'} S_{\mu''}^*(H) \subset P_{\nu_1''}(H),$$

which is impossible since $(e_{n+1}, 0)$ belongs to $Q_\lambda(H)$ but not to $P_i(H)$ for any i. Similarly, μ'' must be trivial, and we have $\mu = \mu'$, $\nu = \nu'$ and $Q_\lambda = Q_{\lambda'}$. But the projection of $Q_\lambda(H)$ in $\mathbb{C}^{n+1}$ completely determines the set $\{\lambda_i\}$ (multiplicities are irrelevant here), so we must have $\{\lambda_i\} = \{\lambda_j'\}$, $q_\lambda = q_{\lambda'}$ and $s_\mu q_\lambda t_\nu = s_{\mu'} q_{\lambda'} t_{\nu'}$, as required. Now suppose (b) happens. Then if i is the last entry in ν'', we have $Q_i Q_{\lambda'} = S_{\nu''} Q_\lambda S_{\mu''}$, and exactly the same reasoning implies that ν'' is trivial. Similarly, $Q_{\lambda'} = Q_\lambda S_{\mu''}$ implies that the range of $Q_{\lambda'}$ is contained in $P_{\mu_1''}(H)$, which is impossible unless μ'' is trivial. Thus $Q_{\lambda'} = Q_\lambda$, and, as above, this implies $s_\mu q_\lambda t_\nu = s_{\mu'} q_{\lambda'} t_{\nu'}$. We have now shown that π is faithful, and the result follows. ∎

We now wish to describe the connection between $C^*(\mathcal{C}_A)$ and the Cuntz-Krieger algebra $\mathcal{O}_A$, which is generated by n partial isometries S_i whose initial and range

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projections Q_i, P_i satisfy $P_jP_i = 0$ for $j \neq i$ and $Q_i = \sum_j A(i,j)P_j$. Now whenever π is a representation of $\mathcal{C}_A$ and $S_i = \pi(s_i)$, our condition (c) implies that $Q_i \geqslant \sum_j A(i,j)P_j$. Our main theorem asserts that, loosely speaking, the element $p = 1 - \sum p_i$ generates an ideal I of $C^*(\mathcal{C}_A)$, whose structure is independent of the matrix A, such that the quotient $C^*(\mathcal{C}_A)/I$ is isomorphic to $\mathcal{O}_A$. For the last part to make sense, we have to assume that A satisfies condition (I) of [7], which ensures that $\mathcal{O}_A$ is independent of the choice of partial isometries S_i ([7], 2.13); for other A, we suggest that $C^*(\mathcal{C}_A)/I$ could be used as a good analogue of $\mathcal{O}_A$. The proof of our theorem is based on that of ([6], 3.1), which deals with a similar problem for the case where $A(i,j) = 1$ for all i,j (see Remark 2.7).

THEOREM 2.6. *The elements of the form $s_\alpha(q_\gamma - \sum_i q_\gamma p_i)t_\beta$ span a closed ideal I in $C^*(\mathcal{C}_A)$, which is isomorphic to the direct sum of at most $2^n - 1$ copies of the algebra $\mathcal{K}$ of compact operators. If A satisfies condition (I) of [7], then $C^*(\mathcal{C}_A)/I$ is isomorphic to $\mathcal{O}_A$.*

PROOF: We shall write $p = 1 - \sum p_i$ and $s_\alpha q_\gamma p t_\beta$ for the generator $s_\alpha(q_\gamma - \sum q_\gamma p_i)t_\beta$; although $C\mathcal{C}_A$ does not really have an identity, this should cause no problems provided we always multiply p by some q_γ. If $I_0 = \operatorname{sp}\{s_\alpha q_\gamma p t_\beta\}$, then I_0 is obviously closed under left multiplication by elements of the form s_μ, and also by elements of the form q_γ, since $q_i s_j = A(i,j)s_j$. A product $t_\nu s_\alpha$ can only be non-zero if either $\nu = (\alpha_1, \cdots, \alpha_j)$ for some $j \leqslant |\alpha|$, or $\alpha = (\nu_1, \cdots, \nu_k)$ for some $k \leqslant |\nu|$. Thus when $t_\nu s_\alpha \neq 0$ we have

$$t_\nu s_\alpha q_\gamma p t_\beta = \begin{cases} t_{\nu'} q_{\nu_k} q_\gamma p t_\beta & \text{for some multi-index } \nu' \text{ (if } |\alpha| < |\nu|) \\ q_{\nu_k} q_\gamma p t_\beta & \text{(if } |\alpha| = |\nu|) \\ q_{\nu_j} s_{\alpha'} q_\gamma p t_\beta & \text{for some multi-index } \alpha' \text{ (if } |\alpha| > |\nu|). \end{cases}$$

The last two certainly belong to I_0, and the first is zero since $t_i = t_i p_i$ implies $t_i p = 0$ for any i. Thus I_0 is closed under left multiplication by any element $s_\mu q_\lambda t_\nu$ of $\mathcal{C}_A$. Exactly the same arguments work on the other side, so I_0 is an ideal in $C\mathcal{C}_A$, and its closure I is an ideal in $C^*(\mathcal{C}_A)$.

We next want to show that, for fixed m, the span of $\{s_\alpha q_\gamma p t_\beta : |\alpha| \leqslant m, |\beta| \leqslant m\}$ is a direct sum of matrix algebras, and for this we need to decompose the projections $q_\gamma p$ into sums of minimal projections. For each nonempty subset J of $\{1, 2, \cdots, n\}$ we define

$$r_J = \left(\prod_{i \in J} q_i\right)\left(\prod_{i \notin J}(1 - q_i)\right)p;$$

again this makes good sense in $\mathcal{C}_A$ because the 1's disappear when we multiply out. The family $\{r_J : \phi \neq J \subset \{1, \cdots, n\}\}$ consists of $2^n - 1$ mutually orthogonal idempotents,

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and we claim that every $q_\lambda p$ is a sum of r_J's. To see this, we first note that for any commuting family $\{x_i : 1 \leqslant i \leqslant k\}$ with $k \geqslant 1$, we have the formal identity

$$\sum_{J\subset\{1,\cdots,k\}} \left(\prod_{i\in J} x_i\right)\left(\prod_{i\notin J}(1-x_i)\right) = 1;$$

this can be checked by computing the coefficient of each $x_K = \prod_{i\in K} x_i$ on the left-hand side. Thus if $\lambda = \{\lambda_1, \cdots, \lambda_l\}$ and $\{i : i \notin \lambda\} = \{i_1, \cdots, i_k\}$, we have

$$\begin{aligned} q_\lambda p &= \left(\prod_{j=1}^{l} q_{\lambda_j}\right) p \\ &= \left(\prod_{j=1}^{l} q_{\lambda_j}\right)\left(\sum_{J\subset\{i_1,\cdots,i_k\}}\left(\prod_{i\in J} q_i\right)\left(\prod_{i\notin J}(1-q_i)\right)\right)p \\ &= \sum_{J\subset\{i_1,\cdots,i_k\}} r_{J\cup\lambda}, \end{aligned}$$

which establishes the claim.

We have now shown that every generator $s_\alpha q_\gamma p t_\beta$ for I is a sum of elements of the form $s_\alpha r_J t_\beta$. Since r_J commutes with each q_i, we have

$$\begin{aligned} (s_\alpha r_J t_\beta)(s_\mu r_K t_\nu) &= \delta_{\beta\mu} s_\alpha r_J q_{\mu_k} r_K t_\nu \qquad \text{(recall } t_i p = p s_j = 0\text{)} \\ &= \delta_{\beta\mu}\delta_{JK} s_\alpha r_J q_{\mu_k} t_\nu \\ &= \begin{cases} \delta_{\beta\mu}\delta_{JK} s_\alpha r_J t_\nu & \text{if } \mu_k \in J \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

If we fix J such that $r_J \neq 0$ and set $I_J^m = \{\alpha = (\alpha_1, \cdots, \alpha_k) : k \leqslant m, \alpha_k \in J\}$, then the previous calculation with $J = K$ says that

$$\{s_\alpha r_J t_\beta : \alpha, \beta \in I_J^m\}$$

is a complete set of matrix units; their span is an algebra $A(m, J)$ isomorphic to $M(n_J^m, \mathbb{C})$, where $n_J^m = \sum_{k=1}^{m} n^{k-1}|J|$ is the number of elements in I_J^m. For fixed J, we have $A(m, J) \subset A(m+1, J)$, and $\bigcup_{m=1}^{\infty} A(m, J)$ is a subalgebra of I whose closure A_J is isomorphic to $\mathcal{K}$. The above calculation also shows that $A(m, J)A(l, K) = 0$ unless $J = K$, and since $\sum A(m, J) = I_0$ is dense in I, it follows that I is the C^*-algebraic direct sum $\oplus\{A_J : r_J \neq 0\}$. Since there are $2^n - 1$ nonempty subsets J of $\{1, \cdots, n\}$, this establishes the first part of the theorem.

Now let π: $C^*(\mathcal{C}_A) \to B(H)$ be a non-degenerate representation with kernel I. Then for any α, β we have $\pi(t_\beta)(1 - \sum \pi(p_i)) = 0$, and

$$\pi(s_\alpha)\Big(1 - \sum \pi(p_i)\Big) = \pi\Big(s_\alpha q_{\alpha_k}\Big(1 - \sum p_i\Big)\Big) = 0$$

because $s_\alpha q_{\alpha_k}(1 - \sum p_i)$ belongs to I. Since π is non-degenerate, it follows that the projection $(1 - \sum \pi(p_i))$ is zero. For each j, we have $\pi(q_j) \geqslant \sum_i A(j,i)\pi(p_i)$, and $\pi(q_j)\pi(p_i) = 0$ if $A(j,i) = 0$; therefore $\sum \pi(p_i) = 1$ implies $\pi(q_i) = \sum_i A(j,i)\pi(p_i)$. Thus the algebra $\pi(C^*(\mathcal{C}_A)) \cong C^*(\mathcal{C}_A)/I$ is isomorphic to $\mathcal{O}_A$ by ([7], 2.13). This completes the proof of the theorem. $\square$

REMARK 2.7. (1) If we try a similar argument for the semigroup O_n of [**17**], all the projections $q_i = t_i s_i$ are 1, and the only r_J which is non-zero is $p = 1 - \sum p_i$, which occurs when J is all of $\{1, \cdots, n\}$. Thus for this semigroup, $C^*(O_n)$ contains an ideal $I \cong \mathcal{K}$, such that $C^*(O_n)/I$ is isomorphic to the Cuntz algebra $\mathcal{O}_n$. This is effectively the argument given in ([4], 3.1), which we have merely adapted to our setting. When $n = 1$, the semigroup O_1 is the bicyclic semigroup we mentioned at the start of Section 1, and the algebra $\mathcal{O}_1 = C^*(O_1)/\mathcal{K}$ is isomorphic to $C(\mathbf{T})$ by Coburn's theorem.

(2) Theorem 2.6 shows that, for fixed n, the only part of $C^*(\mathcal{C}_A)$ which depends on the choice of A is the quotient $\mathcal{O}_A$. In [**7, 6**] the structure of this algebra, and how it depends on A, is discussed in detail. In particular, if A is irreducible (in the sense that for each (i,j) there exists n with $A^n(i,j) \neq 0$), and A is not a permutation matrix, then $\mathcal{O}_A$ is a simple C^*-algebra ([7], 2.14).

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