

AN INVERSE GAMMA ACTIVITY TIME PROCESS WITH NONINTEGER PARAMETERS AND A SELF-SIMILAR LIMIT

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Abstract

We construct a process with inverse gamma increments and an asymptotically self-similar limit. This construction supports the use of long-range-dependent t subordinator models for actual financial data as advocated in Heyde and Leonenko (2005), in that it allows for noninteger-valued model parameters, as is found empirically in model estimation from data.

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1. Introduction

Heyde and Leonenko (2005) and Finlay and Seneta (2006) respectively constructed discrete time t and variance-gamma-distributed (VG-distributed) subordinator models which exhibit long-range dependence (LRD) of squared returns, a desirable property for asset price models. In particular, they considered models of the form

$$\log(P_t) = \log(P_0) + \mu t + \theta T_t + \sigma B(T_t),$$

where P_t is some asset price, $B(t)$ is Brownian motion, $\mu, \theta \in \mathbb{R}$, and $\sigma > 0$. The LRD of squared returns comes from asymptotically self-similar inverse gamma (for the t -model) and gamma (for the VG model) based ‘activity time’ $\{T_t\}$ processes, and in particular is driven by the LRD of the increment processes, denoted by $\tau(t) = T_t - T_{t-1}$, $t = 1, 2, \dots$. Here a continuous-time process $\{Y_t\}$ is said to be self-similar with parameter H if $Y_{ct} \stackrel{D}{=} c^H Y_t$; LRD of a discrete-time stationary process with ultimately nonnegative autocorrelations $\{\gamma_k\}$ is said to hold if $\sum_{k=1}^{\infty} \gamma_k = \infty$; and the activity time T_t , as opposed to the standard clock time t , is the increasing stochastic process over which security prices are taken to evolve. The $\{T_t\}$ processes are scaled such that their increments over unit time have unit expectation, with the $\tau(t)$ taken to be inverse (or reciprocal) gamma $R\Gamma(\nu/2, \nu/2 - 1)$ and gamma $\Gamma(\nu/2, \nu/2)$ distributed, respectively having probability density functions

$$f_{R\Gamma}(x) = \frac{(\nu/2 - 1)^{\nu/2}}{\Gamma(\nu/2)} x^{-\nu/2-1} e^{(1-\nu/2)/x}$$

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and

$$f_{\Gamma}(x) = \frac{(v/2)^{v/2}}{\Gamma(v/2)} x^{v/2-1} e^{-(v/2)x}$$

for $x > 0$. Here if $V \sim \Gamma(\alpha, \beta)$ is gamma distributed (so that $B(V)$ has the VG distribution) then $1/V \sim R\Gamma(\alpha, \beta)$ is inverse gamma distributed (and $B(1/V)$ has the t -distribution).

The activity time processes constructed in Heyde and Leonenko (2005) and Finlay and Seneta (2006) are restricted to integer v however, a condition not consistent with estimation with actual data. Finlay and Seneta (2007) extended $\tau(t)$, the gamma process from which the activity time process $T_t = \sum_{k=1}^t \tau(k)$ is built, to allow for noninteger v . In this note we do likewise for the inverse gamma process.

2. The inverse gamma increment process $\tau(t)$

Let $\{X_t, t \in \mathbb{N}\}$ be the stationary gamma process constructed in Finlay and Seneta (2007, Section 2), which, for $\iota = (v - [v])/2$ ($[v]$ denoting the integer part of v), has a $\Gamma(\iota, \frac{1}{2})$ distribution and bivariate characteristic function given by

$$E(e^{iuX_t + viX_{t+s}}) = ((1 - 2iu)(1 - 2iv))^{-\iota + \iota\rho_2(s)} (1 - 2i(u + v))^{-\iota\rho_2(s)}. \tag{1}$$

Here $\rho_2(s)$ is the autocorrelation function of $\{X_t\}$, and can be taken as any function that satisfies Assumption 1 below.

Assumption 1. $\rho_2(s)$ is a function on $\{0, 1, 2, \dots\}$ satisfying $\rho_2(0) = 1$, $\rho_2(s) \geq 0$, $\rho_2(s) - \rho_2(s + 1) \geq 0$, and $\rho_2(s) - 2\rho_2(s + 1) + \rho_2(s + 2) \geq 0$.

The construction in Finlay and Seneta (2007, Section 2) has the consequence of providing an indirect proof that any function $\rho_2(s)$ satisfying Assumption 1 is an autocorrelation function of a strictly stationary process. The same issue is discussed more generally in Finlay *et al.* (2011).

Next, for $\mathbb{N} = \{1, 2, 3, \dots\}$, let $\{\eta_i(t), t \in \mathbb{N}\}$, $i = 1, \dots, [v]$, with noninteger $v > 4$, be independent and identically distributed stationary Gaussian processes with zero mean, unit variance, and autocorrection function $\rho_1(s)$, $s \in \mathbb{N}$. We assume that $\rho_1(s)$ is of the form

$$\rho_1(s) = (1 + \omega|s|^\alpha)^{(H-1)/\alpha}$$

for $\omega > 0$, $0 < \alpha \leq 2$, and $\frac{1}{2} < H < 1$. That is, we assume that $\rho_1(s)$ belongs to the Cauchy family of autocorrection functions discussed in Gneiting (2000).

Define the independent stationary processes $\{\chi_t^2\}$ and $\{V_t\}$, $t \in \mathbb{N}$, by

$$\chi_t^2 = \frac{\eta_1^2(t) + \dots + \eta_{[v]}^2(t)}{2} \sim \Gamma\left(\frac{[v]}{2}, 1\right), \quad V_t = \frac{X_t}{2} \sim \Gamma(\iota, 1).$$

Then $(\chi_t^2 + V_t) \sim \Gamma(v/2, 1)$ and $\text{cor}(\chi_t^2 + V_t, \chi_{t+s}^2 + V_{t+s}) = ([v]/v)\rho_1^2(s) + (1 - [v]/v)\rho_2(s)$.

We proceed to construct our inverse gamma activity time process $\{T_t\}$ as

$$T_t = \sum_{k=1}^t \tau(k), \quad \tau(t) = \frac{v/2 - 1}{\chi_t^2 + V_t} \sim R\Gamma\left(\frac{v}{2}, \frac{v}{2} - 1\right). \tag{2}$$

This construction ensures that the increments of the activity time process $\{T_t\}$ are inverse gamma distributed and that $E(\tau(t)) = 1$.

Now, from Leonenko (1999, Chapter 2) for example, for $p(x) = x^{[\nu]/2-1}e^{-x}/\Gamma([\nu]/2)$, the density function of χ_t^2 , we can define a complete orthogonal system of functions $\{e_k(x)\}_{k=0}^\infty$ in the Hilbert space $\mathcal{L}_2((0, \infty), p(x) dx)$ by $e_0(x) \equiv 1$ and, for $k > 0$,

$$e_k(x) = L_k^{[\nu]/2-1}(x) \left(\frac{k! \Gamma([\nu]/2)}{\Gamma([\nu]/2 + k)} \right)^{1/2}, \quad \text{where} \quad L_k^\beta(x) = \frac{1}{k!} x^{-\beta} e^x \frac{d^k}{dx^k} \{x^{\beta+k} e^{-x}\} \quad (3)$$

are the generalised Laguerre polynomials of index β . That is, for $G(x) \in \mathcal{L}_2((0, \infty), p(x) dx)$,

$$G(\chi_t^2) = \sum_{k=0}^\infty C_k e_k(\chi_t^2), \quad \text{where} \quad C_k = \int_0^\infty G(x) p(x) e_k(x) dx, \quad (4)$$

and $\sum_{k=0}^\infty C_k^2 = \int_0^\infty G(x)^2 p(x) dx < \infty$. For δ_k^m , the Kronecker symbol, we also have the useful property

$$E(e_k(\chi_t^2) e_m(\chi_{t+s}^2)) = \delta_k^m \rho_1^{2k}(s), \quad (5)$$

so, for example, $\text{cov}(G(\chi_t^2), G(\chi_{t+s}^2)) = \sum_{k=1}^\infty C_k^2 \rho_1^{2k}(s)$.

Now, for fixed $y > 0$ and $\nu > 4$, $(\nu/2 - 1)/(x + y) \in \mathcal{L}_2((0, \infty), p(x) dx)$, so, given V_t , we can expand $(\nu/2 - 1)/(\chi_t^2 + V_t)$ as $\sum_{k=0}^\infty C_k(V_t) e_k(\chi_t^2)$. From (5),

$$\infty > E\left(\frac{\nu/2 - 1}{\chi_t^2 + V_t}\right)^2 = E\left(E\left(\left(\sum_{k=0}^\infty C_k(V_t) e_k(\chi_t^2)\right)^2 \middle| V_t\right)\right) = \sum_{k=0}^\infty E(C_k^2(V_t)).$$

Given the above,

$$\begin{aligned} E\left(\frac{\nu/2 - 1}{\chi_t^2 + V_t} - \sum_{k=0}^n C_k(V_t) e_k(\chi_t^2)\right)^2 &= E\left(E\left(\left(\sum_{k=n+1}^\infty C_k(V_t) e_k(\chi_t^2)\right)^2 \middle| V_t\right)\right) \\ &= \sum_{k=n+1}^\infty E(C_k^2(V_t)), \end{aligned}$$

which goes to 0 as $n \rightarrow \infty$. Thus, we have (in terms of mean-square convergence)

$$\tau(t) = \frac{\nu/2 - 1}{\chi_t^2 + V_t} = \sum_{k=0}^\infty C_k(V_t) e_k(\chi_t^2) \sim R\Gamma\left(\frac{\nu}{2}, \frac{\nu}{2} - 1\right). \quad (6)$$

Our inverse gamma activity time process $\{T_t\}$ will be constructed as the sum of these $\tau(t)$, as given in (2).

3. Main result

Consider the normalized inverse gamma activity time process given by

$$\frac{1}{n^H} (T_{[nt]} - [nt]) = \frac{1}{n^H} \sum_{k=1}^{[nt]} (\tau(k) - 1), \quad (7)$$

where in what follows we will hold ν fixed and let $n \rightarrow \infty$, and, therefore, normalise by n^H (note that ν appears in the limit process—see (15) below—but not in a form that

requires normalisation). From (6) and for $\bar{C}_1 = E(C_1(V_t))$, we can decompose this process into the four terms

$$T_{[nt]} - [nt] = \underbrace{\sum_{m=1}^{[nt]} \sum_{k=2}^{\infty} (C_k(V_m)e_k(\chi_m^2))}_{A_{nt}} + \underbrace{\sum_{m=1}^{[nt]} ((C_1(V_m) - \bar{C}_1)e_1(\chi_m^2))}_{B_{nt}} + \underbrace{\sum_{m=1}^{[nt]} (\bar{C}_1 e_1(\chi_m^2))}_{C_{nt}} + \underbrace{\sum_{m=1}^{[nt]} (C_0(V_m) - 1)}_{D_{nt}},$$

where the equality holds in the sense of mean-square convergence. We then have the following result.

Theorem 1. *For any $\rho_2(s)$ satisfying Assumption 1 and further satisfying $\sum_{s=1}^{\infty} \rho_2(s) < \infty$, A_{nt}/n^H , B_{nt}/n^H , and D_{nt}/n^H converge in mean square to 0 as $n \rightarrow \infty$ and C_{nt}/n^H converges weakly to the sum of $[v]$ independent copies of the H self-similar Rosenblatt process.*

Theorem 1 is the main result of this paper and implies that the process given in (7) converges weakly to a sum of self-similar Rosenblatt processes (the Rosenblatt process is discussed in greater detail below). It complements the result of Finlay and Seneta (2007) for gamma activity time processes and supports the use of long-range-dependent t subordinator models for actual financial data as advocated in Heyde and Leonenko (2005), in that it proves that activity time processes $\{T_t\}$ which display asymptotic self-similarity and have inverse gamma distributed increments (and so imply asset price increments with the t -distribution) exist.

Proof of Theorem 1. First consider

$$A_{nt} = \sum_{m=1}^{[nt]} \sum_{k=2}^{\infty} (C_k(V_m)e_k(\chi_m^2)).$$

For $k > 0$, $E(e_k(\chi_m^2)) = 0$, so $E(A_{nt}) = 0$. Then, since the $\{V_t\}$ and $\{\chi_t^2\}$ are independent, using (5), the Cauchy–Schwarz inequality, and the functional form of $\rho_1^2(s)$, we have, irrespective of $\rho_2(s)$,

$$\begin{aligned} \text{var}\left(\frac{A_{nt}}{n^H}\right) &= \frac{1}{n^{2H}} \sum_{k=2}^{\infty} \sum_{m=1}^{[nt]} \sum_{s=1}^{[nt]} E(C_k(V_m)C_k(V_s))\rho_1^{2k}(|s - m|) \\ &\leq \sum_{k=2}^{\infty} E(C_k^2(V_0))\frac{1}{n^{2H}} \sum_{m=1}^{[nt]} \sum_{s=1}^{[nt]} \rho_1^4(|s - m|) \\ &= k_1^* \left(\frac{[nt]}{n^{2H}} + \frac{2}{n^{2H}} \sum_{s=1}^{[nt]-1} ([nt] - s)(1 + \omega|s|^\alpha)^{4(H-1)/\alpha} \right) \end{aligned}$$

for $k_1^* = \sum_{k=2}^{\infty} E(C_k^2(V_0)) < \infty$. Here the last two lines follow since $\rho_1^{2k}(|s - m|) \leq \rho_1^4(|s - m|)$ for $k \geq 2$, and, for any positive integer a and any function I ,

$$\sum_{m=1}^a \sum_{s=1}^a I(|s - m|) = aI(0) + 2 \sum_{s=1}^{a-1} (a - s)I(s). \tag{8}$$

Now $[nt]/n^{2H} \rightarrow 0$ for $\frac{1}{2} < H < 1$, and, for $\frac{1}{2} < H < 1$, $H \neq \frac{3}{4}$, and $k_2^* < \infty$ a constant, we have

$$\frac{2}{n^{2H}} \sum_{s=1}^{[nt]-1} ([nt] - s)(1 + \omega|s|^\alpha)^{4(H-1)/\alpha} < \omega^{4(H-1)/\alpha} \frac{[nt]}{n^{2H}} \sum_{s=1}^{[nt]} s^{4(H-1)} \tag{9}$$

$$\begin{aligned} &\leq k_2^*(nt)^{1-2H} \int_1^{nt} s^{4(H-1)} ds \\ &= k_2^* \left(\frac{(nt)^{2H-2} - (nt)^{1-2H}}{4H - 3} \right), \end{aligned} \tag{10}$$

which is positive but goes to 0 as $n \rightarrow \infty$. Here (9) follows since, for $s \geq 1, \alpha, \gamma, \omega > 0$, and $\frac{1}{2} < H < 1$,

$$(1 + \omega s^\alpha)^{\gamma(H-1)/\alpha} < \omega^{\gamma(H-1)/\alpha} s^{\gamma(H-1)}. \tag{11}$$

For $H = \frac{3}{4}$, (10) becomes

$$k_2^*(nt)^{-1/2} \int_1^{nt} \frac{ds}{s} = k_2^*(nt)^{-1/2} \log(nt) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Hence, $\text{var}(A_{nt}/n^H) \rightarrow 0$ as $n \rightarrow \infty$ for $\frac{1}{2} < H < 1$ and so A_{nt}/n^H converges to 0 in mean square for each t , irrespective of $\rho_2(s)$.

Next consider

$$B_{nt} = \sum_{m=1}^{[nt]} ((C_1(V_m) - \bar{C}_1)e_1(\chi_m^2)).$$

First note that, for $G(x) = (\nu/2 - 1)/(x + y)$ and $p(x) = x^{[\nu]/2-1} e^{-x} / \Gamma([\nu]/2)$, we can compute $C_1 = C_1(y) = \int_0^\infty G(x)p(x)e_1(x) dx$ from (4) as

$$C_1(y) = \left(\frac{\nu}{2} - 1\right) \sqrt{\frac{[\nu]}{2}} \int_1^\infty e^{y(1-t)} (t^{-[\nu]/2} - t^{-[\nu]/2-1}) dt.$$

Then, from (1), since $V_m(1 - t) = X_m((1 - t)/2)$, we have, for $t, u \geq 1$,

$$0 \leq E(e^{V_m(1-t)+V_s(1-u)}) = (tu)^{-t+\iota\rho_2(|s-m|)}(t+u-1)^{-\iota\rho_2(|s-m|)} \leq (tu)^{-t+\iota\rho_2(|s-m|)}. \tag{12}$$

Hence,

$$\begin{aligned} &0 \leq E(C_1(V_m)C_1(V_s)) \\ &= \left(\frac{\nu}{2} - 1\right)^2 \frac{[\nu]}{2} \int_1^\infty \int_1^\infty E(e^{V_m(1-t)+V_s(1-u)}) \left(\frac{1}{t^{[\nu]/2}} - \frac{1}{t^{[\nu]/2+1}}\right) \\ &\quad \times \left(\frac{1}{u^{[\nu]/2}} - \frac{1}{u^{[\nu]/2+1}}\right) dt du \\ &\leq \left(\frac{\nu}{2} - 1\right)^2 \frac{[\nu]}{2} \int_1^\infty \int_1^\infty (tu)^{-t+\iota\rho_2(|s-m|)} \left(\frac{1}{t^{[\nu]/2}} - \frac{1}{t^{[\nu]/2+1}}\right) \left(\frac{1}{u^{[\nu]/2}} - \frac{1}{u^{[\nu]/2+1}}\right) dt du \\ &= \left(\frac{\nu}{2} - 1\right)^2 \frac{[\nu]}{2} \left(\frac{1}{([\nu]/2 + \iota - \iota\rho_2(|s-m|) - 1)^2} + \frac{1}{([\nu]/2 + \iota - \iota\rho_2(|s-m|))^2} \right. \\ &\quad \left. - \frac{2}{([\nu]/2 + \iota - \iota\rho_2(|s-m|) - 1)([\nu]/2 + \iota - \iota\rho_2(|s-m|))} \right). \end{aligned} \tag{13}$$

Similarly,

$$\begin{aligned}
 (\bar{C}_1)^2 &= (E(C_1(V_m)))^2 \\
 &= \left(\frac{\nu}{2} - 1\right)^2 \frac{[\nu]}{2} \left(\int_1^\infty E(e^{V_m(1-t)}) \left(\frac{1}{t^{[\nu]/2}} - \frac{1}{t^{[\nu]/2+1}}\right) dt\right)^2 \\
 &= \left(\frac{\nu}{2} - 1\right)^2 \frac{[\nu]}{2} \left(\int_1^\infty t^{-\iota} \left(\frac{1}{t^{[\nu]/2}} - \frac{1}{t^{[\nu]/2+1}}\right) dt\right)^2 \\
 &= \left(\frac{\nu}{2} - 1\right)^2 \frac{[\nu]}{2} \left(\frac{1}{([\nu]/2 + \iota - 1)^2} + \frac{1}{([\nu]/2 + \iota)^2} - \frac{2}{([\nu]/2 + \iota - 1)([\nu]/2 + \iota)}\right).
 \end{aligned}
 \tag{14}$$

Now, from (12), (13), and (14),

$$\begin{aligned}
 E(C_1(V_m)C_1(V_s)) - (\bar{C}_1)^2 &= \left(\frac{\nu}{2} - 1\right)^2 \frac{[\nu]}{2} \int_1^\infty \int_1^\infty (tu)^{-\iota} \left(\left(\frac{tu}{t+u-1}\right)^{\iota\rho_2(|s-m|)} - 1\right) \left(\frac{1}{t^{[\nu]/2}} - \frac{1}{t^{[\nu]/2+1}}\right) \\
 &\quad \times \left(\frac{1}{u^{[\nu]/2}} - \frac{1}{u^{[\nu]/2+1}}\right) dt du.
 \end{aligned}$$

Thus, $E(C_1(V_m)C_1(V_s)) - (\bar{C}_1)^2 \geq 0$ since $t, u \geq 1$ and so $tu \geq t+u-1$. Then, for $a > b > 0$ and $b < 1$,

$$0 \leq \frac{1}{(a-b)^2} - \frac{1}{a^2} = b \left(\frac{2a-b}{a^2(a-b)^2}\right) \leq b \left(\frac{2a}{a^2(a-1)^2}\right),$$

so

$$\begin{aligned}
 0 &\leq E((C_1(V_m) - \bar{C}_1)(C_1(V_s) - \bar{C}_1)) \\
 &= E(C_1(V_m)C_1(V_s)) - (\bar{C}_1)^2 \\
 &\leq \left(\frac{\nu}{2} - 1\right)^2 \frac{[\nu]}{2} \left(\frac{1}{([\nu]/2 + \iota - \iota\rho_2(|s-m|) - 1)^2} - \frac{1}{([\nu]/2 + \iota - 1)^2}\right) \\
 &\quad + \frac{1}{([\nu]/2 + \iota - \iota\rho_2(|s-m|))^2} - \frac{1}{([\nu]/2 + \iota)^2} \\
 &\leq k_3^* \rho_2(|s-m|)
 \end{aligned}$$

for k_3^* a constant. Hence, for $B_{nt} = \sum_{m=1}^{[nt]} ((C_1(V_m) - \bar{C}_1)e_1(\chi_m^2))$ and any $\rho_2(s)$ satisfying Assumption 1 and further satisfying $\sum_{s=1}^\infty \rho_2(s) < \infty$, $E(B_{nt}) = 0$, and using (8),

$$\begin{aligned}
 \text{var}\left(\frac{B_{nt}}{n^H}\right) &= \frac{1}{n^{2H}} \sum_{m=1}^{[nt]} \sum_{s=1}^{[nt]} (E((C_1(V_m) - \bar{C}_1)(C_1(V_s) - \bar{C}_1))\rho_1^2(|s-m|)) \\
 &\leq \frac{k_3^*}{n^{2H}} \sum_{m=1}^{[nt]} \sum_{s=1}^{[nt]} \rho_2(|s-m|) \\
 &= \frac{k_3^*}{n^{2H}} \left([nt] + 2 \sum_{s=1}^{[nt]-1} ([nt] - s)\rho_2(s)\right)
 \end{aligned}$$

$$\begin{aligned}
 &< \frac{k_3^*[nt]}{n^{2H}} \left(1 + 2 \sum_{s=1}^{[nt]-1} \rho_2(s) \right) \\
 &\rightarrow 0 \quad \text{as } n \rightarrow \infty,
 \end{aligned}$$

where the last line follows since $\sum_{s=1}^\infty \rho_2(s) < \infty$ by assumption.

For $\rho_2(s) = \rho_1^2(s)$, and, therefore, satisfying Assumption 1 but *not* satisfying $\sum_{s=1}^\infty \rho_2(s) < \infty$,

$$\begin{aligned}
 \text{var} \left(\frac{B_{nt}}{n^H} \right) &= \frac{1}{n^{2H}} \sum_{m=1}^{[nt]} \sum_{s=1}^{[nt]} (\mathbb{E}((C_1(V_m) - \bar{C}_1)(C_1(V_s) - \bar{C}_1)) \rho_1^2(|s - m|)) \\
 &\leq \frac{k_3^*}{n^{2H}} \sum_{m=1}^{[nt]} \sum_{s=1}^{[nt]} \rho_1^4(|s - m|) \\
 &\rightarrow 0 \quad \text{from earlier.}
 \end{aligned}$$

Thus, in either case B_{nt}/n^H converges in mean square to 0.

Next consider $C_{nt} = \sum_{m=1}^{[nt]} (\bar{C}_1 e_1(\chi_m^2))$. Evaluating $e_1(x)$ from (3) as $\sqrt{2/[v]}([v]/2 - x)$, we have, from Taqqu (1975, Theorem 6.1 and Proposition 6.1), irrespective of $\rho_2(s)$,

$$\begin{aligned}
 \frac{1}{n^H} \sum_{m=1}^{[nt]} (\bar{C}_1 e_1(\chi_m^2)) &= \frac{\bar{C}_1}{n^H} \sum_{m=1}^{[nt]} \sqrt{\frac{2}{[v]}} \left(\frac{[v]}{2} - \chi_m^2 \right) \\
 &= \frac{-\bar{C}_1}{n^H \sqrt{2[v]}} \sum_{j=1}^{[v]} \sum_{m=1}^{[nt]} (\eta_j^2(m) - 1) \\
 &\Rightarrow \frac{-\bar{C}_1}{\sqrt{2[v]}} \sum_{j=1}^{[v]} R_j(t), \tag{15}
 \end{aligned}$$

where convergence is in the weak sense and each $R_j(t)$ is an independent copy of the self-similar Rosenblatt process. The Rosenblatt process $R(t)$ is defined in Taqqu (1975); it has strictly stationary increments and is H self-similar, $\mathbb{E}(R(t)) = 0$, and $\mathbb{E}(|R(t)|^\gamma) < \infty$ for $\gamma \leq 1/H$; it is separable and almost surely continuous, and the characteristic function of $R(1)$ admits the representation

$$\phi(u) = \exp \left\{ \frac{1}{2} \sum_{k=2}^\infty \frac{(2iu)^k}{(k+1)!} S_k \right\}$$

for

$$S_k = \int_0^1 \cdots \int_0^1 \frac{dx_1 \cdots dx_k}{(|x_1 - x_2||x_2 - x_3| \cdots |x_{k-1} - x_k||x_k - x_1|)^{1-H}},$$

which is valid for small values of $|u|$.

Finally, consider

$$D_{nt} = \sum_{m=1}^{[nt]} (C_0(V_m) - 1).$$

For D_{nt} , $C_0(y)$ can be evaluated to give

$$C_0(y) = \left(\frac{v}{2} - 1 \right) \int_1^\infty e^{y(1-t)} t^{-[v]/2} dt.$$

Direct calculation shows that $E(C_0(V_m)) = 1$, while $\text{var}(C_0(V_m)) < \infty$ since

$$\begin{aligned} E(C_0(V_m)^2) &= \left(\frac{[v]}{2} - 1\right)^2 \int_0^\infty p^t(y) \left(\int_1^\infty e^{y(1-t)} t^{-[v]/2} dt\right) \left(\int_1^\infty e^{y(1-s)} s^{-[v]/2} ds\right) dy \\ &= \left(\frac{[v]}{2} - 1\right)^2 \int_1^\infty \int_1^\infty (ts)^{-[v]/2} \left(\int_0^\infty p^t(y) e^{y(2-t-s)} dy\right) dt ds \\ &= \left(\frac{[v]}{2} - 1\right)^2 \int_1^\infty \int_1^\infty (ts)^{-[v]/2} (t+s-1)^{([v]-v)/2} dt ds \\ &\leq \left(\frac{[v]}{2} - 1\right)^2 \int_1^\infty \int_1^\infty (t+s-1)^{-v/2} dt ds \\ &= \frac{([v]/2 - 1)^2}{(v/2 - 1)(v/2 - 2)} \\ &< \infty, \end{aligned}$$

where $p^t(y) = y^{t-1}e^{-y}/\Gamma(t)$ is the density function of V_m and $t = (v - [v])/2$ from earlier. Hence, $C_0(y) \in \mathcal{L}_2((0, \infty), p^t(y) dy)$ and similar to before we can write

$$C_0(V_m) = \sum_{k=0}^\infty C_k^t e_k^t(V_m) \quad \text{for } C_k^t = \int_0^\infty C_0(y) p^t(y) e_k^t(y) dy,$$

where

$$e_k^t(y) = L_k^{t-1}(y) \left(\frac{k! \Gamma(t)}{\Gamma(t+k)}\right)^{1/2} \quad \text{and} \quad \sum_{k=0}^\infty (C_k^t)^2 = \int_0^\infty C_0^2(y) p^t(y) dy < \infty.$$

Since $C_0^t = e_0^t(y) = 1$,

$$\frac{1}{n^H} D_{nt} = \frac{1}{n^H} \sum_{m=1}^{[nt]} (C_0(V_m) - 1) = \frac{1}{n^H} \sum_{m=1}^{[nt]} \sum_{k=1}^\infty C_k^t e_k^t(V_m).$$

Now, for $W_i \sim \Gamma(a_i, 1)$, $i = 1, 2, 3$, and each W_i independent, we have (see Griffiths (1969))

$$E(e_k^t(W_1 + W_2) e_m^t(W_2 + W_3)) = \delta_k^m \left(\frac{\Gamma(a_1 + a_2) \Gamma(a_2 + a_3)}{\Gamma(a_1 + a_2 + k) \Gamma(a_2 + a_3 + k)}\right)^{1/2} \frac{\Gamma(a_2 + k)}{\Gamma(a_2)}.$$

From (1) however, $\{V_t, V_{t+s}\} \stackrel{D}{=} \{W_1 + W_2, W_2 + W_3\}$ for W_i as defined above with $a_1 = a_3 = t - \nu\rho_2(s)$ and $a_2 = \nu\rho_2(s)$, so

$$E(e_k^t(V_t) e_m^t(V_{t+s})) = \delta_k^m \rho_2(s) \left(\frac{1 + \nu\rho_2(s)}{1 + t}\right) \dots \left(\frac{k - 1 + \nu\rho_2(s)}{k - 1 + t}\right) \leq \delta_k^m \rho_2(s). \tag{16}$$

Hence, for any $\rho_2(s)$ satisfying Assumption 1 and additionally satisfying $\sum_{s=1}^\infty \rho_2(s) < \infty$, we have

$$\text{var}\left(\frac{D_{nt}}{n^H}\right) \leq \frac{1}{n^{2H}} \left(\sum_{k=1}^\infty (C_k^t)^2\right) \sum_{m=1}^{[nt]} \sum_{s=1}^{[nt]} \rho_2(|m - s|) \rightarrow 0 \quad \text{from earlier.}$$

For $\rho_2(s) = \rho_1^2(s)$, and, therefore, *not* satisfying $\sum_{s=1}^\infty \rho_2(s) < \infty$, we write D_{nt}/n^H as

$$\frac{1}{n^H} D_{nt} = \frac{1}{n^H} \sum_{m=1}^{[nt]} C_1^t e_1^t(V_m) + \frac{1}{n^H} \sum_{m=1}^{[nt]} \sum_{k=2}^\infty C_k^t e_k^t(V_m).$$

Then direct calculation yields $e_1^t(y) = \sqrt{1/t}(t - y)$, so

$$\frac{1}{n^H} \sum_{m=1}^{[nt]} C_1^t e_1^t(V_m) = \frac{-C_1^t}{2\sqrt{tn^H}} \sum_{m=1}^{[nt]} (2V_m - 2t),$$

which, from Finlay and Seneta (2007, Theorem 2), converges in probability to 0. Next, using (8), (11), and (16),

$$\begin{aligned} \text{var} \left(\frac{1}{n^H} \sum_{m=1}^{[nt]} \sum_{k=2}^\infty C_k^t e_k^t(V_m) \right) &\leq \frac{1}{n^{2H}} \left(\sum_{k=2}^\infty (C_k^t)^2 \right) \sum_{m=1}^{[nt]} \sum_{s=1}^{[nt]} \rho_1^2(|m - s|) \\ &< \frac{k_4^*[nt]}{n^{2H}} \left(1 + 2\omega^{2(H-1)/\alpha} \sum_{s=1}^{[nt]-1} s^{2(H-1)} \right) \\ &\sim \frac{2k_4^*nt\omega^{2(H-1)/\alpha}}{n^{2H}} \int_0^{nt} s^{2H-2} ds \\ &= \frac{2k_4^*t^{2H}\omega^{2(H-1)/\alpha}}{2H-1} \\ &< \infty \end{aligned}$$

for $k_4^* = \sum_{k=2}^\infty (C_k^t)^2 < \infty$. Similarly, using (16), we have

$$E(e_k^t(V_i)e_m^t(V_{i+s})) \geq \delta_k^m \rho_1^2(s) \left(\frac{1}{1+t} \right) \cdots \left(\frac{k-1}{k-1+t} \right),$$

so, for $(C_k^{t*})^2 = (C_k^t)^2(1/(1+t)) \cdots ((k-1)/(k-1+t))$ and $k_5^* = \sum_{k=2}^\infty (C_k^{t*})^2 > 0$, from (8),

$$\begin{aligned} \text{var} \left(\frac{1}{n^H} \sum_{m=1}^{[nt]} \sum_{k=2}^\infty C_k^t e_k^t(V_m) \right) &\geq \frac{1}{n^{2H}} \left(\sum_{k=2}^\infty (C_k^{t*})^2 \right) \sum_{m=1}^{[nt]} \sum_{s=1}^{[nt]} \rho_1^2(|m - s|) \\ &> \frac{k_5^*}{n^{2H}} \left([nt] + 2(1 + \omega)^{2(H-1)/\alpha} \sum_{s=1}^{[nt]-1} ([nt] - s)s^{2(H-1)} \right) \\ &\sim \frac{2k_5^*(1 + \omega)^{2(H-1)/\alpha}}{n^{2H}} \int_0^{nt} (nt - s)s^{2H-2} ds \\ &= \frac{2k_5^*t^{2H}(1 + \omega)^{2(H-1)/\alpha}}{2H(2H-1)} \\ &> 0, \end{aligned}$$

where we have used the fact that, for $s \geq 1, \alpha, \gamma, \omega > 0$ and $\frac{1}{2} < H < 1$,

$$(1 + \omega s^\alpha)^\gamma (H-1)/\alpha > (s^\alpha + \omega s^\alpha)^\gamma (H-1)/\alpha = (1 + \omega)^\gamma (H-1)/\alpha s^{\gamma(H-1)},$$

and Cauchy's integral test. Hence, in the case $\rho_2(s) = \rho_1^2(s)$, $0 < \liminf_{n \rightarrow \infty} \text{var}(D_{nt}/n^H) \leq \limsup_{n \rightarrow \infty} \text{var}(D_{nt}/n^H) < \infty$.

Hence, we have proved Theorem 1 and further proved that, when $\rho_2(s) = \rho_1^2(s)$, so that Assumption 1 holds but $\sum_{s=1}^{\infty} \rho_2(s) = \infty$, A_{nt}/n^H and B_{nt}/n^H converge in mean square to 0 as $n \rightarrow \infty$, C_{nt}/n^H converges weakly to the sum of $[v]$ independent copies of the Rosenblatt process, and D_{nt}/n^H has, for each fixed t , variance bounded away from 0 and ∞ .

4. Discussion

We have constructed a long-range dependence (LRD) and asymptotically self-similar activity time process $\{T_t\}$ which has inverse gamma distributed increments with noninteger distributional parameters. This process naturally extends that considered in Heyde and Leonenko (2005) to the context of a subordinator model for financial assets with marginal t -distribution, which was restricted to integer parameter values, and, thus, supports the use of the t -model for actual financial data.

We also note that there are other methods of constructing activity time processes with LRD and an asymptotically self-similar limit. For example, Leonenko *et al.* (2012) constructed gamma and inverse Gaussian processes, based on a superposition of Ornstein–Uhlenbeck-type processes, which converge to Brownian motion (in the case of a finite superposition) or fractional Brownian motion (in the case of an infinite superposition). Finlay and Seneta (2007) also reviewed a number of different approaches to constructing activity time processes.

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