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ON THE UNITARY EQUIVALENCE OF CERTAIN CLASSES OF NON-NORMAL OPERATORS. I

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1. Introduction. The following (so-called unitary equivalence) problem is of paramount importance in the theory of operators: given two (bounded linear) operators A_1, A_2 on a (complex) Hilbert space $\mathfrak{H}$, determine whether or not they are unitarily equivalent, i.e., whether or not there is a unitary operator U on $\mathfrak{H}$ such that $U^*A_1U = A_2$. For normal operators this question is completely answered by the classical multiplicity theory [**7**; **11**]. Many authors, in particular, Brown [**3**], Pearcy [**9**], Deckard [**5**], Radjavi [**10**], and Arveson [**1**; **2**], considered the problem for non-normal operators and have obtained various significant results. However, most of their results (cf. [**13**]) deal only with operators which are of type I in the following sense [**12**]: an operator, A, is of type I (respectively, II_1, II_∞, III) if the von Neumann algebra generated by A is of type I (respectively, II_1, II_∞, III). For non-normal operators of type I the problem is already known to be difficult, and the known results are far from exhaustive. In this paper we prove some interesting results for operators of more general type.

The problem of unitary equivalence is closely connected with the following problem of algebraic equivalence: given two operators A_1, A_2 on the Hilbert space $\mathfrak{H}$, and denoting by $\mathfrak{A}_1$, $\mathfrak{A}_2$, respectively, the von Neumann algebras generated by A_1, A_2, respectively, determine whether or not they are algebraically equivalent, i.e., whether or not there is an (algebraic $*$-) isomorphism ϕ of $\mathfrak{A}_1$ onto $\mathfrak{A}_2$ such that $\phi(A_1) = A_2$. In fact, if $\mathfrak{A}_1 = \mathfrak{A}_2$ is a factor, it is well-known that the two concepts coincide. So we concentrate on the algebraic equivalence.

Let us outline at this point a "sieving" programme for the above problem of algebraic equivalence, and put the present work into perspective. For simplicity, let us call two operators equivalent when they are algebraically equivalent in the above sense. Firstly, we examine whether $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are isomorphic. If they are not, then A_1 and A_2 are non-equivalent; if they are, we can assume that $\mathfrak{A}_1 = \mathfrak{A}_2$, and proceed to the second stage. Secondly, assuming that A_2 generates the same von Neumann algebra, $\mathfrak{A}$, as A_1, and denoting by $\mathfrak{M}_1$ (respectively, $\mathfrak{M}_2$) the von Neumann algebra generated by the real part, $\operatorname{Re} A_1$ (respectively, $\operatorname{Re} A_2$), of A_1 (respectively, A_2), we

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examine whether $[\mathfrak{A}, \mathfrak{M}_1]$ and $[\mathfrak{A}, \mathfrak{M}_2]$ are equivalent, i.e., whether there is an automorphism ϕ of $\mathfrak{A}$ such that $\phi(\mathfrak{M}_1) = \mathfrak{M}_2$. When $\mathfrak{A}$ is a type I factor, the classical multiplicity theory (see [**11**]) provides a solution to this question. The case where $\mathfrak{A}$ is a type II factor has been examined, and some results have been obtained by Bures [**4**]. If the answer to this question is negative, then A_1 and A_2 are non-equivalent; otherwise, we can assume that $\mathfrak{M}_1 = \mathfrak{M}_2$ and proceed to the third stage. Thirdly, assume that A_2 generates the same von Neumann algebra $\mathfrak{A}$ as A_1 and that $\operatorname{Re} A_1$ generates the same von Neumann algebra $\mathfrak{M}$ as $\operatorname{Re} A_2$. We examine whether $\operatorname{Im} A_2 = \alpha(\operatorname{Im} A_1)$ for some $\alpha \in A(\mathfrak{A}; \mathfrak{M})$, where $\operatorname{Im} A$ is the imaginary part of A and $A(\mathfrak{A}; \mathfrak{M})$ is the group of all automorphisms of $\mathfrak{A}$ which leave $\mathfrak{M}$ invariant. The computation of $A(\mathfrak{A}; \mathfrak{M})$ is in general very difficult and only fragmentary information is available (see § 3 below). If the answer to the above question is negative, then A_1 and A_2 are non-equivalent; otherwise, we can assume that $\operatorname{Im} A_1 = \operatorname{Im} A_2$ and proceed to the final stage. Finally, assuming the same conditions as in the third stage, and in addition that $\operatorname{Im} A_1 = \operatorname{Im} A_2 = T$, we examine whether there is an automorphism ϕ of $\mathfrak{A}$ such that $\phi(\operatorname{Re} A_1) = \operatorname{Re} A_2$ and $\phi(T) = T$. Obviously, A_1 and A_2 are equivalent if and only if the answer to the above question is affirmative. To settle this question one needs to compute the group $A(\mathfrak{M}, \mathfrak{A}; \mathfrak{N})$ of all automorphisms of $\mathfrak{M}$ which extend to automorphisms of $\mathfrak{A}$ keeping $\mathfrak{N}$ pointwise fixed, where $\mathfrak{N}$ denotes the von Neumann algebra generated by T. For a large class of $[\mathfrak{A}, \mathfrak{M}, \mathfrak{N}]$, this $A(\mathfrak{M}, \mathfrak{A}; \mathfrak{N})$ is determined in § 5 below.

We have not pretended that this programme is an esay one; indeed all questions listed above are very difficult. After all, we cannot and do not expect a simple solution to the general problem.

As an application of our results, we shall construct in a subsequent paper numerous examples of non-equivalent operators (of type II_1, II_∞ and III). In fact, for a large class of $[\mathfrak{C}, \mathfrak{A}]$, where $\mathfrak{C}$ is thick in $\mathfrak{A}$ ($\mathfrak{A}$ can be a factor of type II_1, II_∞, III, etc.) (see [**4**]), we construct a family (A_i) of pair-wise inequivalent operators such that $\mathfrak{R}(A_i) = \mathfrak{A}$, $\mathfrak{R}(\operatorname{Re} A_i) = \mathfrak{C}$, the $\operatorname{Im} A_i$'s are identical, and the $\operatorname{Re} A_i$'s are pair-wise unitarily equivalent (where $\mathfrak{R}(A)$ denotes the von Neumann algebra generated by A).

We now give a summary of the contents. In § 2, we introduce the basic set up: $\mathfrak{A}$ is the von Neumann algebra constructed from a free and ergodic C-system $[\bar{\mathfrak{M}}, \mathfrak{K}, \mathfrak{G}, g \mapsto U_g]$ according to von Neumann and Dixmier [**13, 8**], $\mathfrak{M} = \bar{\mathfrak{M}} \otimes I$, and $\mathfrak{N} = \mathfrak{R}(U_g \otimes V_g : g \in \mathfrak{G})$ (for details of notations, cf. § 2 below). In § 3 we compute the group $A(\mathfrak{A}; \mathfrak{M}, \mathfrak{N})$ of all automorphsims of $\mathfrak{A}$ which keep $\mathfrak{M}$ pointwise fixed and keep $\mathfrak{N}$ invariant. This result indicates that $A(\mathfrak{A}; \mathfrak{M})$ can be rather complicated. (Note that $A(\mathfrak{A}; \mathfrak{M}, \mathfrak{N}) \subset A(\mathfrak{A}; \mathfrak{M})$.) In § 4, we present simple examples of operators which are distinguishable (up to unitary equivalence) by means of the calculation of $A(\mathfrak{A}; \mathfrak{M}, \mathfrak{N})$ in § 3. In § 5, we compute $A(\mathfrak{M}, \mathfrak{A}; \mathfrak{N})$, the importance of which was indicated in the preceding paragraphs. It turns out that $A(\mathfrak{M}, \mathfrak{A}; \mathfrak{N})$ is essentially the commutant $\mathfrak{G}'$ of $\mathfrak{G}$, which is studied in a separate paper [**14**]. Then in § 6 we apply the results of [**14**] to operators.

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2. The basic set up. We start with a free and ergodic C-system $[\bar{\mathfrak{M}}, \mathfrak{K}, \mathfrak{G}, g \mapsto U_g]$; more precisely, we have $\bar{\mathfrak{M}}$ a maximal abelian von Neumann algebra on a Hilbert space $\mathfrak{K}$, $\mathfrak{G}$ a free and ergodic group of automorphisms of $\bar{\mathfrak{M}}$, and $g \mapsto U_g$ is a unitary representation of $\mathfrak{G}$ on K such that $U_g(M)U_g^* = g(M)$ for all $M \in \bar{\mathfrak{M}}$. Let $\mathfrak{K}_{\mathfrak{G}}$ denote the Hilbert space with an orthonormal basis $(\phi_g)_{g\in\mathfrak{G}}$ indexed on $\mathfrak{G}$, let V_g be the unitary operator on $\mathfrak{K}_{\mathfrak{G}}$ which maps ϕ_h to ϕ_{gh}, and let $\mathfrak{H} = \mathfrak{K} \otimes \mathfrak{K}_{\mathfrak{G}}$. Let $\mathfrak{M} = \bar{\mathfrak{M}} \otimes I$, I the identity on $\mathfrak{K}_{\mathfrak{G}}$, let $\mathfrak{N}$ be the von Neumann algebra $\mathfrak{N}(\{U_g \otimes V_g : g \in G\})$ generated by $\{U_g \otimes V_g : g \in \mathfrak{G}\}$ in $\mathscr{L}(\mathfrak{H})$, and let $\mathfrak{A}$ be the von Neumann algebra $\mathfrak{A}[\bar{\mathfrak{M}}, \mathfrak{K}, \mathfrak{G}, g \mapsto U_g] = \mathfrak{R}(\mathfrak{M}, \mathfrak{N})$ generated by $\mathfrak{M} \cup \mathfrak{N}$. (Thus, $\mathfrak{A}$ is the algebra constructed from $[\bar{\mathfrak{M}}, \mathfrak{K}, \mathfrak{G}, g \mapsto U_g]$ according to von Neumann and Dixmier.) Note that we always have $(U_g \otimes V_g)\mathfrak{M}(U_g \otimes V_g)^* = \mathfrak{M}$, and that $\mathfrak{M} \cap \mathfrak{N} = \mathbf{C}$. As the C-system is free and ergodic, $\mathfrak{M}$ is maximal abelian in $\mathfrak{A}$ and $\mathfrak{A}$ is a factor. In this paper, we shall consider operators A_1, A_2 on $\mathfrak{H}$ with $\mathfrak{R}(A_1) = \mathfrak{R}(A_2) = \mathfrak{A}$, and such that $(\mathfrak{R}(\operatorname{Re} A_1), \mathfrak{R}(\operatorname{Re} A_2))$ fits in $[\mathfrak{A}, \mathfrak{M}]$, or that $(\mathfrak{R}(\operatorname{Im} A_1), \mathfrak{R}(\operatorname{Im} A_2))$ fits in $[\mathfrak{A}, \mathfrak{N}]$ in the following sense.

Definition 2.1. Let $\mathfrak{A} \supset \mathfrak{M}$ be von Neumann algebras. Then $(\mathfrak{E}, \mathfrak{F})$ fits in $[\mathfrak{A}, \mathfrak{M}]$ if $\mathfrak{E}$, $\mathfrak{F}$ are von Neumann subalgebras of $\mathfrak{M}$ such that for any automorphism ϕ of $\mathfrak{A}$ with $\phi(\mathfrak{E}) = \mathfrak{F}$, $\phi(\mathfrak{M}) = \mathfrak{M}$. (Obviously, $(\mathfrak{E}, \mathfrak{F})$ fits in $[\mathfrak{A}, \mathfrak{M}]$ when $\mathfrak{E}' \cap \mathfrak{A} = \mathfrak{F}' \cap \mathfrak{A} = \mathfrak{M}$; i.e., when $\mathfrak{E}$, $\mathfrak{F}$ are thick [**4**].)

3. The calculation of $A(\mathfrak{A}; \mathfrak{M}, \mathfrak{N})$**.** We begin with the definition of $A(\mathfrak{A}; \mathfrak{M}, \mathfrak{N})$.

Definition 3.1. We denote by $A(\mathfrak{A}; \mathfrak{M}, \mathfrak{N})$ the group of automorphisms of $\mathfrak{A}$ which keep $\mathfrak{M}$ pointwise fixed and leave $\mathfrak{N}$ invariant. Clearly, $A(\mathfrak{A}; \mathfrak{M}, \mathfrak{N})$ is a subgroup of the group $A(\mathfrak{A}; \mathfrak{M})$ of all automorphisms of $\mathfrak{A}$ which leave $\mathfrak{M}$ invariant, mentioned in § 1.

LEMMA 3.1. *For each $\sigma \in A(\mathfrak{A}; \mathfrak{M}, \mathfrak{N})$ there is a character $c : \mathfrak{G} \to \mathbf{C}$ such that for all $g \in \mathfrak{G}$,*

$$\sigma(U_g \otimes V_g) = c(g)(U_g \otimes V_g).$$

Proof. As $\sigma \in A(\mathfrak{A}; \mathfrak{M}, \mathfrak{N})$, it keeps $\mathfrak{M}$ pointwise fixed and leaves $\mathfrak{N}$ invariant. Now for every $M \in \mathfrak{M}$,

$$\begin{aligned}\sigma(U_g \otimes V_g)M\sigma(U_g \otimes V_g)^* &= \sigma((U_g \otimes V_g)M(U_g \otimes V_g)^*)\\ &= (U_g \otimes V_g)M(U_g \otimes V_g)^*.\end{aligned}$$

Hence,

$$(U_g \otimes V_g)^*\sigma(U_g \otimes V_g) \in \mathfrak{M}' \cap \mathfrak{N} = \mathfrak{M} \cap \mathfrak{N} = \mathbf{C}.$$

Therefore, there is a complex number $c(g)$ such that

$$\sigma(U_g \otimes V_g) = c(g)(U_g \otimes V_g).$$

It follows readily that the mapping $g \mapsto c(g)$ is a character on $\mathfrak{G}$. This completes the proof.

Definition 3.2. Suppose that $\mathfrak{A}$ is a von Neumann algebra on the Hilbert space $\mathfrak{H}$, and that U is a unitary operator on $\mathfrak{H}$. Suppose that $U\mathfrak{A}U^* = \mathfrak{A}$. Then U is said to induce the automorphism of $\mathfrak{A}$ given by

$$A \in \mathfrak{R} \mapsto UAU^* \in \mathfrak{A}.$$

LEMMA 3.2. *For each character $c : \mathfrak{G} \to \mathbf{C}$, there is a unitary operator W_c on $\mathfrak{K}_{\mathfrak{G}}$ such that the unitary $I \otimes W_c$ induces an automorphism $\sigma_c \in A(\mathfrak{A}; \mathfrak{M}, \mathfrak{N})$ with*

$$\sigma_c(U_g \otimes V_g) = c(g)(U_g \otimes V_g), g \in \mathfrak{G}.$$

Proof. Let c be a character on $\mathfrak{G}$. Define a unitary operator W_c on $\mathfrak{K}_{\mathfrak{G}}$ by: $W_c(\phi_g) = c(g)\phi_g$, $g \in \mathfrak{G}$. Then by a direct and simple calculation, one shows that $I \otimes W_c$ commutes with $\mathfrak{M}$, and that the unitary $I \otimes W_c$ induces an automorphism $\sigma \in A(\mathfrak{A}; \mathfrak{M}, \mathfrak{N})$ with $\sigma_c(U_g \otimes V_g) = c(g)(U_g \otimes V_g)$, $g \in \mathfrak{G}$. This completes the proof.

Definition 3.3. For a character c on $\mathfrak{G}$, let σ_c denote the automorphism of $\mathfrak{A}$ induced by the unitary $I \otimes W_c$ in Lemma 3.2.

THEOREM 3.3. $A(\mathfrak{A}; \mathfrak{M}, \mathfrak{N}) = \{\sigma_c : c \text{ a character on } G\}$.

Proof. By Lemma 3.2, each $\sigma_c \in A(\mathfrak{A}; \mathfrak{M}, \mathfrak{N})$. By Lemma 3.1, for each $\sigma \in A(\mathfrak{A}; \mathfrak{M}, \mathfrak{N})$, there is a character c on $\mathfrak{G}$ such that $\sigma|_{\mathfrak{N}} = \sigma_c|_{\mathfrak{N}}$. But $\sigma|_{\mathfrak{M}} = \sigma_c|_{\mathfrak{M}}$, both being the identity map on $\mathfrak{M}$. Since $\mathfrak{A} = \mathfrak{R}(\mathfrak{M}, \mathfrak{N})$, $\sigma = \sigma_c$. This completes the proof.

4. Operators distinguishable by means of $A(\mathfrak{A}; \mathfrak{M}, \mathfrak{N})$. The following direct consequence of Theorem 3.3 is useful in operator theory.

THEOREM 4. *Let A_1 and A_2 be two operators on $\mathfrak{H}$ such that*

$$\mathfrak{R}(A_1) = \mathfrak{R}(A_2) = \mathfrak{A}, \quad \operatorname{Re} A_1 = \operatorname{Re} A_2, \quad \mathfrak{R}(\operatorname{Re} A_1) = \mathfrak{M},$$

and $(\mathfrak{R}(\operatorname{Im} A_1), \mathfrak{R}(\operatorname{Im} A_2))$ fits in $[\mathfrak{A}, \mathfrak{N}]$ (cf. Definitions 2.1, 2.2). Then the following statements are equivalent:

(i) *A_1 and A_2 are unitarily equivalent;*

(ii) *A_1 and A_2 are (algebraically) equivalent;*

(iii) $\operatorname{Im} A_2 = \sigma_c(\operatorname{Im} A_1)$ *for some character c on G.*

(*For the definition of σ_c, cf. definition* 3.3.)

Proof. The theorem follows directly from Theorem 3.3 and the observations that A_1 and A_2 are equivalent if and only if there is a $\sigma \in A(\mathfrak{A}; \mathfrak{M}, \mathfrak{N})$ such that $\operatorname{Im} A_2 = \sigma(\operatorname{Im} A_1)$, and that each σ_c is implemented by a unitary operator on $\mathfrak{H}$.

We now illustrate Theorem 4.1 by the following simple example. Let $\bar{\mathfrak{M}}$ be $L_\infty[0, 1]$ acting by multiplication on $\mathfrak{K} = L_2[0, 1]$ (cf. [**14**]). Let D be the group of dyadic rationals in $[0, 1]$ under addition mod 1. For each $d \in D$, let τ_d be the automorphism of $\bar{\mathfrak{M}}$ given by: $\tau_d M_f = M_{d[f]}$, where $M_f \in \bar{\mathfrak{M}}$ is the multiplication by $f \in L_\infty[0, 1]$, and $d[f]$ is the function in $L_\infty[0, 1]$ given by: $d[f](y) = f(y - d)$, $y \in [0, 1]$. Let $\mathfrak{G} = \{\tau_d : d \in D\}$. Let U_d be the unitary operator on K given by:

$$(U_d h)(x) = h(x - d), h \in L_2[0, 1], x \in [0, 1].$$

Then [**14**] the system $(\bar{\mathfrak{M}}, \mathfrak{K}, \mathfrak{G}, \tau_d \mapsto U_d)$ is an abelian, free and ergodic C-sustem. Let g be a strictly monotone, continuous and real-valued function defined on $[0, 1]$. Then $M_g \otimes I$ is self-adjoint, and it generates $\mathfrak{M}$ (i.e., $\mathfrak{M} = \mathfrak{R}(M_g \otimes 1)$. Suppose that $\sum_{d \in D} \alpha_d (U_d \otimes V_d)$ and $\sum_{d \in D} \beta_d (U_d \otimes V_d)$ are self-adjoint, and suppose that each of them generates $\mathfrak{N}$. Then by Theorem 4.1, we have that the operators

$$M_g \otimes I + i \textstyle\sum_{d \in D} \alpha_d (U_d \otimes V_d)$$

and

$$M_g \otimes I + i \textstyle\sum_{d \in D} \beta_d (U_d \otimes V_d)$$

are unitarily equivalent if and only if for some character χ on D,

$$\textstyle\sum_{d \in D} \alpha_d (U_d \otimes V_d) = \sigma_\chi (\sum_{d \in D} \beta_d (U_d \otimes V_d)),$$

i.e., if and only if $\alpha_d = \chi(d) \beta_d$, for all $d \in D$.

5. The calculation of $A(\mathfrak{M}, \mathfrak{A}; \mathfrak{N})$**.** By definition, $A(\mathfrak{M}, \mathfrak{A}; \mathfrak{N})$ is the group of all automorphisms of $\mathfrak{M}$ which extend to automorphisms of $\mathfrak{A}$ keeping $\mathfrak{N}$ pointwise fixed. For a subset S of automorphisms of $\bar{\mathfrak{M}}$, let $\bar{S}$ be the set of all automorphisms α of $\mathfrak{M}$ such that for some $s \in S$, $\alpha(M \otimes I) = s(M) \otimes I$, for all $M \in \bar{\mathfrak{M}}$. In the proof of $A(\mathfrak{M}, \mathfrak{A}; \mathfrak{N}) = (\mathfrak{G}')^-$ in Theorem 5.2 below, we need the following lemma.

LEMMA 5.1. *For each $\alpha \in \mathfrak{G}'$, there is a unitary operator W_α on $\mathfrak{H}$ such that $W_\alpha(M \otimes I) W_\alpha^* = \alpha(M) \otimes I$, for all $M \in \bar{\mathfrak{M}}$, and $W_\alpha(U_g \otimes V_g) W_\alpha^* = U_g \otimes V_g$, for all $g \in \mathfrak{G}$.*

Proof. Suppose that $\alpha \in \mathfrak{G}'$. Since $\bar{\mathfrak{M}}$ is maximal abelian in $\mathscr{L}(\mathfrak{K})$, by [**9**, p. 241] there is a unitary operator Y on $\mathfrak{K}$ such that $YMY^* = \alpha(M)$, for all $M \in \bar{\mathfrak{M}}$. Define a unitary operator W_α on H by:

$$W_\alpha(x \otimes \phi_g) = (U_g Y U_g^* x) \otimes \phi_g, x \in \mathfrak{K}, g \in \mathfrak{G}.$$

Then

$$W_\alpha^*(x \otimes \phi_g) = (U_g Y^* U_g^* x) \otimes \phi_g.$$

The proof is completed by a direct and simple calculation.

THEOREM 5.2. $A(\mathfrak{M}, \mathfrak{A}; \mathfrak{N}) = (\mathfrak{G}')^-$.

Proof. Suppose that $\alpha \in A(\mathfrak{M}, \mathfrak{A}; \mathfrak{N})$. Then α extends to an automorphism $\tilde{\alpha}$ of $\mathfrak{A}$ with $\tilde{\alpha}|\mathfrak{M} = \alpha$ and $\tilde{\alpha}(U_g \otimes V_g) = U_g \otimes V_g$, for all $g \in \mathfrak{G}$. Let $\bar{\alpha}$ be the automorphism of $\bar{\mathfrak{M}}$ such that for all $M \in \bar{\mathfrak{M}}$, $\bar{\alpha}(\mathfrak{M}) \otimes I = \alpha(M \otimes I)$. Then it follows from a direct calculation that $\bar{\alpha} \in \mathfrak{G}'$ and so $\alpha \in (\mathfrak{G}')^-$.

Suppose, on the other hand, that $\alpha \in (\mathfrak{G}')^-$. Let $\bar{\alpha}$ be as above. Then $\bar{\alpha} \in \mathfrak{G}'$. So, by Lemma 5.1, there is a unitary operator W_α on $\mathfrak{H}$ such that

$$W_\alpha(M \otimes I)W_\alpha^* = \bar{\alpha}(M) \otimes I = \alpha(M \otimes I),$$

and

$$W_\alpha(U_g \otimes V_g)W_\alpha^* = U_g \otimes V_g.$$

Thus, we see that α extends to an automorphism of $\mathfrak{A}$ keeping $\mathfrak{N}$ pointwise fixed.

Remark. It is clear that the results of this section are true for any C-system (i.e., the C-system need not be ergodic or free).

6. Operators distinguishable by $\mathfrak{G}'$. We now state the results of § 5 in the form most suitable for applications in operatory theory.

THEOREM 6. *Suppose that A_1, A_2 are two operators on a Hilbert space $\mathfrak{H}$ such that $\mathfrak{R}(A_1) = \mathfrak{R}(A_2) = \mathfrak{A}[\bar{\mathfrak{M}}, \mathfrak{K}, \mathfrak{G}, g \mapsto U_g]$, for some ergodic and abelian C-system $[\bar{\mathfrak{M}}, \mathfrak{K}, \mathfrak{G}, g \mapsto U_g]$, that $(\mathfrak{R}(\mathrm{Re}\, A_1), \mathfrak{R}(\mathrm{Re}\, A_2))$ fits in $[\mathfrak{A}, \bar{\mathfrak{M}} \otimes I]$, that $\mathrm{Im}\, A_1 = \mathrm{Im}\, A_2 = T$, and that $\mathfrak{R}(T) = \mathfrak{R}(U_g \otimes V_g : g \in \mathfrak{G})$ (cf. Definitions 2.1, 2.2 for notations). Then A_1 and A_2 are unitarily equivalent if and only if they are (algebraically) equivalent, and that is the case if and only if there is an $\alpha \in (\mathfrak{G}')^-$ such that $\alpha(\mathrm{Re}\, A_1) = \mathrm{Re}\, A_2$, where $(\mathfrak{G}')^- = \{\alpha \in A(\mathfrak{M} \otimes I)$: for some $s \in \mathfrak{G}'$, $\alpha(M \otimes I) = s(M) \otimes I$, for all $M \in \bar{\mathfrak{M}}\}$.*

Proof. The present theorem follows directly from Lemma 5.1, Theorem 5.2, and Definition 2.1.

Remark. In a separate paper we have determined $\mathfrak{G}'$ for a large class of $\mathfrak{G}$. Combining the results of that paper and Theorem 6, we have the following very interesting theorem.

THEOREM 6.2. *Suppose that*

(i) *$\bar{\mathfrak{M}}$ is $L_\infty[0, 1]$ acting on $\mathfrak{K} = L_2[0, 1]$, D is a dense subgroup of* $[0, 1]$ *under the addition* mod 1, *$\mathfrak{G}$ is the group of all automorphisms on $\bar{\mathfrak{M}}$ induced by the translation in* $[0, 1]$ *by $d \in D$, and $g \in \mathfrak{G} \mapsto U_g$ is the usual* (*cf.* **[8]**) *unitary representation of $\mathfrak{G}$ on $\mathfrak{K}$; or*

(ii) *$\bar{\mathfrak{M}}$ is $L_\infty(\mathbf{R})$ acting on $\mathfrak{K} = L_2(\mathbf{R})$, D is a dense subgroup of $\mathbf{R}$, $\mathfrak{G}$ is the group of all automorphisms on $\bar{\mathfrak{M}}$ induced by the translation in $\mathbf{R}$ by $d \in D$, and $g \in \mathfrak{G} \mapsto U_g$ is the usual unitary representation of G on $\mathfrak{K}$; or*

(iii) *$\bar{\mathfrak{M}}$ is $L_\infty(\mathbf{R})$ acting on $\mathfrak{K} = L_2(\mathbf{R})$, $\mathfrak{G}$ is the group of all automorphisms s_r on $\bar{\mathfrak{M}}$ given by:*

$$(s_r f)(x) = f(r^{-1}x), f \in L_\infty(\mathbf{R}), x \in \mathbf{R},$$

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for $r \in \mathbf{R}$ *with* $r \neq 0$, *and* $g \in \mathfrak{G} \mapsto U_g$ *is the usual unitary representation of* $\mathfrak{G}$ *on* $\mathfrak{K}$; *or*

(iv) $\mathfrak{M}$ *is* $L_\infty(X_1, \times X_2, S_1 \times S_2, \mu_1 \times \mu_2)$ *acting on*

$$\mathfrak{K} = L_2(X_1 \times X_2, S_1 \times S_2, \mu_1 \times \mu_2),$$

$\mathfrak{G}$ *is the group of all automorphisms* $\alpha_{(g_1,g_2)}$, $(g_1, g_2) \in \mathfrak{G}_1 \times \mathfrak{G}_2$, *of* $\mathfrak{M}$ *given by:*

$$(\alpha_{(g_1,g_2)}f)(x, y) = f(g_1(x), g_2(y)),\ (x, y) \in X_1 \times X_2, f \in L_\infty(X_1 \times X_2),$$

where each $G_i (i = 1, 2)$ *is a countable, abelian and ergodic group of automorphisms on* $\mathfrak{M}_i$ $(= L_\infty(X_i, S_i, \mu_i)$ *acting on* $L(X_i, S_i, \mu_i))$ *such that each element* $g_i \in \mathfrak{G}_i$ *and each* $g_i' \in \mathfrak{G}_i'$ *are induced by point transformations, denoted again by* g_i, g_i', *respectively, of* X_i. *Suppose that each* $\mathfrak{G}_i$ *has a unitary representation* $g_i \mapsto U_{g_i}$ *on* $L_2(X_i, S_i, \mu_i)$. *Let* $U_{(g_1,g_2)}$, $(g_1, g_2) \in \mathfrak{G}_1 \times \mathfrak{G}_2$, *be the unitary operator on* $L_2(X_1 \times X_2, S_1 \times S_2, \mu_1 \times \mu_2)$ *given by:*

$$U_{(g_1,g_2)}f_{1,2} = (U_{g_1}f_1)(U_{g_2}f_2),$$

where $f_1 \in L_2(X_1, S_1, \mu_1)$, $f_2 \in L_2(X_2, S_2, \mu_2)$ *and* $f_{1,2}(x, y) = f_1(x)f_2(y)$, *for all* $(x, y) \in X_1 \times X_2$. *Let* $\mathfrak{G}$ *have the unitary representation* $\alpha_{(g_1,g_2)} \mapsto U_{(g_1,g_2)}$.

Let the Hilbert space $\mathfrak{H} = \mathfrak{K} \otimes \mathfrak{K}_\mathfrak{G}$ *and the unitaries* V_g $(g \in \mathfrak{G})$ *be as defined in* § 2. *Suppose that* A_1, A_2 *are operators on* $\mathfrak{H}$ *such that* $\mathfrak{R}(A_1) = \mathfrak{R}(A_2) = \mathfrak{A}[\mathfrak{M}, \mathfrak{K}, \mathfrak{G}, g \mapsto U_g]$, $\operatorname{Im} A_1 = \operatorname{Im} A_2$, $\mathfrak{R}(\operatorname{Im} A_1) = \mathfrak{R}(\{U_g \otimes V_g : g \in G\})$, *and that* $(\mathfrak{R}(\operatorname{Re} A_1), \mathfrak{R}(\operatorname{Re} A_2))$ *fits in* $[\mathfrak{A}, \mathfrak{M} \otimes I]$. *Denote* $\operatorname{Re} A_i$ *by* $M_{f_i} \otimes I$, M_{f_i} *being the multiplication by the essentially bounded measurable function* f_i. *Then* A_1 *and* A_2 *are unitarily equivalent if and only if they are (algebraically) equivalent, and they are unitarily equivalent if and only if one of the following conditions holds:*

(i) $f_2 = \tau_r(f_1)$ *for some* $r \in [0, 1]$, *where* τ_r *is the translation* mod 1 *by* r;

(ii) $f_2 = \tau_r(f_1)$ *for some* $r \in \mathbf{R}$, *where* τ_r *is the translation by* r;

(iii) $f_2 = s_r(f_1)$ *for some non-zero real number* r, *where* s_r *is defined by:*

$$(s_r f)(x) = f(r^{-1}x),\ x \in \mathbf{R};$$

(iv) $f_2 = \alpha_{(g_1'g_2')}(f_1)$ *for some* $(g', g') \in G_1' \times G_2'$, *where*

$$(\alpha_{(g_1'g_2')}f)(x, y) = f(g_1'(x), g_2'(y)).$$

Proof. The present theorem follows readily from Theorem 8.1, and [**14**, Theorems 1, 2, 3].

We now illustrate Theorem 6.2 by the following simple example. Let $\mathfrak{M}$ be $L_\infty(\mathbf{R})$ acting by multiplication on $L_2(\mathbf{R})$, D the dyadic rationals in $\mathbf{R}$, and $\mathfrak{G}$ the group of all automorphisms of $\mathfrak{M}$ induced by translations (in $\mathbf{R}$) by $d \in D$. Suppose that $\sum_{d \in D} \alpha_d (U_d \otimes V_d)$, $\alpha_d \in \mathbf{C}$, is self-adjoint, and it generates $\mathfrak{R}(\{U_d \otimes V_d : d \in D\})$. Let f_1, f_2 be two strictly monotone, continuous, bounded and real-valued functions defined on $\mathbf{R}$. Then by Theorem 8.2 we have:

$$M_{f_1} \otimes I + i \textstyle\sum_{d \in D} \alpha_d (U_d \otimes V_d) \quad \text{and} \quad M_{f_2} \otimes I + i \sum_{d \in D} \alpha_d (U_d \otimes V_d)$$

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are unitarily equivalent if and only if for some $r \in \mathbf{R}$:

$$f_2(x) = f_1(x - r) \; a.e., x \in \mathbf{R}.$$

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