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Nest Representations of TAF Algebras

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Abstract. A nest representation of a strongly maximal TAF algebra A with diagonal D is a representation π for which Lat $\pi(A)$ is totally ordered. We prove that ker π is a meet irreducible ideal if the spectrum of A is totally ordered or if (after an appropriate similarity) the von Neumann algebra $\pi(D)''$ contains an atom.

1 Introduction

Irreducible *-representations and their kernels, the primitive ideals, play a fundamental role in the theory of C*-algebras. For example, the structure of the lattice of all ideals in a C*-algebra is determined by the space of primitive ideals with the hull-kernel topology (there is a bijection between the lattice of ideals and the lattice of closed subsets of the primitive ideal space) and every ideal is the intersection of those primitive ideals which contain it.

Recent work by Michael Lamoureux [6], [7], [8] has shown that a similar situation prevails in a number of non-self-adjoint operator algebra settings. One motivation for investigating non-self-adjoint algebras arises from dynamical systems. While C*-crossed products constructed from dynamical systems are very useful in the study of dynamical systems, some essential information may be lost. (There are different dynamical systems which give rise to isomorphic C*-crossed products.) In the case of free, discrete systems, the remedy is to look instead at the semi-crossed product (a non-self-adjoint algebra) associated with the system (see [1], [11], [12]). In this context there is a bijection between the isomorphism classes of (free, discrete) dynamical systems and their associated semi-crossed products. This correspondance is established via an analysis of the space of ideals of the operator algebra.

In attempting to extend this program to other dynamical systems, Lamoureux drew attention to nest representations and their kernels (which he called *n-primitive ideals*). A *nest representation* π of an operator algebra A is simply a continuous algebra homomorphism of A into some $\mathcal{B}(\mathcal{H})$ with the property that the lattice of projections invariant under $\pi(A)$ is totally ordered. Nest representations do not arise naturally as a concept in the C*-algebra context. For one thing, most representations arising in C*-algebra theory are *-representations. A projection is invariant under a *-representation if, and only if, it is reducing; consequently, for *-representations, the family of nest representations reduces to the family of irreducible representations. Even if one looks at representations which are not *-representations, the n-primitive ideals in a C*-algebra are just the primitive ideals.

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(Nest representations are necessarily topologically cyclic—this is valid for nest representations of any Banach algebra with a bounded approximate identity [6]—and Haagerup [4] has shown that a cyclic representation of a C*-algebra is similar to a *-representation.)

In the non-self-adjoint context, the story is quite different. Non-self-adjoint algebras may lack primitive ideals, but n-primitive ideals generally abound. Lamoureux has shown that, at least in the presence of certain hypotheses, n-primitive ideals in semi-crossed products provide information about arc spaces in dynamical systems, so that the semi-crossed product essentially determines the dynamical system. He has also shown that, in a variety of contexts, the n-primitive ideals carry a topology and that the lattice of closed sets in this topology is isomorphic to the lattice of ideals in the original algebra. Also, every ideal is equal to the intersection of the n-primitive ideals which contain it. Thus, for general operator algebras, n-primitive ideals are often a suitable replacement for the primitive ideals of C*-algebra theory.

The topology which Lamoureux puts on the n-primitive ideals is, of course, the hull-kernel topology. In order to show that the hull-kernel operation yields a topology, Lamoureux uses a technical property for ideals which is related to meet irreducibility and which implies meet irreducibility. (An ideal $\mathcal{I}$ is *meet irreducible* if, for any ideals $\mathcal{J}$ and $\mathcal{K}$, $\mathcal{I} = \mathcal{J} \cap \mathcal{K} \implies \mathcal{I} = \mathcal{J}$ or $\mathcal{I} = \mathcal{K}$.) Meet irreducibility arises in [7] in connection with semi-crossed products and dynamical systems.

The following four results from [8] indicate that meet irreducibility will be closely related to n-primitivity in diverse operator algebra contexts.

(1) Let $\mathcal{I}$ be a closed, two-sided ideal in a separable C*-algebra. Then $\mathcal{I}$ is n-primitive $\iff$ $\mathcal{I}$ is primitive $\iff$ $\mathcal{I}$ is prime $\iff$ $\mathcal{I}$ is meet irreducible. (Some of this has, of course, been known for a long while.)

(2) Let $\mathcal{I}$ be a closed, two-sided ideal in $\mathrm{Alg}(\mathcal{N}) \cap \mathcal{K}$, where $\mathcal{N}$ is a nest of closed subspaces in some Hilbert space $\mathcal{H}$, $\mathrm{Alg}(\mathcal{N})$ is the associated nest algebra consisting of all operators which leave invariant each subspace in $\mathcal{N}$, and $\mathcal{K}$ is the algebra of all compact operators acting on $\mathcal{H}$. Then, $\mathcal{I}$ is n-primitive $\iff$ $\mathcal{I}$ is meet irreducible $\iff$ $\mathcal{I}$ is the kernel of the compression map of $\mathrm{Alg}(\mathcal{N}) \cap \mathcal{K}$ to some interval from $\mathcal{N}$.

(3) Let $\mathcal{I}$ be a closed ideal in the disk algebra, $A(\mathbb{D})$, the algebra of continuous functions on the unit disk of $\mathbb{C}$ which are analytic in the interior. Then, $\mathcal{I}$ is n-primitive $\iff$ $\mathcal{I}$ is meet irreducible $\iff$ $\mathcal{I}$ is either primary or zero. (In this context, it is not true that every ideal is the intersection of meet irreducible ideals.)

(4) Let $A = T_{n_1} \oplus \cdots \oplus T_{n_k}$ be a direct sum of upper triangular n_j by n_j matrix algebras and let $\mathcal{I}$ be a two-sided ideal in A. Then, $\mathcal{I}$ is n-primitive $\iff$ $\mathcal{I}$ is meet irreducible.

Meet irreducible ideals in the context of triangular AF algebras (TAF algebras) were investigated in [3]. In particular, it was proven that every meet irreducible ideal in a strongly maximal TAF algebra is n-primitive [3, Theorem 2.4]. Once again, the lattice of all ideals is isomorphic to the lattice of closed sets in the space of meet irreducible ideals with the hull-kernel topology and every ideal is an intersection of meet irreducible ideals. The converse, "every n-primitive ideal is meet irreducible," was left open, however; it is the purpose of this note to investigate this converse in the context of strongly maximal TAF algebras. In Section 2 we describe a broad class of algebras (those characterized by "totally ordered spectrum") for which the converse always holds. In Section 3 we show that the converse is valid

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for any nest representation which is similar to a representation π which is a *-representation on the diagonal D of the algebra and for which the von Neumann algebra generated by $\pi(D)$ contains an atom. These two sections may be read independently of each other.

1.1 Notation

We now establish notation and terminology. Let B be an AF C*-algebra and let D be a canonical masa in B. This implies that there is a sequence of finite dimensional C*-algebras B_i and embeddings $\phi_i \colon B_i \to B_{i+1}$ such that $B = \varinjlim(B_i, \phi_i)$ and $D = \varinjlim(D_i, \phi_i)$, where each $D_i = D \cap B_i$ is a masa in B_i and each ϕ_i maps the normalizer of D_i into the normalizer of D_{i+1}. In particular, the D-normalizing partial isometries in B have a linear span which is dense in B (w *normalizes* D if $wDw^* \subseteq D$ and $w^*Dw \subseteq D$; the set of D-normalizing partial isometries in B is denoted by $N_D(B)$). A TAF *algebra* A with diagonal D is a subalgebra of B such that $A \cap A^* = D$. It follows that $A = \varinjlim(A_i, \phi_i)$, where $A_i = A \cap B_i$, for all i. Each A_i is necessarily triangular in B_i with diagonal D_i; if, in addition, each A_i is maximal triangular then we say that A is *strongly maximal*. This is equivalent to requiring that $A + A^*$ be dense in B.

AF C*-algebras are groupoid C*-algebras with groupoids which are especially tractable: the groupoids are topological equivalence relations. As such, the C*-algebra can be identified with an algebra of functions on the groupoid. Subalgebras such as TAF algebras and ideals determine and are determined by appropriate substructures of the groupoid. This provides a coordinatization for TAF algebras and their ideals.

The following is a brief sketch of this coordinatization for strongly maximal TAF algebras. For a more thorough description, see [13] or [9]. Let (D, A, B) be a triple in which B is an AF C*-algebra, D is a canonical masa, and A is a strongly maximal TAF subalgebra of B with diagonal D. We need to describe the *spectral triple*, (X, P, G) associated with (D, A, B). The first ingredient X is the usual spectrum of the abelian C*-algebra D (so that $D \cong C(X)$). Note that, in the present context, X will be a Cantor space. The algebra B is generated by partial isometries which normalize D. Each normalizing partial isometry induces a partial homeomorphism of X into itself (a homeomorphism of a closed subset of X onto another closed subset). The union of the graphs of these homeomorphisms is an equivalence relation; when this set is topologized so that the graph of each normalizing partial isometry is open and closed, one obtains the groupoid G. The spectrum P of A is the union of the graphs of the normalizing partial isometries which lie in A.

If $x \in X$, the equivalence class in G which contains x is referred to as the *orbit* of x (since it consists of all the images of x under homeomorphisms associated with D-normalizing partial isometries) and is denoted by orb_x. When $(x, y) \in P$, we shall often write $x \preceq y$; when A is strongly maximal, P induces a total order on each orbit.

Now suppose that π is a nest representation of a TAF algebra A. Since D is an abelian C*-algebra, results in [5] imply that the restriction of π to D is similar to a *-representation. (For the limited domain in which we need this theorem, abelian C*-algebras, Kadison attributes this fact to unpublished 1952 lecture notes of Mackey.) Thus any nest representation of a TAF algebra is similar to another nest representation whose restriction to the diagonal is a *-representation. Accordingly, we assume throughout this paper that any nest representation π of a TAF algebra A acts as a *-representation on the diagonal, D.

2 Totally Ordered Spectrum

In this section, A will be a strongly maximal TAF algebra whose C*-envelope B is simple. This implies that the orbit of each element in X is dense in X. The diagonal of A is denoted by D. The spectral triple for (D, A, B) is (X, P, G). We assume that X has a total order $\preceq$ which agrees on each equivalence class with the total order induced by P. X has a minimal and a maximal element, which we denote by 0 and 1, respectively. If $a \in X$, we say that a has a *gap above* if a has an immediate successor and that a has a *gap below* if a has an immediate predecessor.

Elements of B will be viewed as continuous functions in $C_0(G)$; elements of A are continuous functions whose supports lie in P. Of course, not all elements of $C_0(G)$ are elements of B, but all those with compact support certainly are in B. When elements of B are viewed as functions, the multiplication is given by a convolution formula with respect to a Haar system consisting of counting measure on each equivalence class from G.

If Y is a clopen subset of X, then E_Y denotes the projection in D corresponding to Y (*i.e.*, the characteristic function of Y). If a is an element of X, $[0, a]$ denotes the set $\{x \in X \mid 0 \preceq x \preceq a\}$. When $a \neq 1$, $[0, a]$ is clopen if, and only if, a has a gap above. This occurs precisely when the characteristic function of $[0, a]$ is continuous and hence a projection (not the identity) in D.

In the situation where X carries a total order compatible with P, the meet irreducible ideal sets have been completely described in Theorem 3.1 in [3]. For the convenience of the reader, we restate that theorem here. For each pair of elements a and b in X, define two subsets of P:

$$\sigma_{a,b} = \{(x, y) \in P \mid x \prec a \text{ or } b \prec y\},$$
$$\tau_{a,b} = \sigma_{a,b} \cup \{(a, b)\}.$$

The set $\sigma_{a,b}$ is an ideal set in P; the set $\tau_{a,b}$ is an ideal set provided that $(a, b) \in P$ and $\tau_{a,b}$ is an open subset of P. (We always assume that these two conditions hold.)

Theorem 2.1 *Assume that A is a strongly maximal TAF algebra with simple C*-envelope and totally ordered spectrum. The following is a complete list of all the meet irreducible ideal sets in P:*

1) $\sigma_{a,b}$, if $(a, b) \in P$;
2) $\sigma_{a,b}$, if $(a, b) \notin P$ and either a has no gap above or b has no gap below;
3) $\tau_{a,b}$, where either a has no gap above or b has no gap below.

Theorem 2.2 *Let π be a continuous nest representation of A, where A is a strongly maximal TAF algebra with simple C*-envelope and totally ordered spectrum. Then the kernel of π is a meet irreducible ideal in A.*

Proof Let π be a continuous nest representation of A acting on the Hilbert space $\mathcal{H}$. Let σ be the ideal set in P which corresponds to the ideal $\ker \pi$. Through a series of facts, we will show that σ is one of the meet irreducible ideal sets listed in Theorem 2.1; consequently, $\ker \pi$ is meet irreducible.

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Fact 1 *Suppose that a has a gap above or, equivalently, that $E_{[0,a]}$ is a projection in A not equal to the identity. Then $E_{[0,a]}$ is invariant for A and hence $\pi(E_{[0,a]}) \in \operatorname{Lat}\pi(A)$.*

Proof Let $f \in A$; view f as a C_0 function on G. For any $(x, y) \in G$,

$$
\begin{aligned}
fE_{[0,a]}(x,y) &= \sum_{z \in \mathrm{orb}_x} f(x,z)E_{[0,a]}(z,y) \\
&= \begin{cases} f(x,y), & \text{if } y \preceq a \\ 0, & \text{otherwise} \end{cases} \\
&= E_{[0,a]}(x,x) f(x,y) E_{[0,a]}(y,y) \\
&= E_{[0,a]} f E_{[0,a]}(x,y).
\end{aligned}
$$

We use the fact that $(x, y) \in \operatorname{supp} f$ implies that $x \preceq y$ in X. ■

Fact 2 *If a has a gap above and if there is a point $(a, c) \in P \setminus \sigma$, then $\pi(E_{[0,a]}) \neq 0$ and $\pi(E_{[0,a]})$ is a non-trivial invariant projection for $\pi(A)$.*

Remark If we were not assuming that π is normalized to a *-representation of D, then $\pi(E_{[0,a]})$ would be a non-trivial idempotent whose range is an invariant subspace for $\pi(A)$.

Proof Since $(a, a) \circ (a, c) = (a, c)$ and σ is an ideal set, $(a, a) \in P \setminus \sigma$. If $\pi(E_{[0,a]}) = 0$, then $E_{[0,a]} \in \ker\pi$; hence, $\operatorname{supp} E_{[0,a]} \subseteq \sigma$. Thus $(a, a) \in \sigma$, a contradiction. ■

Fact 3 *If b has a gap below and if there is a point $(c, b) \in P \setminus \sigma$, then $\pi(E_{[b,1]}) \neq 0$.*

Proof Essentially the same as for Fact 2: $(c, b) \circ (b, b) = (c, b)$, so $(b, b) \in P \setminus \sigma$. ■

Fact 4 *Assume a has a gap above, b has a gap below, $\pi(E_{[0,a]}) \neq 0$, and $\pi(E_{[b,1]}) \neq 0$. If $E_{[0,a]} f E_{[b,1]} \in \ker\pi$ for all $f \in N_D(A)$ (or, for all f in a set with linear span dense in A), then π is not a nest representation.*

Proof Since $\ker\pi$ is closed, $E_{[0,a]} f E_{[b,1]} \in \ker\pi$, for all $f \in A$. As a consequence, we have $\pi(E_{[0,a]})\pi(f)\pi(E_{[b,1]}) = 0$, for all $f \in A$. Let $\mathcal{K}_1$ be the range of $\pi(E_{[0,a]})$ and $\mathcal{K}_2$ be the norm closure of $\pi(A)\pi(E_{[b,1]})\mathcal{H}$. Then $\mathcal{K}_1$ and $\mathcal{K}_2$ are non-zero invariant subspaces for $\pi(A)$ and $\mathcal{K}_1 \cap \mathcal{K}_2 = (0)$. So $\operatorname{Lat}\pi(A)$ is not totally ordered by inclusion; π is not a nest representation. ■

In what follows we use the standard identification of X with the diagonal of P, *i.e.*, with $\{(x, x) \mid x \in X\}$.

Fact 5 *If π is a nest representation, then $(P \setminus \sigma) \cap X$ is an interval in X.*

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Proof Suppose not. Then there are three points a, b, c with $a \prec b \prec c$ in X such that $(a,a) \in P \setminus \sigma$, $(b,b) \in \sigma$, and $(c,c) \in P \setminus \sigma$. It follows that if $(x,y) \in P$ with $x \preceq b$ and $b \preceq y$, then $(x,y) \in \sigma$. (Since σ is open there is a neighborhood of (b,b) which is contained in σ; use the assumption that all orbits are dense and the ideal set property for σ.)

Choose α and β so that $a \preceq \alpha \prec b$, $b \prec \beta \preceq c$, α has a gap above, and β has a gap below. (Since X is a totally ordered Cantor set, either a itself has a gap above or there are infinitely many points between a and b which have a gap above; this guarantees the existence of α. The existence of β is handled analogously.) Then $(a,a) \in \operatorname{supp} E_{[0,\alpha]}$; hence $\pi(E_{[o,\alpha]}) \neq 0$ and the range of $\pi(E_{[0,\alpha]})$ is in $\operatorname{Lat} \pi(A)$. Also, $(b,b) \in \operatorname{supp} E_{[\beta,1]}$, so $\pi(E_{[\beta,1]}) \neq 0$. For any $g \in A$,

$$\begin{aligned}(x,y) \in \operatorname{supp} E_{[0,\alpha]} g E_{[\beta,1]} &\implies x \preceq \alpha \prec b \text{ and } b \prec \beta \preceq y \\ &\implies (x,y) \in \sigma.\end{aligned}$$

Consequently, $E_{[0,\alpha]} g E_{[\beta,1]} \in \ker \pi$. Fact 4 now implies that π is not a nest representation, contradicting the hypothesis. ■

Assume π is a nest representation and that a is the left endpoint of $(P \setminus \sigma) \cap X$ and b is the right endpoint of $(P \setminus \sigma) \cap X$. Each of a and b may or may not be elements of $(P \setminus \sigma) \cap X$. However, if a has a gap above, then without loss of generality, we may assume that $a \in (P \setminus \sigma) \cap X$ (simply replace a by its immediate successor, if necessary). Similarly, if b has a gap below, we may assume that $b \in (P \setminus \sigma) \cap X$.

Fact 6 *If $a \prec \alpha \preceq \beta \prec b$ and $(\alpha,\beta) \in P$, then $(\alpha,\beta) \in P \setminus \sigma$.*

Proof Suppose that $a \prec \alpha \preceq \beta \prec b$, $(\alpha,\beta) \in P$, and $(\alpha,\beta) \in \sigma$. Choose $\bar\alpha$ and $\bar\beta$ so that $a \prec \bar\alpha \preceq \alpha \preceq \beta \preceq \bar\beta \prec b$, $\bar\alpha$ has a gap above, and $\bar\beta$ has a gap below. [Exceptions: if $\alpha = \operatorname{succ} a$, choose $\bar\alpha = a$ and note that this element is not in $\sigma \cap X$; if $\beta = \operatorname{pred} b$, choose $\bar\beta = b$ and note that this is not in $\sigma \cap X$.]

It follows that $\pi(E_{[0,\bar\alpha]})$ is non-zero and has range in $\operatorname{Lat} \pi(A)$ and that $\pi(E_{[\bar\beta,1]}) \neq 0$. Let $g \in A$ be arbitrary. If (x,y) is in $\operatorname{supp} E_{[0,\bar\alpha]} g E_{[\bar\beta,1]}$, then $x \preceq \bar\alpha \preceq \alpha$ and $\beta \preceq \bar\beta \preceq y$; hence $(x,y) \in \sigma$. Thus $E_{[0,\bar\alpha]} g E_{[\bar\beta,1]} \in \ker \pi$, for all g. Fact 4 now implies that π is not a nest representation, a contradiction. ■

Conclusion A *If π is a nest representation and a and b are the endpoints of the interval $(P \setminus \sigma) \cap X$, then*

$$\{(x,y) \in P \mid x \prec a \text{ or } b \prec y\} \subseteq \sigma \subseteq \{(x,y) \in P \mid x \preceq a \text{ or } b \preceq y\}.$$

Proof The conclusion follows from the following implications for a point $(x,y) \in P$:

$$\begin{aligned} x \prec a &\implies (x,x) \in \sigma \implies (x,y) \in \sigma \\ b \prec y &\implies (y,y) \in \sigma \implies (x,y) \in \sigma \\ a \prec x \preceq y \prec b &\implies (x,y) \in P \setminus \sigma. \end{aligned}$$

■

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Given a and b in X, let

$$H = \{(a, y) \in P \mid a \preceq y \preceq b\}, \text{ and}$$
$$V = \{(x, b) \in P \mid a \preceq x \preceq b\}.$$

Fact 7 *If a has no gap above, then $H \cap \sigma \subseteq \{(a, b)\}$. If b has no gap below, then $V \cap \sigma \subseteq \{(a, b)\}$.*

Proof This follows immediately from the fact that σ is an open subset of P. ■

Fact 8 *If a has a gap above, then either $H \cap \sigma = H$ or $H \cap \sigma \subseteq \{(a, b)\}$.*

Proof Assume the contrary. Then $(a, a) \notin \sigma$ and there is β such that $a \prec \beta \prec b$ and $(a, \beta) \in \sigma$. Consider two cases. First, assume that $\beta = \operatorname{pred} b$. In this case, since b has a gap below, we also have available the assumption that $(b, b) \notin \sigma$. (See the comments after Fact 5.) Since $(a, a) \notin \sigma$ and $(b, b) \notin \sigma$, both $\pi(E_{[0,a]})$ and $\pi(E_{[b,1]})$ are non-zero. Furthermore, for any $g \in A$, $\operatorname{supp} E_{[0,a]} g E_{[b,1]} \subseteq \sigma$. Indeed, if $(x, y) \in \operatorname{supp} E_{[0,a]} g E_{[b,a]}$, then $x \preceq a$ and $b \preceq y$. If either $x \prec a$ or $b \prec y$, the $(x, y) \in \sigma$. If $x = a$ and $y = b$, then $(a, b) = (x, y) \in P$; since $(a, \beta) \in \sigma$, we also have $(a, b) \in \sigma$. Fact 4 implies that π is not a nest representation, a contradiction.

In the alternative case, $\beta \prec b$ and there is $\bar{\beta}$ such that $\beta \prec \bar{\beta} \prec b$ and $\bar{\beta}$ has a gap below. Since $(\bar{\beta}, \bar{\beta}) \notin \sigma$, $\pi(E_{[\bar{\beta},1]}) \neq 0$. As before, $\pi(E_{[0,a]}) \neq 0$. Let $(x, y) \in \operatorname{supp} E_{[0,a]} g E_{[\bar{\beta},1]}$, where g is any element of A. If $x \prec a$ then $(x, y) \in \sigma$. If $x = a$ then $\beta \prec \bar{\beta} \preceq y$, whence $(x, y) = (a, y) \in \sigma$. Once again, Fact 4 yields a contradiction. ■

Fact 9 *If b has a gap below, then either $V \cap \sigma = V$ or $V \cap \sigma \subseteq \{(a, b)\}$.*

Proof The idea behind the proof is essentially the same as in the proof of Fact 8. This time, if the conclusion does not hold, then $(b, b) \notin \sigma$ and there is α such that $a \prec \alpha \prec b$ and $(\alpha, b) \in \sigma$. If $\alpha = \operatorname{succ} a$ then take $\bar{\alpha} = a$; otherwise, take $\bar{\alpha}$ so that $a \prec \bar{\alpha} \prec \alpha$ and $\bar{\alpha}$ has a gap above. Now apply Fact 4 to $E_{[0,\bar{\alpha}]}$ and $E_{[b,1]}$. ■

Conclusion B *Conclusion A and Facts 8 and 9 imply that σ has one of the following two forms:*

$$\sigma_{a,b} = \{(x, y) \in P \mid x \prec a \text{ or } b \prec y\} \quad \textit{or}$$
$$\tau_{a,b} = \sigma_{a,b} \cup \{(a, b)\}.$$

Note that the latter is a possibility only if $(a, b) \in P$ and $\tau_{a,b}$ is open.

Fact 10 *If a has a gap above and b has a gap below and either $\sigma = \sigma_{a,b}$ with $(a, b) \notin P$ or $\sigma = \tau_{a,b}$, then π is not a nest representation.*

Proof Apply Fact 4 to $E_{[0,a]}$ and $E_{[b,1]}$. ■

This effectively ends the proof of Theorem 2.2. If π is a nest representation, then σ is one of the ideal sets listed in Theorem 2.1. Since these are all meet irreducible, we have proven that $\ker \pi$ is meet irreducible. ■

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3 Nest Representations with Atoms

In this section we give a condition on a nest representation which guarantees that $\ker\pi$ is meet irreducible. This condition requires that π be a *-representation on the diagonal of the strongly maximal TAF algebra. As pointed out in the introduction, any nest representation is similar to one with this property; consequently, we assume throughout this section that the restriction of π to D is a *-representation.

Recall that, in a von Neumann algebra $\mathcal{D}$, a projection E is said to be an *atom* if E majorizes no proper (nonzero) subprojection. $\mathcal{D}$ is *atomic* if, for any projection $P \in \mathcal{D}$, $P = \bigvee\{E \in \mathcal{D} : E \text{ is an atom and } E \leq P\}$.

If π is a nest representation of a strongly maximal TAF algebra A with diagonal D such that the von Neumann algebra $\pi(D)''$ contains an atom, then $\ker\pi$ is meet irreducible. This is established in Theorem 3.9, for which we give two proofs. The first proof depends on Theorem 2.1 in [3]; the alternative proof is independent of this theorem. Both proofs require the fact (established in Proposition 3.5) that if $\pi(D)''$ contains an atom, then it is an atomic von Neumann algebra. The alternative proof can be read immediately after Proposition 3.5; from this point on it uses the inductivity of ideals rather than the spectrum characterization of meet irreducible ideals from [3]. In fact, a reader willing to assume that $\pi(D)''$ is atomic can read the alternative proof to Theorem 3.9 immediately after Lemma 3.1. This provides a much shorter route to a somewhat weaker theorem. The alternative proof can be found at the end of this section.

If $\pi(D)''$ is atomic, then Corollary 3.10 implies that $\operatorname{Lat}\pi(A)$ is a purely atomic nest. The reverse implication is false: Example I.3 in [10] provides an example of a *-extendible representation of a standard limit algebra for which $\operatorname{Lat}\pi(A) = \{0, I\}$ and $\pi(D)$ is weakly dense in a continuous masa. Since $\ker\pi = \{0\}$ in this example, $\ker\pi$ is meet irrreducible; we conjecture that $\ker\pi$ is meet irreducible any time that $\operatorname{Lat}\pi(A)$ contains an atom. Proposition 3.5 establishes a dichotomy: either $\pi(D)''$ is atomic or else $\pi(D)''$ contains no atoms at all. This raises the natural question: is the same dichotomy valid for $\operatorname{Lat}\pi(A)$? This dichotomy certainly does not hold for general nest representations (those which are not *-representations on the diagonal); the failure is a consequence of the similarity theory for nests. The proof of Theorem 2.4 in [3] applied to a refinement algebra provides an example of a nest representation π of a TAF agebra whose nest is purely atomic, order isomorphic to the Cantor set, and such that the atoms are ordered as the rationals. There exist nests which are not purely atomic but which are order isomorphic to the Cantor nest and which have (rank one) atoms ordered as the rationals; the similarity theorem for nests [2] gives the existence of an invertible operator which carries the invariant subspace nest for π onto a nest of the second type. The composition of π with this similarity yields a nest representation of a strongly maximal TAF algebra (the refinement algebra) which has atoms but is not purely atomic.

As before, we let X denote the spectrum of the diagonal D of a TAF algebra A. This is a zero dimensional topological space and the clopen sets form a basis for the topology. When e is a projection in D, we let $\hat{e}$ denote the spectrum of e in X (*i.e.*, the support set of e viewed as an element of $C(X)$).

Now suppose that π is a *-representation of the diagonal D of a TAF algebra and let E be the spectral measure associated with π. E is a regular, projection valued measure defined on the Borel sets of X which "agrees" with π on clopen subsets in the sense that $E(\hat{e}) = \pi(e)$,

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where e is any projection in D and $\hat{e}$ is its support in X. If $\mathcal{D}$ is the von Neumann algebra generated by $\pi(D)$, then any projection P in $\mathcal{D}$ is of the form $E(S)$, where S is a Borel subset of X.

When S is a singleton $\{x\}$ we shall write E_x in place of $E(\{x\})$. If e_n is a decreasing sequence of projections in $\mathcal{D}$ such that $\bigcap \hat{e}_n = \{x\}$, then, by the regularity of E, $E_x = \bigwedge \pi(e_n)$. In particular, $\bigwedge \pi(e_n) = \bigwedge \pi(f_n)$ for any two decreasing sequences of projections in $\mathcal{D}$ with $\bigcap \hat{e}_n = \{x\}$ and $\bigcap \hat{f}_n = \{x\}$.

If there is a projection e in $\ker \pi$ with $x \in \hat{e}$, then clearly $E_x = 0$. In this case, if $\hat{e}_n$ is any decreasing sequence of clopen sets with $\bigcap \hat{e}_n = \{x\}$, then, since $\{\hat{e}_n\}$ is a neighborhood basis for x, we have $\pi(e_n) = 0$ for all large n. The alternative is that $\pi(e) \neq 0$ for any projection with $x \in \hat{e}$; in particular, for any decreasing sequence e_n with $\bigcap \hat{e}_n = \{x\}$, $\pi(e_n) \neq 0$ for all n. The projection $E_x = \bigwedge \pi(e_n)$ may or may not be 0; it is, however, independent of the choice of decreasing clopen sets with intersection $\{x\}$.

Note also that if $x, y \in X$ and $x \neq y$, then $E_x E_y = 0$.

Lemma 3.1 *Let D be the diagonal of a TAF algebra; let $\pi\colon D \to \mathcal{B}(\mathcal{H})$ be a *-representation; let E be the spectral measure for π; and let $\mathcal{D}$ be the von Neumann algebra generated by $\pi(D)$. For any $x \in X$, if $E_x \neq 0$ then E_x is an atom of $\mathcal{D}$. Conversely, if E_0 is an atom for $\mathcal{D}$, then there is a unique element $x \in X$ such that $E_0 = E_x$.*

Proof Any projection in $\mathcal{D}$ has the form $E(S)$ for some Borel subset S of X. Given $x \in X$, if $x \in S$ then $E_x \leq E(S)$ and if $x \notin S$ then $E_x E(S) = 0$. This shows that when $E_x \neq 0$, it is an atom of $\mathcal{D}$.

Now suppose that E_0 is an atom of $\mathcal{D}$. Let S be such that $E_0 = E(S)$. It is evident that there is at most one point $x \in S$ such that $E_x \neq 0$; we need to prove the existence of such a point.

Since X is a Cantor set, we can find, for each $n \in \mathbb{N}$, 2^n disjoint clopen sets $\hat{e}^n_k$, $k = 1, \dots, 2^n$, whose union is X, with the further property that any decreasing sequence of these sets has one-point intersection. Since E_0 is an atom of $\mathcal{D}$, for each n there is a unique integer k_n in $\{1, \dots, 2^n\}$ such that $E_0 \leq \pi(\hat{e}^n_{k_n})$. Let x be such that $\bigcap_n \hat{e}^n_{k_n} = \{x\}$. Clearly, $E_0 \leq E_x$. But E_x is an atom when it is non-zero; hence $E_0 = E_x$. The uniqueness of x follows immediately form the orthogonality of E_x and E_y when $x \neq y$. ∎

Notation Let $\pi\colon A \to \mathcal{B}(\mathcal{H})$ be a representation of a TAF algebra A (which acts as a *-representation on the diagonal D). For $\xi \in \mathcal{H}$ and $x \in X$ let $\mathcal{M}_\xi$ denote the smallest π-invariant subspace which contains ξ and $\mathcal{M}_x$ denote the smallest π-invariant subspace which contains $\operatorname{Ran} E_x$.

Since the linear span of the D-normalizing partial isometries is dense in A, $\mathcal{M}_\xi$ is the closed linear span of $\{\pi(v)\xi \mid v \in N_D(A)\}$ and $\mathcal{M}_x$ is the closed linear span of $\{\pi(v)\xi \mid v \in N_D(A), \xi \in E_x\}$.

We now need to investigate the manner in which D-normalizing partial isometries act on atoms of $\mathcal{D}$. Throughout the remainder of this section, A will denote a TAF algebra (we will add the hypothesis that A is strongly maximal later); π will denote a nest representation of A acting on a Hilbert space $\mathcal{H}$; D will denote the diagonal of A (with spectrum X); and $\mathcal{D} = \pi(D)''$.

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Lemma 3.2 *Let $v \in N_D(A)$ and let $x \in X$. If $x \notin \widehat{v^*v}$, then $\pi(v)E_x = 0$. If $x \in \widehat{v^*v}$, there exists $y \in X$ such that $(y, x) \in \hat{v}$ and $\pi(v)E_x = E_y\pi(v)$. In particular, if $y \neq x$,* $\operatorname{Ran} \pi(v)E_x \perp \operatorname{Ran} E_x$.

Proof Let $\{e_n\}$ be a decreasing sequence of projections in D with $\bigcap \hat{e}_n = \{x\}$. If $x \notin \widehat{v^*v}$, then $ve_n = 0$ for large n, in which case $\pi(v)E_x = \pi(v)\pi(e_n)E_x = \pi(ve_n)E_x = 0$.

Now suppose that $x \in \widehat{v^*v}$ and let y be such that $(y, x) \in \hat{v}$. With e_n as above, let $f_n = ve_nv^*$, so that $\bigcap \hat{f}_n = \{y\}$ and $\bigwedge \pi(f_n) = E_y$. Then $ve_n = f_nv$ for large n; hence $\pi(v)\pi(e_n) = \pi(f_n)\pi(v)$ and, taking strong limits, $\pi(v)E_x = E_y\pi(v)$. It follows that $\operatorname{Ran}\big(\pi(v)E_x\big) \subset \operatorname{Ran} E_y$ and, hence, that E_x and $\pi(v)E_x$ have orthogonal ranges when $y \neq x$. ■

Remark If $v \in N_D(A)$ and $(y, x) \in \hat{v}$, then $\pi(v)E_x = E_y\pi(v)E_x$. It is possible that $\pi(v)E_x = 0$ even when $\pi(v) \neq 0$ and $E_x \neq 0$.

Lemma 3.3 *Let $x \in X$. If $E_x \neq 0$, then E_x is a rank-one atom.*

Proof Let $x \in X$ and assume that $\operatorname{Ran} E_x$ contains two unit vectors ξ and ζ such that $\xi \perp \zeta$. Let $v \in N_D(A)$. Since $\pi(v)E_x = \pi(v)\pi(e)E_x$ for any projection e for which $x \in \hat{e}$, we may, by a suitable restriction, reduce to considering two cases: when $\hat{v}$ is contained in the diagonal $\{(z, z) \mid z \in X\}$ of G and when $\hat{v}$ is disjoint from the diagonal. When $\hat{v}$ is contained in the diagonal, v is a projection in D and $\pi(v)$ either dominates E_x or is orthogonal to E_x. In this case, either $\pi(v)\xi = \xi$ and $\pi(v)\zeta = \zeta$ or $\pi(v)\xi = 0$ and $\pi(v)\zeta = 0$. When $\hat{v}$ is disjoint form the diagonal of G, there is $y \in X$ such that $y \neq x$ and $(y, x) \in \hat{v}$. In this case, $\pi(v)\xi \in \operatorname{Ran} E_y$ and $\pi(v)\zeta \in \operatorname{Ran} E_y$. In particular, $\pi(v)\xi \perp \zeta$ and $\pi(v)\zeta \perp \xi$ for all $v \in N_D(A)$.

It now follows that $\xi \in \mathcal{M}_\xi$ and $\xi \notin \mathcal{M}_\zeta$ while $\zeta \in \mathcal{M}_\zeta$ and $\zeta \notin \mathcal{M}_\xi$. But π is a nest representation and $\mathcal{M}_\xi$ and $\mathcal{M}_\zeta$ are π-invariant; hence one must contain the other. Thus, the rank of E_x is at most 1. ■

Corollary 3.4 *Let $u, w \in N_D(A)$ and assume that u and w have a common subordinate. Let (y, x) be a point in P such that $(y, x) \in \hat{u} \cap \hat{w}$. Then $\pi(u)E_x = \pi(w)E_x$ and, if E_x, $E_y \neq 0$,* $\operatorname{Ran} \pi(u)E_x = \operatorname{Ran} E_y$.

Proof Let e be a (nonzero) projection in D such that $e \leq \min\{u^*u, w^*w\}$ and $x \in \hat{e}$. (One could just take $e = u^*uw^*w$.) Then $ue = we$. Since $\pi(e)$ dominates E_x,

$$\pi(u)E_x = \pi(u)\pi(e)E_x = \pi(ue)E_x = \pi(we)E_x = \pi(w)\pi(e)E_x = \pi(w)E_x.$$

For the second assertion, suppose E_x and $E_y \neq 0$. Now by Lemma 3.2, for any $v \in N_D(A)$, either $\pi(v)E_y$ is zero or else $\pi(v)E_y$ is contained in $\operatorname{Ran} E_z$ for some $z \in X$ with $(z, y) \in \hat{v}$. But as $y \prec x$, $\operatorname{Ran} \pi(v)E_y \perp \operatorname{Ran} E_x$. We have seen that $\mathcal{M}_y$, the smallest π-invariant subspace containing $\operatorname{Ran} E_y$, is disjoint from $\operatorname{Ran} E_x$; since π is a nest representation, it follows that $\mathcal{M}_y \subset \mathcal{M}_x$. Thus, for some v, $\operatorname{Ran} \pi(v)E_x \cap \operatorname{Ran} E_y \neq (0)$. Since the ranges of E_x and E_y are one-dimensional, $\operatorname{Ran} \pi(v)E_x = \operatorname{Ran} E_y$. For such a v we have $(y, x) \in \hat{v}$. By the first paragraph, v can be replaced by any u with $(y, x) \in \hat{u}$. ■

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Proposition 3.5 *$E_0 = \bigvee\{E_x \mid x \in X\}$ is either 0 or I.*

Proof Let $E_1 = I - E_0$. We shall show that both $E_0\mathcal{H}$ and $E_1\mathcal{H}$ are invariant under π. Since π is a nest representation, this means that one of them must be zero.

If $E_0\mathcal{H}$ is not invariant, then for some $v \in N_D(A)$ and $\xi \in E_0\mathcal{H}$, $E_1\pi(v)\xi \neq 0$. Now $\xi = \sum\{E_x\xi \mid x \in X\}$, so there exists $x \in X$ with $E_1\pi(v)E_x\xi \neq 0$. By Lemma 3.2, $\pi(v)E_x = E_y\pi(v)E_x$, for some y. Since $\pi(v)E_x \neq 0$ it follows that $E_y \neq 0$; hence E_y is an atom in $\mathcal{D}$. As E_1 majorizes no atoms, $E_1E_y = 0$. On the other hand, $0 \neq E_1\pi(v)E_x = E_1E_y\pi(v)E_x$, so that $E_1E_y \neq 0$. This contradiction shows that $E_0\mathcal{H}$ is invariant.

If $E_1\mathcal{H}$ is not invariant, there exists a vector $\xi \in E_1\mathcal{H}$ and a D-normalizing partial isometry v in A such that $E_0\pi(v)E_1\xi \neq 0$. Since E_0 is the sum of the atoms it majorizes, $E_y\pi(v)E_1\xi \neq 0$ for some $y \in X$. By Lemma 3.2, there is an element $x \in X$ with $E_y\pi(v) = E_y\pi(v)E_x$, whence $E_y\pi(v)E_xE_1 \neq 0$. In particular, $E_xE_1 \neq 0$, which contradicts the fact that E_1 majorizes no atoms. Thus, $E_1\mathcal{H}$ is also invariant. ■

Remark Let $\mathcal{D}$ be the von Neumann algebra generated by $\pi(D)$, where D is the diagonal of the TAF algebra A. According to Proposition 3.5, if $\pi\colon A \to \mathcal{B}(\mathcal{H})$ is a nest representation, then either $\mathcal{D}$ is generated by its atoms (that is, $\mathcal{D}$ is purely atomic), or else it has no atoms. (This, of course, presumes that the restriction of π to D is a *-representation.)

Lemma 3.6 *If x and y are two points of X such that E_x and E_y are both nonzero, then x and y belong to the same orbit in X.*

Proof If x and y are not in the same orbit, it follows that $\operatorname{Ran} E_x \perp \mathcal{M}_y$ and $\operatorname{Ran} E_y \perp \mathcal{M}_x$ (Lemma 3.2). Since $\operatorname{Ran} E_y \subseteq \mathcal{M}_y$ and $\operatorname{Ran} E_x \subseteq \mathcal{M}_x$, $\mathcal{M}_x$ and $\mathcal{M}_y$ are not lineraly ordered, a contradiction. ■

Lemma 3.7 *Assume, further, that A is a strongly maximal TAF algebra. If, for some $x \in X$, $E_x \neq 0$, then $J = \{z \mid E_z \neq 0\}$ is an interval in the orbit of x.*

Proof If $E_x \neq 0$, Lemma 3.6 implies that J is contained in the orbit of x. If $E_z = 0$ for all $z \neq x$, we are done. Suppose then, that $E_y \neq 0$ for some $y \neq x$ in the orbit of x. Without loss of generality we may assume that $y \prec x$. Let $v \in N_D(A)$ be such that $(y, x) \in \hat{v}$. Then $\mathcal{M}_y \subset \mathcal{M}_x$ and $\operatorname{Ran} \pi(v)E_x = \operatorname{Ran} E_y$ (Corollary 3.4). In particular, $\pi(v)E_x \neq 0$.

Let z be a point in the orbit of x with $y \prec z \prec x$. Since A is strongly maximal, there exist $u, w \in N_D(A)$ with $(y, z) \in \hat{u}$ and $(z, x) \in \hat{w}$. Now $\pi(v)E_x = \pi(uw)E_x = \pi(u)\pi(w)E_x$; hence $\pi(w)E_x \neq 0$. As $\operatorname{Ran} \pi(w)E_x \subset \operatorname{Ran} E_z$, it follows that $E_z \neq 0$ and, hence, $z \in J$. This shows that J is an interval. ■

Corollary 3.8 *If $\mathcal{D}$ has an atom and J is the interval obtained in Lemma 3.7, then $\bigvee\{E_x \mid x \in J\} = I$ and $\mathcal{D}$ is a masa in $\mathcal{B}(\mathcal{H})$.*

Proof The first assertion follows from Lemma 3.1 and Proposition 3.5, since all atoms have the form E_x, for some $x \in X$. Since the set $\{E_x \mid x \in X\}$ is a collection of commuting, rank-one atoms whose ranges span $\mathcal{H}$, the von Neumann algebra which they generate is a masa in $\mathcal{B}(\mathcal{H})$. ■

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Theorem 3.9 *Let A be a strongly maximal TAF algebra and $\pi\colon A \to \mathcal{B}(\mathcal{H})$ be a nest representation for which the von Neumann algebra $\mathcal{D}$ generated by $\pi(D)$ contains an atom. Then the kernel of π is a meet irreducible ideal in A.*

Proof By hypothesis, $\mathcal{D}$ contains an atom, necessarily of the form E_x, for some $x \in X$. By Lemma 3.7, there is a nonempty interval J in an orbit in X with the following property: for any D-normalizing partial isometry v, $\pi(v) \neq 0$ if, and only, if $J \times J$ intersects $\hat{v}$. In other words, the complement of the spectrum of the ideal $\ker\pi$ contains $(J \times J) \cap P$. But the complement is a closed set, so it contains $\overline{(J \times J) \cap P}$. On the other hand, if v is a D-normalizing partial isometry with $\hat{v}$ disjoint from $\overline{(J \times J) \cap P}$, then $\pi(v) = 0$. Thus,

$$\widehat{\ker\pi} = P \setminus \overline{(J \times J) \cap P}.$$

By [3, Theorem 2.1], $\ker\pi$ is a meet-irreducible ideal. ■

Corollary 3.10 *Let A, π, and J be as above and let $\mathcal{N} = \operatorname{Lat}\pi(A)$.*

1) *If F is decreasing subset of J, then $P = \sum\{E_x \mid x \in F\} \in \mathcal{N}$. On the other hand, if $P \in \mathcal{N}$, then $F = \{x \mid 0 \neq E_x \leq P\}$ is a decreasing subset of J. This correspondence between decreasing subsets of J and projections in $\mathcal{N}$ is a bijection.*
2) *$\mathcal{D} = \operatorname{Alg}\mathcal{N} \cap (\operatorname{Alg}\mathcal{N})^*$ equals the von Neumann algebra generated by $\mathcal{N}$.*

Proof The first assertion is clear. For the second, first note that, since $\mathcal{D}$ is a masa, $\mathcal{D} = \operatorname{Alg}\mathcal{N} \cap (\operatorname{Alg}\mathcal{N})^*$. Let $x \in J$, let $P = \sum\{E_y \mid y \in J \text{ and } y \preceq x\}$, and let $P_- = \sum\{E_y \mid y \in J \text{ and } y \prec x\}$. Then P_- is the immediate predecessor of P in $\mathcal{N}$ and $E_x = P - P_-$. Thus, every atom from $\mathcal{D}$, and hence $\mathcal{D}$ itself, is contained in the von Neumann algebra generated by $\mathcal{N}$. The reverse inclusion is obvious. ■

There is an alternate proof for Theorem 3.9 based on a presentation for A rather than on the spectrum of A and Theorem 2.1 in [3]. This proof is dependent only on the preliminary results through Proposition 3.5; in fact, if the reader is willing to assume that $\pi(D)''$ is purely atomic, then only Lemma 3.1 is needed. In the alternate proof, we view A as the union of an ascending chain of subalgebras each of which is star extendibly isomorphic to a maximal triangular subalgebra of a finite dimensional C*-algebra. Also we may assume that a system of matrix units for each A_k has been selected in such a way that each matrix unit in A_k is a sum of matrix units in A_{k+1}; this gives a matrix unit system for A. Another fact from the lore of direct limit algebras that we need is that ideals are inductive: if I is an ideal in A, then I is the closed union of the ideals $I_k = I \cap A_k$ in A_k.

Alternate Proof of Theorem 3.9 Assume that π is a nest representation but that $\ker\pi$ is not meet irreducible. Let I and J be two ideals in A such that $I \cap J = \ker\pi$ and $I \cap J$ differs from both I and J. By the inductivity of ideals, there exist matrix units $u_I \in I \setminus J$ and $u_J \in J \setminus I$. These matrix units must lie in some A_k; since we may replace the sequence A_k by a subsequence, we may assume that u_I and u_J lie in A_1.

Since $u_I \notin J$, $\pi(u_I) \neq 0$. By Proposition 3.5,

$$\pi(u_I) = \sum_{x,y} E_x \pi(u_I) E_y,$$

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where the sum is taken over all pairs of atoms and is convergent in the strong operator topology. Consequently, there exist points x_I and y_I in X such that $E_{x_I}\pi(u_I)E_{y_I} \neq 0$. For each k, let e_k^I and f_k^I be the unique diagonal matrix units in A_k such that $x_I \in \hat{e}_k^I$ and $y_I \in \hat{f}_k^I$. Since $u_I \in I_k$, $e_k^I u_I f_k^I \in I_k$, for all k. On the other hand, $\pi(e_k^I u_I f_k^I) \neq 0$, so $e_k^I u_I f_k^I \notin J_k$, for all k.

Let x_J and y_J in X and e_k^J and f_k^J in A_k be analogously defined for the ideal J. This time, $e_k^J u_J f_k^J \in J_k$ and $e_k^J u_J f_k^J \notin I_k$.

We shall show (after possibly reversing the roles of I and J) that for infinitely many k, $e_k^I A_k f_k^J \subseteq \ker\pi$ and $f_k^J A_k e_k^I \subseteq \ker\pi$. This leads to a contradiction with the hypothesis that π is a nest representation and so shows that $\ker\pi$ is necessarily meet irreducible. Indeed, from

$$\pi(e_k^I)\pi(A_k)\pi(f_k^J) = 0, \quad \text{all } k,$$
$$\pi(f_k^J)\pi(A_k)\pi(e_k^I) = 0, \quad \text{all } k,$$

it follows that

$$E_{x_I}\pi(A)E_{y_J} = 0,$$
$$E_{y_J}\pi(A)E_{x_I} = 0,$$

and hence that $E_{x_I} \perp \mathcal{M}_{y_J}$ and $E_{y_J} \perp \mathcal{M}_{x_I}$, where $\mathcal{M}_{y_J}$ and $\mathcal{M}_{x_I}$ are the smallest π-invariant subspaces containing E_{y_J} and E_{x_I} respectively. But then $\mathcal{M}_{y_J}$ and $\mathcal{M}_{x_I}$ are not related by inclusion, a contradiction.

Each finite dimensional algebra A_k is a direct sum of T_n's and the matrix units e_k^I and f_k^I are in the same summand, as are e_k^J and f_k^J. If these two summands differ, then $e_k^I A_k f_k^J = 0$ and $f_k^J A_k e_k^I = 0$. Should this occur for infinitely many k, then we are done. So we need consider only the case in which, for all k, all of e_k^I, e_k^J, f_k^I and f_k^J are in the same summand in A_k.

If e and f are diagonal matrix units (minimal diagonal projections) in A_k, let $\mathrm{m}(e, f)$ be the matrix unit in $C^*(A_k)$ with initial projection f and final projection e (if there is such a matrix unit). If $\mathrm{m}(e, f) \in A_k$, then $e \preceq f$ in the diagonal order on minimal diagonal projections. We shall need the following property of ideals in A_k: if $e_1 \preceq e_2 \preceq f_2 \preceq f_1$ and if $\mathrm{m}(e_2, f_2)$ is in an ideal, then $\mathrm{m}(e_1, f_1)$ is also in the ideal.

Since e_k^I and e_k^J are in the same T_n-summand of A_k, they are related in the diagonal order. By interchanging I and J and passing to a subsequence, if necessary, we may assume that $e_k^I \preceq e_k^J$, for all k. The facts concerning the membership of $e_k^I u_I f_k^I$ and $e_k^J u_J f_k^J$ in I_k and J_k may be rephrased as

$$\mathrm{m}(e_k^I, f_k^I) \in I_k \text{ and } \mathrm{m}(e_k^I, f_k^I) \notin J_k,$$
$$\mathrm{m}(e_k^J, f_k^J) \notin I_k \text{ and } \mathrm{m}(e_k^J, f_k^J) \in J_k.$$

As a consequence $f_k^I \prec f_k^J$, for all k. (If $f_k^J \preceq f_k^I$ for some k, then $e_k^I \preceq e_k^J \preceq f_k^J \preceq f_k^I$. Since $\mathrm{m}(e_k^J, f_k^J) \in J_k$, we have $\mathrm{m}(e_k^I, f_k^I) \in J_k$, a contradiction.) But now,

$$\mathrm{m}(e_k^I, f_k^I) \in I_k \text{ and } f_k^I \prec f_k^J \Longrightarrow \mathrm{m}(e_k^I, f_k^J) \in I_k$$
$$\mathrm{m}(e_k^J, f_k^J) \in J_k \text{ and } e_k^I \preceq e_k^J \Longrightarrow \mathrm{m}(e_k^I, f_k^J) \in J_k.$$

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Thus $m(e^I_k, f^J_k) \in I_k \cap J_k \subseteq \ker\pi$; hence $e^I_k A_k f^J_k \subseteq \ker\pi$. Also, since $e^I_k \preceq f^I_k \prec f^J_k$, $f^J_k A_k e^I_k = \{0\} \subseteq \ker\pi$. As pointed out earlier, this implies that π is not a nest representation; so, when π is a nest representation with an atomic lattice, $\ker\pi$ is meet irreducible. ∎

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