The density theorem
for loop near-rings

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An algebraic formulation of the density theorem for loop near-
rings is presented, which generalizes the same result for near-
rings.

1.   Introduction

The algebraic and topological formulations of the density theorem are
quite well known in ring theory. These results have been extended by
several authors for primitive near-rings; in particular Betsch [1] proved
a density theorem for 0-primitive near-rings. The aim of this paper is to
generalize Betsch's result [1, 2.14] to the case of algebraic systems known
as loop near-rings.

2.   Preliminaries

For the definitions of loops and normal subloops see [2]. We recall
[3, Definition 1.1] that a loop near-ring $N = (N, +, \cdot, 0)$ is a system
where

(1) $(N, +)$ is an additive loop with identity 0 , which we
        denote by N^+ ;

(2) $(N, \cdot)$ is a semigroup;

(3) $a \cdot (b+c) = a \cdot b + a \cdot c$ for all a, b, c in N ; and

(4) $0 \cdot a = 0$, for all $a \in N$.

For any a belonging to an additive loop we shall denote the unique right and left additive inverses of a by a_r and a_l, respectively. Now it is easy to verify that $a0 = 0$, $(ab)_r = ab_r$, $(ab)_l = ab_l$, for all a, b in N.

Throughout this paper N always stands for a loop near-ring. The identity element of $(N, +)$ will be denoted by 0.

An N-loop G is an additive loop $(G, +, \overline{0})$ together with a mapping $(g, n) \to gn$ of $G \times N \to G$ such that for all $n, m \in N$, $g \in G$,

   (1) $g(n+m) = gn + gm$,

   (2) $g(nm) = (gn)m$.

Then we have $g0 = \overline{0}$, $(gn)_r = gn_r$, $(gn)_l = gn_l$, for all $g \in G$ and $n \in N$. We abbreviate $(G, +, 0)$ by G. A subloop Δ of an N-loop G is called an N-subloop of G, provided that $\Delta N \subseteq \Delta$. The N-subloops in N^+ are called N-loop modules of N. N-loop homomorphisms and N-loop kernels are defined in the usual way. The quotient N-loop of an N-loop G by an N-loop kernel K will be denoted by G/K, where $(g+K)n = gn + K$, for all $g+K \in G/K$ and $n \in N$.

A nonempty subset K of an N-loop G is an N-loop kernel of G if and only if

   (1) $(K, +)$ is a normal subloop of G,

   (2) for all $g \in G$, $k \in K$, $n \in N$, $(g+k)n + gn_r \in K$ [3,
         Theorem 1.11].

N-loop kernels of N^+ are called right ideals. A right ideal L of N is an ideal if and only if $NL \subseteq L$. Intersection of any family of N-loop kernels of an N-loop G is an N-loop kernel of G [3, Theorem 1.14]. The set of all N-loop kernels of an N-loop forms a commutative semigroup under addition. If G is an N-loop, then for any $g \in G$, $gN = \{gn \mid n \in N\}$ is an N-subloop of G, and $A(g) = \{n \in N \mid gn = \overline{0}\}$ is a right ideal of N; $A(G) = \bigcap_{g \in G} A(g)$ is an ideal of N. N-loops of type V, V-primitive loop near-rings, V-primitive ideals for $V = 0, 1, 2$ are defined in the usual way as for near-rings. For further results and information see [3].

3. Algebraic formulation of the density theorem

Our proof of the density theorem (3.16) will be based on Lemmas 3.12 and 3.15. We shall obtain these lemmas as consequences of the fundamental result (Lemma 3.9), which generalizes the same result for near-rings due to Wielandt [1, Lemma 2.9]. We start with the following

THEOREM 3.1. *Let G be an N-loop with an N-generator g , and L a right ideal of N ; then $gL = \{ga \mid a \in L\}$ is an N-loop kernel.*

We first divide this theorem into several lemmas and prove one after the other.

LEMMA 3.2. *gL is a subloop of G .*

Proof. Let $ga, gb \in gL$. Then $ga + gb = g(a+b) \in gL$ and $g0 = \overline{0}$ gL . Since $ga, gb \in G$, there exist unique elements x_1 and x_2 in G such that $gb = ga + x_1 = x_2 + ga$. Since $a, b \in L$ and since L is a subloop of N^+ , there exist unique elements a_1, a_2 in L such that $b = a + a_1 = a_2 + a$. Hence $gb = ga + ga_1 = ga_2 + ga$. Therefore, by the uniqueness of x_1, x_2 , we have $x_1 = ga_1$ and $x_2 = ga_2$. Hence $x_1, x_2 \in gL$. Therefore gL is a subloop of G .

LEMMA 3.3. *gL is a normal subloop of G .*

Proof. Let $x \in G$ and $ga \in gL$. Then $x = gn$, where $n \in N$. Consider $x + ga = gn + ga = g(n+a)$. Since L is a normal subloop of N^+ , $n + L = L + n$ for all $n \in N$. But $n+a \in n+L$. Hence $n + a = b + n$, where $b \in L$. Then

$$x + ga = g(n+a) = g(b+n) = gb + gn = gb + x .$$

Therefore $x+ga \in gL+x$. Hence $x + gL \subseteq gL + x$. Similarly it can be proved that $gL + x \subseteq x + gL$. Therefore $x + gL = gL + x$ for all $x \in G$. Let $x, y \in G$ and $ga \in gL$. Then $x = gn_1$, $y = gn_2$, where $n_1, n_2 \in N$. Now $(x+y) + ga = (gn_1+gn_2) + ga = g((n_1+n_2)+a)$. Since L is a normal subloop of N^+ , $(n_1+n_2) + L = n_1 + (n_2+L)$. Now $(n_1+n_2) + a \in (n_1+n_2) + L$. Hence $(n_1+n_2) + a = n_1 + (n_2+b)$, where $b \in L$. Therefore

$$(x+y) + ga = g\big((n_1+n_2)+a\big) = g\big(n_1+(n_2+b)\big) = gn_1 + (gn_2+b) = x + (y+gb) \ .$$

Hence $(x+y) + ga \in x + (y+gL)$. Therefore $(x+y) + gL \subseteq x + (y+gL)$. The other inclusion is also verified. Therefore $(x+y) + gL = x + (y+gL)$ for all $x, y \in G$. By a similar argument it can be shown that for all $x, y \in G$, $gL + (y+x) = (gL+y) + x$. Therefore gL is a normal subloop of G .

   LEMMA 3.4. gL *is an* N-*loop kernel of* G .

   Proof. Let $ga \in gL$, $x \in G$, and $n \in N$. Let $x = gm$ where $m \in N$. Now $(x+ga)n + xn_r = (gm+ga)n + (gm)n_r = g\big((m+a)n+mn_r\big) \in gL$, since L is a right ideal of N and $a \in L$.

   By Lemmas 3.1 and 3.2, gL is a normal subloop of G . Therefore gL is an N-loop kernel of G .

   COROLLARY 3.5. *If* G *is an* N-*loop of type* 1 *and* L *a right ideal of* N *, then for each* $g \in G$ *, either* $gL = \{\overline{0}\}$ *or* $gL = G$.

   Proof. If $gL \neq \{\overline{0}\}$, then $gN \neq \{\overline{0}\}$ and hence $gN = G$. Therefore g is an N-generator of G . Since G is irreducible and since $\{\overline{0}\} \neq gL$ is an N-loop kernel of G , we have $gL = G$.

   COROLLARY 3.6. *If* G *is an* N-*loop of type* 1 *with* $A(G) = \{0\}$ *and* $L \neq \{0\}$ *a right ideal of* N *, then there exists an element* $g \in G$ *such that* $gL = G$.

   Proof. Since $A(G) = \{0\}$, there exists an element $g \in G$ such that $gL \neq \{\overline{0}\}$. Hence $gL = G$ (Corollary 3.5).

   LEMMA 3.7. *Let* G *be an* N-*loop with* $A(G) = \{0\}$ *. If* N^+ *is not associative then* G *is not associative.*

   Proof. Since N^+ is not associative, there exist $x, y, z \in N$ such that $(x+y) + z \neq x + (y+z)$. Since $A(G) = \{0\}$, there exists a $g \in G$ such that $g\big((x+y)+z\big) \neq g\big(x+(y+z)\big)$; for otherwise

$$g\big((x+y)+z\big) = g\big(x+(y+z)\big)$$

for all $g \in G$ implies that $g\big((x+y)+z\big) + g\big(x+(y+z)\big)_r = \overline{0}$ for all $g \in G$, and hence $\big((x+y)+z\big) + \big(x+(y+z)\big)_r \in A(G) = \{0\}$. Therefore $\big((x+y)+z\big) + \big(x+(y+z)\big)_r = 0 = \big(x+(y+z)\big) + \big(x+(y+z)\big)_r$; hence by

cancellation in N^+ , $(x+y) + z = x + (y+z)$, a contradiction. Therefore,
for some $g \in G$, $g(x+(y+z)) \neq g((x+y)+z)$; that is,
$gx + (gy+gz) \neq (gx+gy) + gz$. Therefore G is not associative.

   LEMMA 3.8 [4, Proposition 2.1]. *Let A , B, K be normal subloops of
an additive loop G ; then the additive loop*

$$\overline{G} := (A+K) \cap (B+K)/(A \cap B) + K$$

is an abelian group.

   LEMMA 3.9. *Let N be a loop near-ring and let A , B, K be N-loop
kernels of some N-loop G . Then the N-loop*

$$\overline{G} := (A+B) \cap (B+K)/(A \cap B) + K$$

*is an abelian group, and for all $\overline{g} \in \overline{G}$, $n \in N$ the mapping
$\overline{G} \to \overline{G} : \overline{g} \to \overline{gn}$ is an endomorphism of $(\overline{G}, +)$.*

   Proof. Let $E = (A \cap B) + K$ and $H = (A+K) \cap (B+K)$; then
$\overline{G} := H/E$ is an abelian group (Lemma 3.8). Now it is enough to show that
for all $x, y \in H$ and $n \in N$, $(x+y)n + E = (xn+yn) + E$. Let
$x, y \in H$; then $x \in A+K$ and $y \in B+K$; therefore $x = a + p$,
$y = b + q$ for some $a \in A$, $b \in B$, and $p, q \in K$. Since E is a
normal subloop, $x + E = (a+p) + E = a + (p+E) = a + E$, since $p \in K$
implies $p \in E$; hence $p + E = E$. Similarly we get $y + E = b + E$.
Since $a \in A$ and A is an N-loop kernel of G , we have
$(a+b)n + bn_r \in A$, and also $an \in A$. Therefore

$$((a+b)n+bn_r) + A = A = \overline{0} + A = (bn+bn_r) + A ;$$

since G/A is a loop and the cancellation laws hold good in a loop, we
have $(a+b)n + A = bn + A = A + bn = (A+an) + bn = A + (an+bn)$. Since
$(a+b)n \in (a+b)n + A$, we have $(a+b)n \in A + (an+bn)$. Since A is a
normal subloop of G , we have

$(a+b)n + (an+bn)_r \in \{A+(an+bn)\} + (an+bn)_r$
$$= A + \{(an+bn)+(an+bn)_r\} = A + \overline{0} = A .$$

By a similar argument we get $(a+b)n + (an+bn)_r \in B$. Therefore
$(a+b)n + (an+bn)_r \in A \cap B \subseteq E$. Hence

$$\{(a+b)n+(an+bn)_r\} + E = E = \overline{0} + E = \{(an+bn)+(an+bn)_r\} + E .$$

Since G/E is a loop, we get $(a+b)n + E = (an+bn) + E$. Therefore $(x+y)n + E = (xn+yn) + E$, since $x + E = a + E$ and $y + E = b + E$. Hence the result.

COROLLARY 3.10. *Under the assumptions and with the notation of Lemma 3.9, we get* $\overline{N} = N/A(\overline{G})$ *is a ring.*

Proof. $\overline{G}$ is an abelian group (Lemma 3.9). Further, $\overline{G}$ is a faithful $\overline{N}$-loop. Since addition in $\overline{G}$ is associative, addition in $\overline{N}$ is also associative (Lemma 3.7); consequently $(\overline{N}, +)$ is a group, and hence $\overline{N}$ is a near-ring. Hence $\overline{G}$ is a faithful $\overline{N}$-group and $\overline{G}$ is abelian. Further, for all $\overline{n} \in \overline{N}$, $\overline{g} \rightarrow \overline{gn}$ is an endomorphism of $(\overline{G}, +)$ (Lemma 3.9). Therefore $\overline{N}$ is a ring [1, Lemma 1.5].

COROLLARY 3.11. *Let* N *be a loop near-ring with identity and let no nonzero epimorphic image of* N *be a ring. Then the lattice of right ideals of* N *is distributive.*

The proof is similar as in the case of near-rings [1, Corollary 2.1]. As a further consequence of Lemma 3.9 we obtain:

LEMMA 3.12. *Let* N *be a loop near-ring and* G *an* N-*loop which is monogenic with generator* g $(G = gN)$ *and faithful. Let* B *and* C *be right ideals of* N *with the property that* $B + A(g) = C + A(g) = N$ *and* $B \cap C \subseteq A(g)$. *Then* N *is a ring.*

Proof. We form the N-loop

$$\overline{G} := [B+A(g)] \cap [C+A(g)]/(B \cap C) + A(g) = N/A(g) ,$$

which is N-isomorphic to G. By Lemma 3.9, $\overline{G}$ is an abelian group, and hence G is an abelian group; therefore N is a near-ring and N induces only endomorphisms of $(G, +)$ on G. Since G is faithful, N must be a ring [1, Lemma 1.5].

Let N be a 0-primitive loop near-ring. G will denote a faithful N-loop of type 0. We denote the set $\{g \in G \mid gN = G\}$ by C. That is, C is the set of all N-generators of G, and $C \neq \emptyset$.

LEMMA 3.13. *Let* G *be an* N-*loop of type* 0 *and* $g \in C$. *Then* $A(g)$ *is a maximal right ideal of* N , *and the quotient loop* $N/A(g)$ *is* N-*loop isomorphic to* G .

The proof is easy and will be omitted.

COROLLARY 3.14. *If* g, g_1 $\in$ C *and if* $A(g) \supseteq A(g_1)$, *then*
$A(G) = A(g_1)$.

The proof is easy and will be omitted.

LEMMA 3.15. *Let* N *be a loop near-ring which is not a ring and is*
0-*primitive on* G . *Let* g_1, g_2, ..., g_s $\in$ C . *Assume* $A(g_i) \neq A(g_j)$ *for*
$i \neq j$. *Then for* $t = 1, 2, ..., s-1$ *the following statement is true:*

$$S(t) = \bigcap_{i=1}^{t} A(g_i) \nsubseteq A(g_k) \quad \text{for all} \quad t < k \leq s \ .$$

The proof of this lemma is similar as in the case of near-rings [*1*,
Lemma 2.13].

THEOREM 3.16. *Let* N *be a* 0-*primitive loop near-ring which is not*
a ring, and G *an* N-*loop of type* 0 *with* $A(G) = \{0\}$. *Let*
g_1, g_2, ..., g_k $\in$ C *such that* $A(g_i) \neq A(g_j)$ *for* $i \neq j$. *For arbitrary*
elements g_1', g_2', ..., g_k' *of* G *there exists an* $n \in N$ *such that*
$g_i' = g_i n$ *for* $i = 1, 2, ..., k$.

Proof. Since g_1 $\in$ C, there exists an element n_t $\in$ N such that
$g_1 n_1 = g_1'$. So the result is true for $k = 1$. Assume the result for
$i = 1, 2, ..., t$, where $1 \leq t \leq k-1$. Then there exists an n_t $\in$ N such
that $g_i n_t = g_i'$ for $i = 1, 2, ..., t$. We have

$$K := \bigcap_{i=1}^{t} A(g_i) \nsubseteq A(g_{t+1})$$

(Lemma 3.15). Hence $g_{t+1} K \neq \{0\}$, and since $g_{t+1} K$ is an N-loop kernel
of G , $g_{t+1} K = G$. Therefore there exists an element $k \in K$, such that
$g_{t+1}' = g_{t+1} k$. Also there exists a $k' \in K$, such that $g_{t+1} n_t = g_{t+1} k'$.
Let $n_{t+1} = (n_t + k_r') + k$. Then

$$g_i n_{t+1} = (g_i n_t + g_i k_r') + g_i k = g_i n_t = g_i'$$

for $i = 1, 2, ..., t$;

$$g_{t+1}n_{t+1} = \left(g_{t+1}n_t + g_{t+1}k'_r\right) + g_{t+1}k$$
$$= \left(g_{t+1}k' + g_{t+1}k'_r\right) + g_{t+1}k = g_{t+1}k = g'_{t+1} \; .$$

Hence $g_i n_{t+1} = g'_i$ for $i = 1, 2, \ldots, t+1$.

By induction the result follows.

References

[1]   Gerhard Betsch, "Primitive near-rings", *Math. Z.* **130** (1973), 351-361.

[2]   Richard Hubert Bruck, *A survey of binary systems*, 2nd edition
          (Ergebnisse der Mathematik und ihrer Grenzgebiete, **20**.   Springer-
          Verlag, Berlin, Heidelberg, New York, 1966).

[3]   D. Ramakotaiah and C. Santhakumari, "On loop near-rings", *Bull.*
          Austral. Math. Soc. **19** (1978), 417-435.

[4]   C. Santhakumari, "On quotient loops of normal subloops", *Canad. Math.*
          Bull. (to appear).

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