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EXISTENCE OF SOLUTIONS OF A CLASS OF STOCHASTIC VOLTERRA INTEGRAL EQUATIONS WITH APPLICATIONS TO CHEMOTHERAPY

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Abstract

In this paper we establish the existence of solutions of a more general class of stochastic integral equation of Volterra type. The main tools used here are the measure of noncompactness and the fixed point theorem of Darbo. The results generalize the results of Tsokos and Padgett [9] and Szynal and Wedrychowicz [7]. An application to a stochastic model arising in chemotherapy is discussed.

1. Introduction

The mathematical description of various processes in physical, biological and engineering sciences give rise to random or stochastic integral equations. Theoretical treatments of such problems can be found in [5, 8, 9].

The most important problem examined up to now is that concerning the existence of solutions of considered equations. It is solved mostly by the Banach fixed point principle, the Schauder fixed point theorem, and successive approximations [5, 6, 8, 9]. In this paper we use the notion of measure of noncompactness in a Banach space and the fixed point theorem of Darbo type ([1, 2]). This approach allows us to weaken the conditions of [7]. An example of the equation considered in this paper is in chemotherapy of a two-organ biological system. First, we discuss a stochastic model for chemotherapy in a single organ biological system. A closed system with a simplified heart, one organ or capillary bed, was described in Tsokos and Padgett [9]. The heart is considered as a mixing chamber of constant volume, and the recirculation of blood in the system is with constant rate of flow. Also, it is assumed that the injection of drug is given at the entrance of the heart producing a known concentration

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in the blood plasma. As the blood flows through the organ the particles of drug are assumed to enter the extracellular space only by the process of diffusion through the capillary walls. Bellman, Jacquez and Kalaba [4] developed a deterministic model, but the concentration of drug in the plasma in given areas of the system is more realistically considered as a random function of time due to the random nature of the diffusion process.

The injection is given at the entrance of the heart resulting in a concentration $u_I(t)$, $0 \le t \le t_I$ of drug in plasma entering the heart, where t_I is the duration of the injection. Let $\tau > 0$ be the time required for the blood to flow from the exit of the organ to the entrance of the heart. The concentration of drug in plasma entering the heart at time $t > \tau$, $u_R(t; w)$ is a random variable, and is given by

$$u_R(t; w) = \begin{cases} 0, & t < 0, \\ u_I(t), & 0 \le t \le \tau, \\ u_I(t) + u(e, t - \tau; w), & t \ge \tau, \end{cases}$$

where $u_I(t) = 0$ if $t < t_I$, e is the length of the capillary in the organ, $u(e, t; w)$ is the concentration of drug in plasma leaving the organ at time t, and c is the constant flow rate of plasma in the capillary bed.

The concentration of drug in plasma leaving the heart $u_L(t; w)$ satisfies the integral equation, see [3, 4, 9]:

$$u_L(t; w) = (c/V^*) \int_0^t \{u_R(s; w) - u_L(s; w)\}\, ds, \quad t \ge 0. \tag{1}$$

The concentration of drug entering the organ at time t is given by

$$u(0, t; w) = \begin{cases} 0, & 0 \le t < \tau, \\ u_L(t - \tau; w), & t \ge \tau. \end{cases}$$

The concentration of drug leaving the heart $u_L(t; w)$ satisfies the semi-stochastic integral equation

$$u_L(t; w) = G(t) + \int_0^t k\,(s, u_L(s; w))\, ds, \tag{2}$$

where

$$G(t) = \int_0^{T(t)} (c/V^*) u_L(s)\, ds, \tag{3}$$

$$T(t) = \begin{cases} t, & 0 \le t < t_I, \\ t_I, & t \ge t_I, \end{cases}$$

V^* is the constant volume of the heart, and

$$k(s, u_L(s;w); w) = -(c/V^*)\,[u_L(s;w) - u(e, s-t; w)].$$

Tsokos and Padgett [9] proved that a semi-random solution of (2) exists, that is a deterministic solution in $0 \le t < \tau$ and a random solution in $\tau \le t \le M$, where $\tau < M < \infty$, $t_I < M$, whereas Szynal and Wedrychowicz [7] studied the problem under less restrictive conditions.

Now let us discuss the two-organ biological system. Let $u_j(s, t; w)$ denote the random concentration of drug in organ j at point s in the capillary at time t, for $j = 1, 2$. Let c_j be the constant volume flow rate of blood in organ j, $j = 1, 2$, and $c = c_1 + c_2$. Then c is the total constant-volume flow rate of blood in the system.

The concentration of drug entering the heart $u_R(t; w)$ after time τ is a random variable for each $t > \tau$, and is given by (refer Tsokos and Padgett [9]):

$$u_R(t;w) = \begin{cases} 0, & t < 0, \\ u_I(t), & 0 \le t < \tau, \\ u_I(t) + \dfrac{c_1 u_1(e, t-\tau; w) + c_2 u_2(e, t-\tau; w)}{c}, & \tau \le t \le M, \end{cases} \tag{4}$$

where $u_j(e, t; w)$ is the concentration of drug in plasma leaving organ j (at the e end) at time t, $j = 1, 2$, and $u_I(t) = 0$ for $t > t_I$.

Substituting (4) in (1), we have

$$\begin{aligned} u_L(t;w) &= \frac{c}{V^*}\int_0^t \left[u_I(s) + \frac{c_1 u_1(e, s-\tau; w) + c_2 u_2(e, s-\tau; w)}{c} - u_L(s;w)\right] ds \\ &= \frac{c}{V^*}\int_0^t u_I(s)\,ds - \frac{c}{V^*}\int_0^t \left\{u_L(s;w) - \sum_{j=1}^{2} c_j u_j(e, s-\tau; w)\right\} ds \\ &= G(t) + \sum_{j=1}^{2}\int_0^t k_j\,(s, u_L(s;w); w)\, ds, \end{aligned} \tag{5}$$

where $k_j\,(s, u_L(s;w); w) = -(c/V^*)\left\{u_L(s;w)/2 - c_j u_j(e, s-\tau; w)\right\}$. We shall prove that a semi-random solution of (5) exists under milder conditions than those of Tsokos and Padgett [9]. Note that (5) has a deterministic solution when $0 \le t \le \tau$.

The aim of this paper is to prove an existence theorem for a more general stochastic integral equation of the form

$$x(t;w) = h(t;w) + \sum_{j=1}^{m}\int_0^t K_j(u, x(u;w); w)\,du, \tag{6}$$

where $t > 0$ and

(i) $w \in \Omega$, the sample space of the complete probability measure space (Ω, A, P),
(ii) $x(t; w)$ is the unknown random function for $t \in \mathbb{R}_+$,
(iii) $h(t; w)$ is the known random variable, for $t \in \mathbb{R}_+$,
(iv) $K_j(u, x(u; w); w)$ are stochastic kernels defined for $0 \le u \le t < \infty$ and $w \in \Omega$, $j = 1, 2, \ldots, m$.

2. Mathematical preliminaries

We denote by $L^2(\Omega, A, P)$ the space of A-measurable square integrable maps $x(t; w)$ with

$$\|x(t)\|_{L^2} = \left(\int_\Omega |x(t; w)|^2 \, dP(w)\right)^{1/2}.$$

We now give the following definitions.

DEFINITION 1. We shall call $x(t; w)$ a random solution of the stochastic integral equation (6) if for every fixed $t \in \mathbb{R}_+$, $x(t; w) \in L^2(\Omega, A, P)$ and satisfies (6).

Throughout this paper X will denote an infinite dimensional real Banach space with norm $\|\cdot\|$ and the zero element 0. $V(x, r)$ stands for the closed ball centered at x of radius r. Denote by M_X the family of all nonempty bounded subsets of X, and by N_X the family of all relatively compact and nonempty subsets of X.

The following axioms defining a measure of noncompactness are taken from Banas and Goebel [2].

DEFINITION 2. A nonempty family $B \subset N_X$ is said to be a kernel (of a measure of noncompactness), provided it satisfies the conditions

(a) $U \in B \Longrightarrow \bar{U} \in B$;
(b) $U \in B$, $V \subset U$, $V \neq \emptyset \Longrightarrow V \in B$;
(c) $U, V \in B \Longrightarrow \alpha U + (1 - \alpha) V \in B, \quad \alpha \in [0, 1]$;
(d) $U \in B \Longrightarrow \text{Conv } U \in B$;
(e) B^C (the sub-family of B consisting of all closed sets) is closed in N^C with respect to the topology generated by the Hausdorff metric.

DEFINITION 3. The function $\mu : M_X \to [0, +\infty)$ is said to be a measure of noncompactness with kernel B ($\ker \mu = B$) if it satisfies the conditions

(i) $\mu(U) = 0 \Longleftrightarrow U \in B$;
(ii) $\mu(U) = \mu(\bar{U})$;
(iii) $\mu(\text{Conv } U) = \mu(U)$;

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(iv) $U \subset V \Longrightarrow \mu(U) \le \mu(V)$;

(v) $\mu(\alpha U + (1-\alpha)V) \le \alpha\mu(U) + (1-\alpha)\mu(V), \quad \alpha \in [0, 1]$;

(vi) if $U_n \in M_X$, $\bar{U}_n = U_n$ and $U_{n+1} \subset U_n$, $n = 1, 2, \dots$, and if $\lim_{n\to\infty} \mu(U_n) = 0$, then $U = \bigcap_{n=1}^{\infty} U_n \neq \emptyset$.

If a measure of noncompactness μ in addition satisfies the following two conditions:

(vii) $\mu(U + V) \le \mu(U) + \mu(V)$;

(viii) $\mu(\alpha U) = |\alpha|\mu(U), \quad \alpha \in \mathbb{R}$;

it will be sub-linear.

Let $M \subset X$ be a nonempty set and let μ be a measure of noncompactness on X.

DEFINITION 4. We say that a continuous mapping $T : M \to X$ is a contraction with respect to μ (μ-concentration) if for any set $U \in M_X$ its image $TU \in M_X$, and there exists a constant $k \in [0, 1)$ such that

$$\mu(TU) \le k\mu(U).$$

We shall use the following modified version of the fixed-point theorem of Darbo type.

THEOREM 1. *Let C be a nonempty, closed, convex and bounded set of X and let $T : C \to C$ be an arbitrary μ-contraction. Then T has at least one fixed point in C and the set* Fix $T = \{x \in C : Tx = x\}$ *of all fixed points of T belongs to* ker μ.

Let $p(\cdot)$ be a positive continuous function defined on $[0, \infty)$ such that

$$\lim_{T\to\infty} \sup_{t \ge T} p(t) = p_0.$$

Let C_p denote the space of all continuous maps $x(t; .)$ from $\mathbb{R}_+$ into $L^2(\Omega, A, P)$ with the topology defined by the norm

$$\|x\|_p = \sup\{p(t)\|x(t)\|_{L^2} : t \ge 0\} < \infty.$$

The space C_p with norm $\|\cdot\|_p$ is a real Banach space, see [1].

Now for $x \in C_p$, $U \in M_{C_p}$, $T > 0$ and $\epsilon > 0$, we put

$$\beta^T(x, \epsilon) = \sup\{\|p(t)x(t) - p(s)x(s)\|_{L^2} : t, s \in [0, T], |t - s| \le \epsilon\};$$

$$\beta_0^T(U, \epsilon) = \sup\left\{\beta^T(x, \epsilon) : x \in U\right\};$$

$$\beta_0^T(U) = \lim_{\epsilon\to 0} \beta^T(U, \epsilon);$$

$$\beta_0(U) = \lim_{T\to\infty} \beta_0^T(U);$$

$$a(U) = \lim_{T\to\infty} \sup_{x\in U} \sup_{t\geq T} \|x(t)\|_{L^2} p(t);$$

$$b(U) = \lim_{T\to\infty} \sup_{x\in U} \sup_{s,t\geq T} \{\|p(t)x(t) - p(s)x(s)\|_{L^2}\};$$

$$\mu_0(U) = \beta_0(U) + a(U) + \sup\{p(t)m(U(t)) : t \geq 0\};$$

$$\mu_1(U) = \beta_0(U) + b(U) + \sup\{p(t)m(U(t))) : t \geq 0\};$$

where m is a sub-linear measure of noncompactness on $M_{L^2(\Omega,A,P)}$ and

$$U(t) = \left\{x(t) \in L^2(\Omega, A, P) : x \in U\right\}.$$

The functions μ_0 and μ_1 define sub-linear measures of noncompactness on M_C. It is also known that $\ker \mu_0$ is the set of all sets $U \in M_C$ such that the functions belonging to U are equicontinuous on any compact subset of $\mathbb{R}_+$ and

$$\lim_{t\to\infty} p(t)\|x(t)\|_{L^2} = 0$$

uniformly with respect to $x \in U$. Further properties of μ_0 and μ_1 can be found in [1] and [2].

3. Main results

Now we shall make the following assumptions concerning (6).

Let $t \in \mathbb{R}_+$ be fixed. We assume that the functions $x(t;w)$, $h(t;w)$, and $K_j(u,z;w)$, $j = 1, 2, \dots, m$ in (6) are real valued, x and h are product measurable on $\mathbb{R}_+ \times \Omega$, and $K_j(u,z;w)$ are product measurable on $\mathbb{R}_+ \times \mathbb{R} \times \Omega$ for each z.

Let $A(t;w) \in L_\infty(\Omega, A, P)$ and

$$\||A(t)|\| = \text{P-}\operatorname*{ess\,sup}_{w\in\Omega} |A(t;w)|.$$

THEOREM 2. *Assume that the functions K_j, $j = 1, 2, \dots, m$, and h in* (6) *satisfy the following conditions*:

(i) $K_j : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *are sub-linear. That is there exists nonnegative functions A_j and B_j belonging to $L_\infty(\Omega, A, P)$ such that $|K_j(t, x(t;w);w)| \leq A_j(t;w)|x(t;w)| + B_j(t;w)$, $j = 1, 2, \dots, m$;*

(ii) $M_j = \sup\left\{p(t)\int_0^t (\||A_j(u)|\|/p(u))\,du : t \in \mathbb{R}_+\right\}$, $j = 1, 2, \dots, m$ *and* $0 \leq M < 1$ *where* $M = \sum_{j=1}^m M_j$;

(iii) $N_j = \sup\left\{p(t)\int_0^t (\||B_j(u)|\|/p(u))\,du : t \in \mathbb{R}_+\right\}$, $j = 1, 2, \dots, m$ *and* $N = \sum_{j=1}^m N_j$;

(iv) $\lim_{t\to\infty} p(t)\|h(t)\|_{L^2} = 0$;

(v) $\lim_{t\to\infty} p(t) \int_0^t |||B_j(u)||| \, du = 0, \quad j = 1, 2, \ldots, m$;

(vi) $\lim_{t\to\infty} p(t)\|K_j(t, x(t)) - K_j(t, y(t))\|_{L^2} = 0$, $j = 1, 2, \ldots, m$ *uniformly with respect to* x *and* y *belonging to* $V(0, r)$, *where* $r = \big(\|h\|_p + N\big)/(1 - M)$;

(vii) *the mapping* $z(t; w) \to K_j(t, z(t; w))$ *from* $C_p \to C_p$ *is continuous in the topology generated by* $\|\cdot\|_p$, $j = 1, 2, \ldots, m$;

(viii) *there exist* L_j *satisfying* $0 \le L < 1$, *where* $\sum_{j=1}^m L_j = L$,

$$m\left(\int_0^t K_j(u, U(u))\, du\right) \le L_j m(U(t)),$$

$U(t) = \big\{x(s) \in L^2(\Omega, A, P),\ s \ge 0,\ x \in U \subset V(0, r) : \ p(t)\|x(t)\|_{L^2} \le \|U\|_p\big\}$.
Then there exists at least one solution $x \in C_p$ *of* (6) *such that*

$$\lim_{t\to\infty} p(t)\|x(t)\|_{L^2} = 0.$$

PROOF. Define F on C_p by

$$(Fx)(t; w) = h(t; w) + \sum_{j=1}^m \int_0^t K_j(u, x(u; w); w)\, du. \tag{7}$$

For $x \in C_p$, by assumptions (i)–(iii) of the theorem, we get

$$\begin{aligned} p(t)\|(Fx)(t)\|_{L^2} &\le p(t)\left[\|h(t)\|_{L^2} + \sum_{j=1}^m \int_0^t \|K_j(u, x(u))\|_{L^2}\, du\right] \\ &\le \|h\|_p + \|x\|_p p(t) \sum_{j=1}^m \int_0^t \big(|||A_j(u)|||/p(u)\big)\, du \\ &\quad + p(t) \sum_{j=1}^m \int_0^t |||B_j(u)|||\, du, \end{aligned}$$

that is

$$\|Fx\|_p \le \|h\|_p + \|x\|_p M + N. \tag{8}$$

Therefore F maps C_p into C_p. Moreover we note that $F : V(0, r) \to V(0, r)$ for $r = (\|h\|_p + N)/(1 - M)$. We now prove that the map F is continuous in the ball $V(0, r)$. Let $x, y \in V(0, r)$. By (vi) for any given $\epsilon_j > 0$, $j = 1, 2, \ldots, m$, we can choose $T > 0$ such that

$$p(t)\left\|K_j(t, x(t)) - K_j(t, y(t))\right\|_{L^2} < \epsilon_j, \quad \text{when } t > T. \tag{9}$$

Furthermore, we can assume without loss of generality that there exists $T > 0$ such that $|||A_j(t)||| > 1, j = 1, 2, \ldots, m$, where $t \geq T$ and

$$|||A_j|||^T = \min\left\{|||A_j(u)||| : 0 \leq u \leq T\right\} > 0, \qquad j = 1, 2, \ldots, m,$$
$$|||A_j|||^T = \min\left\{|||A_j||| : 1 \leq j \leq m\right\}.$$

Put $p_T = \max\{p(u) : 0 \leq u \leq T\}$. We have for $t \geq T$

$$\begin{aligned} &p(t)\,\|(Fx)(t) - (Fy)(t)\|_{L^2} \\ &\quad\leq p(t)\sum_{j=1}^{m}\int_0^t \left\|K_j(u, x(u)) - K_j(u, y(u))\right\|_{L^2} du \\ &\quad\leq p(t)\sum_{j=1}^{m}\frac{p_T}{|||A_j|||^T}\int_0^T \frac{|||A_j(u)|||}{p(u)}\left\|K_j(u, x(u)) - K_j(u, y(u))\right\|_{L^2} du \\ &\qquad + p(t)\sum_{j=1}^{m}\int_T^t \frac{|||A_j(u)|||}{p(u)}p(u)\left\|K_j(u, x(u)) - K_j(u, y(u))\right\|_{L^2} du. \end{aligned}$$

Then we have

$$\sup_{t\geq T} p(t)\|(Fx)(t) - (Fy)(t)\|_{L^2} \leq \sum_{j=1}^{m}\left[\frac{M_j p_T}{|||A_j|||_T \epsilon_j} + \epsilon_{j'}\right] \tag{10}$$

whenever $\|x - y\| < \delta$. Also for $\epsilon_{1j} > 0$ we have

$$\sup_{0\leq t\leq T} p(t)\|(Fx)(t) - (Fy)(t)\|_{L^2} < \sum_{j=1}^{m}\epsilon_{1j} \text{ whenever } \|x - y\|_p < \delta. \tag{11}$$

Hence, by (10) and (11), for given $\epsilon > 0$, $\|Fx - Fy\|_p < \epsilon$ whenever $\|x - y\|_p < \delta$, $x, y \in V(0, r)$. Therefore F is continuous on the ball $V(0, r)$.

Let $\epsilon > 0$, $T > 0$ be given and $t, s \in [0, T]$, $|t - s| < \epsilon$. By (7) for $0 \leq s \leq t$ and $x \in U \subset V(0, r)$, we have

$$\begin{aligned} &\|p(t)(Fx)(t) - p(s)(Fx)(s)\|_{L^2} \\ &\quad\leq |p(t) - p(s)|\|h(t)\|_{L^2} + p(s)\|h(t) - h(s)\|_{L^2} \\ &\qquad + |p(t) - p(s)|\sum_{j=1}^{m}\left\|\int_0^s K_j(u, x(u))\,du\right\|_{L^2} + p(t)\sum_{j=1}^{m}\left\|\int_s^t K_j(u, x(u))\,du\right\|_{L^2} \end{aligned}$$

By using (i)

$$\begin{aligned} &|p(t) - p(s)|\left\|\int_0^s K_j(u, x(u))\,du\right\|_{L^2} \\ &\quad\leq T|p(t) - p(s)|\left[\|x\|_p \max\left\{\left(|||A_j(u)|||/p(u)\right) : 0 \leq u \leq T\right\}\right. \\ &\qquad \left. + \max\left\{|||B_j(u)||| : 0 \leq u \leq T\right\}\right]. \end{aligned} \tag{13}$$

Similarly we get

$$p(s)\left\|\int_s^t K_j(u,x(u))\,du\right\|_{L^2} \le |t-s|p(s)\left(r\max\left\{\||A_j(u)|\|p(u):0\le u\le T\right\} + \max\left\{\||B_j(u)|\|/p(u):0\le u\le T\}\right)\right). \tag{14}$$

We need to recall the definition of the modulus of continuity which is defined for all real functions u as

$$v_T(u;\epsilon)=\sup\{|u(t)-u(s)|:t,s\in[0,T],|t-s|<\epsilon\},\quad \epsilon>0. \tag{15}$$

Now, by the properties of p, h and (15), we get

$$\lim_{\epsilon\to 0} v_T(h;\epsilon)=0,\quad j=1,2,\dots,m,\quad \lim_{\epsilon\to 0} v_T(p;\epsilon)=0. \tag{16}$$

Therefore by (12) to (14) and (16) we get for $U\subset V(0,r)$

$$\beta_0(FU)=0. \tag{17}$$

Fix now $U\subset V(0,r)$. We prove that

$$a(FU)\le Ma(U). \tag{18}$$

It is clear, by the definition of the integral, that for any given $\alpha_j>0$ there exists a positive integer $n_j=n_j(\alpha_j)$ such that, for $n\ge n_j$,

$$\left|\int_0^t \||A_j(u)|\|\,(\|x(u)\|_{L^2}/p(u))\,p(u)\,du - \sum_{k=0}^{n-1}(t/n)\||A_j(kt/n)|\|\,(\|x(kt/n)\|_{L^2}/p(kt/n))\,p(kt/n)\right|<\alpha_j.$$

Let $T<t$. Put $k_{j^*}=\max\{k:\ 0\le k\le n,\ kt/n<T\}$. Then we have

$$\begin{aligned}\sum_{j=1}^m\int_0^t &\||A_j(u)|\|(\|x(u)\|_{L^2}/p(u))p(u)\,du\\ &\le\sum_{j=1}^m\Bigg[\alpha_j+\sum_{k=0}^{k_{j^*}}(t/n)\||A_j(kt/n)|\|(\|x(kt/n)\|_{L^2}/p(kt/n))p(kt/n)\\ &\quad+\sum_{k=k_{j^*}+1}^{n-1}(t/n)\||A_j(kt/n)|\|(\|x(kt/n)\|_{L^2}/p(kt/n))p(kt/n)\Bigg]\\ &=\sum_{j=1}^m\alpha_j+I_1+I_2.\end{aligned}\tag{19}$$

Now for any given $\alpha'' > 0$,

$$I_1 \leq \sum_{j=1}^{m}\Big[k_j \cdot \max\{p(kt/n)\|x(kt/n)\|_{L^2} : kt/n < T\} \times \max\{\||A_j(u)|\|/p(\tau) : 0 \leq \tau \leq T\}n^{-1}\Big] < \alpha'', \quad \text{for sufficiently large } n. \tag{20}$$

Similarly for any given $\alpha''' > 0$,

$$I_2 \leq \sup\{p(t)\|x(t)\|_{L^2} : t \geq T\}\left(\sum_{j=1}^{m}\int_0^t \||A_j(u)|\|/p(u))\,du + \alpha'''\right) \tag{21}$$

for sufficiently large n. Therefore, by (19)–(21), we get

$$\begin{aligned} p(t)\|(Fx)(t)\|_{L^2} &\leq p(t)\|h(t)\|_{L^2} + M\sup\{p(t)\|x(t)\|_{L^2} : t \geq T\} + p(t)(\alpha' + \alpha'' + r\alpha''') \\ &\quad + p(t)\sum_{j=1}^{m}\int_0^t \||B_j(u)|\|\,du. \end{aligned}$$

Thus, by assumptions (iv) and (v), we obtain

$$\begin{aligned} &\lim_{T\to\infty}\sup_{x\in U}\{\sup\{p(t)\|(Fx)(t)\|_{L^2} : t \geq T\}\} \\ &\quad \leq (\alpha' + \alpha'' + r\alpha''')p_0 + M\lim_{T\to\infty}\sup_{x\in U}\{\sup\{p(t)\|x(t)\|_{L^2} : t \geq T\}\}. \end{aligned}$$

Let $\alpha' \to 0$, $\alpha'' \to 0$ and $\alpha''' \to 0$. Then we get (18). Finally by (17) and (18) we obtain $\mu_0(FU) \leq D\mu_0(U)$ where $D = \max\{M, L\}$. This proves that F is a μ_0 contraction. Now by Theorem 1 the proof is complete.

4. Example

We shall apply the above theorem to (5). Suppose that the function in (5) satisfy the following conditions:

(i) $k_j : \mathbb{R}_+ \times \mathbb{R} \times \Omega \to \mathbb{R}$ are sub-linear,

$$|k_j(s, uL(s; w); w)| \leq |c/V^*|(|u_L(s; w)| + |u_j(e, s; w)|),$$

where $u_j(e, s; w) \in L_\infty(\Omega, A, P)$, $j = 1, 2$;

(ii) $M = \sup\left\{p(t)|c/V^*|\int_0^t(1/p(s))\,ds : t \in \mathbb{R}_+\right\}$, $\quad 0 \leq M \leq 1$;

(iii) $N_j = \sup\left\{p(t)|c/V^*|\int_0^t \||u_j(e, s)|\|\,ds : t \in \mathbb{R}_+\right\} < \infty$, $\quad N = N_1 + N_2$;

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(iv) $\lim_{t\to\infty} p(t)|G(t)| = 0$;

(v) $\lim_{t\to\infty} |c/V^*| \int_0^t \||u_j(e,s)|\| \, ds = 0, \quad j = 1, 2$

(vi) $\lim_{t\to\infty} p(t)\|k_j(t, u_L(t)) - k_j(t, v_L(t))\|_{L^2} = 0$ for $j = 1, 2$ uniformly with respect to u_L and $v_L \in V(0, r)$ where $r = (G + N)/(1 - M)$, $G = \sup\{p(t)|G(t)| : t \in \mathbb{R}_+\}$;

(vii) the mapping $u_L(s; w) \to k_j(s, u_L(s; w))$ from $C_p(\mathbb{R}_+, L^2(\Omega, A, P); p)$ into $C_p(\mathbb{R}_+, L^2(\Omega, A, P); p)$ is continuous in the topology generated by the norm $\|\cdot\|_p$;

(viii) there exists L_j, $j = 1, 2$, $0 \le L_1 + L_2 < 1$ such that

$$m\left(\int_0^t k_j(s, U(s); w)\, ds\right) \le L_j m(U(t)),$$

$$U(t) = \left\{u_L(s) \in L^2(\Omega, A, P), s \ge 0, u_L \in U \subset V(0, r) : p(t)\|u_L(t)\|_{L^2} \le \|U\|_p\right\},$$

where $r = (G + N)/(1 - M)$.

Then by Theorem 2 there exists at least one solution $u_L \in C_p$ of (5) such that $\lim_{t\to\infty} p(t)\|u_L(t)\|_{L^2} = 0$.

REMARK. Equation (5) corresponding to a two-organ biological system can be extended to an m-organ biological system which is still more realistic. In an m-organ biological system, the concentration of drug in plasma leaving the heart $u_L(t; w)$, satisfies the integral equation

$$u_L(t; w) = G(t) + \sum_{j=1}^{m} \int_0^t k_j\ (s, u_L(s; w); w)\ ds.$$

By the main theorem this equation has a solution.

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