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ON THE EINSTEIN–KÄHLER METRIC AND THE HOLONOMY OF A LINE BUNDLE

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Abstract In this paper we give a relation between the Futaki invariant for a compact complex manifold M and the holonomy of a determinant line bundle over a loop in the base space of any principal G-bundle, where G is the identity component of the maximal compact subgroup of the complex Lie group consisting of all biholomorphic automorphisms of M. Using the property of the Futaki invariant, we show that the holonomy is an obstruction to the existence of the Einstein–Kähler metrics on M. Our main result is Theorem 2.1.

Keywords: Einstein–Kähler metric; holonomy; Futaki invariant; eta invariant; index theorem

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1. Introduction

Let M be a compact connected complex m-dimensional manifold, $H(M)$ the finite-dimensional complex Lie group consisting of all biholomorphic automorphisms of M, and $h(M)$ its Lie algebra consisting of all holomorphic vector fields on M. A Kähler metric with Kähler form ω is called an Einstein–Kähler metric if the Ricci form of ω is a constant multiple of ω. In [**4**] Futaki defined a Lie algebra homomorphism $f : h(M) \to \boldsymbol{C}$, which is called the Futaki invariant, and proved that $f(X) = 0$ for any $X \in h(M)$ if M admits an Einstein–Kähler metric. Let ∇ be a type $(1,0)$ connection of the holomorphic tangent bundle TM with its connection form θ and $L(X)$ the $gl(m;\boldsymbol{C})$-valued 0-form defined by $L(X) = L_X - \nabla_X$ for $X \in h(M)$. Then, by multiplying a constant factor to the Futaki invariant in [**4**], the Futaki invariant is expressed as follows:

$$f(X) = \int_M c_1^{m+1}\left(\frac{\mathrm{i}}{2\pi}(L(X) + \Theta)\right), \tag{1.1}$$

where c_1 is the first Chern polynomial and Θ is the curvature form of θ (see [**5**, Proposition 2.3]).

Now let G be the identity component of the maximal compact subgroup of $H(M)$, ϑ its Lie algebra, and $\pi : P \to B$ any principal G-bundle over a smooth manifold B with a connection. Let h be any G-invariant Hermitian metric of TM. Then we can assume that the type $(1,0)$ connection ∇ is a G-invariant unitary connection. Let L be the virtual holomorphic G-bundle over M defined by

$$L = \otimes^{m+1}(K_M^{-1} - \tau), \tag{1.2}$$

where K_M^{-1} is the anticanonical bundle of M and τ is the trivial complex line bundle over M with the trivial G-action. Then the metric h and the standard metric of τ define a G-invariant metric h^L of L, and the connection ∇ and the trivial connection of τ define a G-invariant unitary connection ∇^L of L. Moreover, since the complex manifold M has the natural spin$^{\mathrm{c}}$-structure, the half spinor bundles $S^{\pm}$ on M are defined and the L-valued spin$^{\mathrm{c}}$-Dirac (Dolbeault) operator,

$$D^L : \Gamma(S^+ \otimes L) \to \Gamma(S^- \otimes L),$$

on M is defined by using the metrics h, h^L and the connections ∇, ∇^L. Here we can define a smooth fibration of manifolds $F \to B$ with fibre M by $F = P \times_G M$. Since G preserves all structures defining D^L, we can define a locally constant family of spin$^{\mathrm{c}}$-Dirac operators $D^F := P \times_G D^L$ parametrized by B. Let ζ be the determinant line bundle of the family D^F. Then it is clear that $\zeta = P \times_G K$, where K is a one-dimensional complex G-module defined by $K = \wedge^k((\ker D^L)^*) \otimes \wedge^l \ker((D^L)^*)$, where k and l are the dimensions of $\ker D^L$ and $\ker((D^L)^*)$, respectively. Hence, the connection in P defines the connection of ζ and the holonomy $\mathrm{hol}_\zeta(\gamma)$ of ζ around any loop γ in B is defined.

2. Main result

The next theorem is our main result.

Theorem 2.1. *Let γ be any loop in B and b any point on γ. Assume that a horizontal lift of γ in P connects a point $p \in \pi^{-1}(b)$ with the point $p \exp X \in \pi^{-1}(b)$ for $X \in \vartheta$. Then the following equality holds:*

$$\mathrm{hol}_\zeta(\gamma) = \mathrm{e}^{-2\pi \mathrm{i} f(X)}.$$

Proof. The strategy for the proof is as follows. First we will give a relation between the holonomy $\mathrm{hol}_\zeta(\gamma)$ and the eta invariant of $M \times S^1$ with respect to a metric corresponding to the holonomy by using Witten's holonomy formula. Then we will show that the Futaki invariant $f(X)$ is equal to the integral of the Chern form on $M \times D^2$ whose boundary is $M \times S^1$ by means of direct calculation. We will finally show that the eta invariant is equal to the integral by using the Atiyah–Patodi–Singer Theorem.

First note that $f(X) \in \boldsymbol{R}$. We can demonstrate this fact as follows. Since both the G-action and the connection ∇ preserve h, it follows that $L_X h = 0$ and $\nabla_X h = 0$. Hence it follows that $L(X)h = 0$, and therefore $L(X)$ is skew-Hermitian with respect to the metric h and has only pure imaginary eigenvalues, as does Θ. Hence it follows that

$$f(X) = \int_M c_1^{m+1}\left(\frac{\mathrm{i}}{2\pi}(L(X) + \Theta)\right) \in \boldsymbol{R}. \tag{2.1}$$

Now let $S^1 = \boldsymbol{R}/\boldsymbol{Z}$ be the circle with coordinate t $(0 \leqslant t \leqslant 1)$, g_0 the metric on S^1 which comes from the standard metric on $\boldsymbol{R}$, W the product space $W = M \times S^1$, and $q_S : W \to S^1$ the natural projection. Then the horizontal subspace $q_S^* TS^1$ of the fibration $q_S : W \to S^1$, which is different from the obvious horizontal subspace, is defined by the vector field $Y := X + (\partial/\partial t)$, where $X \in \vartheta$ is identified with the real vector field corresponding to X. Hence we can define the product Riemannian metric of $TW = TM \oplus q_S^* TS^1$ by $h \oplus (g_0/\varepsilon^2)$, where ε is an arbitrary positive constant. Let ζ^W be the determinant line bundle of the trivial family $D^L \times S^1$ parametrized by S^1, and let $\mathrm{hol}_{\zeta^W}(S^1)$ be the holonomy of ζ^W around S^1 with respect to the connection in W defined by $q_S^* TS^1$. Since the horizontal curve $\tilde{\gamma} = \{(\exp sX \cdot p, b+s) \mid 0 \leqslant s \leqslant 1\}$ in W connects the point (p,b) with the point $(\exp X \cdot p, b)$ for any point $(p,b) \in W$ $(p \in M,\ b \in S^1)$, $\mathrm{hol}_{\zeta^W}(S^1)$ equals $\exp X|_K \in \boldsymbol{C}$, which coincides with $\mathrm{hol}_\zeta(\gamma)$. Hence it follows that

$$\mathrm{hol}_\zeta(\gamma) = \mathrm{hol}_{\zeta^W}(S^1). \tag{2.2}$$

Here the connections ∇, ∇^L and the horizontal subspace $q_S^* TS^1$ define unitary connections ∇^W of TW and ∇' of the virtual bundle $L^W := L \times S^1$ over W. Let A_ε^W be the L^W-valued self-adjoint Dirac operator on W defined by using the connections ∇^W, ∇', the metric $h \oplus (g_0/\varepsilon^2)$, and the spin$^{\mathrm{c}}$-structure of TW defined by the natural spin$^{\mathrm{c}}$-structure of TM and the unique trivial spin$^{\mathrm{c}}$-structure of TS^1, η_ε^W the eta invariant of A_ε^W, $d_\varepsilon^W := \dim\ker A_\varepsilon^W$ and $\xi_\varepsilon^W := \frac{1}{2}(\eta_\varepsilon^W + d_\varepsilon^W)$. Then the next equality follows from the Witten's holonomy formula [**3**, Theorem 3.16]:

$$\mathrm{hol}_{\zeta^W}(S^1) = \lim_{\varepsilon \to +0} (-1)^{\mathrm{Index}(D^L)} \mathrm{e}^{-2\pi\mathrm{i}\xi_\varepsilon^W}, \tag{2.3}$$

where $\mathrm{Index}(D^L)$ is the Atiyah–Singer index of D^L.

Let θ^W denote the connection form of ∇^W. Then we can see that

$$\theta^W = q_W^* \theta + L(X)\,\mathrm{d}t,$$

where $q_W : W \to M$ is the natural projection because

$$\nabla^W_{\partial/\partial t} Z = \nabla^W_Y Z - \nabla^W_X Z = \frac{\mathrm{d}}{\mathrm{d}s}[\exp(-sX)_* Z]_{s=0} - \nabla_X Z = L(X)(Z)$$

for any $Z \in h(M)$. Now let $I = [1,2]$ be an interval with coordinate r, C the cylinder defined by $C = I \times S^1 = \{(r,t) \mid 1 \leqslant r \leqslant 2,\ 0 \leqslant t \leqslant 1\}$, and V the product space $V = M \times C$. Then the boundary of V consists of two components $W_1 = M \times \{1\} \times S^1$ and $W_2 = M \times \{2\} \times S^1$. Let $\varphi(r)$ be a smooth function such that $0 \leqslant \varphi(r) \leqslant 1$, $\varphi(r) = 0$ for $r \in [1, \frac{4}{3}]$, $\varphi(r) = 1$ for $r \in [\frac{5}{3}, 2]$, and $(z_1, z_2, \dots, z_m)$ a local holomorphic coordinate on M. Let Y^r denote the vector field on V defined by

$$Y^r = \varphi(r)X + \frac{\partial}{\partial t}.$$

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Then a complex structure J^V and a Hermitian metric h^V on V is defined by using the complex structure J and the Hermitian metric h on M as follows:

$$J^V\Big(\frac{\partial}{\partial z_i}\Big) = J\Big(\frac{\partial}{\partial z_i}\Big), \qquad J^V\Big(\frac{\partial}{\partial r}\Big) = \varepsilon Y^r,$$
$$h^V\Big(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\Big) = h\Big(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\Big), \qquad h^V\Big(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\Big) = \varepsilon^2 h^V(Y^r, Y^r) = 1,$$
$$h^V\Big(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial r}\Big) = h^V\Big(\frac{\partial}{\partial z_i}, Y^r\Big) = h^V\Big(\frac{\partial}{\partial r}, Y^r\Big) = 0.$$

Then we can define a unitary connection θ^V of TV by

$$\theta^V = q_V^*\theta + \varphi(r)L(X)\,\mathrm{d}t,$$

where $q_V : V \to M$ is the natural projection. Note that the restrictions of the metric and the connection on V to W_2 coincide with those of W and hence W_2 is identified with W. Now the curvature form Θ^V of θ^V is computed as

$$\Theta^V = \mathrm{d}\theta^V + \theta^V \wedge \theta^V = q_V^*\Theta + \varphi'(r)L(X)\,\mathrm{d}r \wedge \mathrm{d}t \qquad (\mathrm{mod}\ \mathrm{d}z_i \wedge \mathrm{d}t),$$

and hence it follows from (1.1) that

$$\begin{aligned}\int_V c_1^{m+1}(TV, \Theta^V) &= \int_V \Big(\mathrm{Tr}\Big(\frac{\mathrm{i}}{2\pi}\Theta^V\Big)\Big)^{m+1} \\ &= \int_M (m+1)\Big(\mathrm{Tr}\Big(\frac{\mathrm{i}}{2\pi}\Theta\Big)\Big)^m \mathrm{Tr}\Big(\frac{\mathrm{i}}{2\pi}L(X)\Big) \int_1^2 \varphi'(r)\,\mathrm{d}r \int_0^1 \mathrm{d}t \\ &= \int_M c_1^{m+1}\Big(\frac{\mathrm{i}}{2\pi}(L(X)+\Theta)\Big) = f(X), \end{aligned} \tag{2.4}$$

where $c_1(TV, \Theta^V)$ is the first Chern form with respect to Θ^V.

Now let U be the product space $U = M \times D^2$. Then the product complex structure of the complex structures on M and D^2 define a complex structure on U, which coincides with J^V near the boundary $\partial U = W_1$. We give a rotationally symmetric Hermitian metric h^D on D^2 which is a product metric of $(1-\delta, 1] \times S^1$ near the boundary $\partial D^2 = \{1\} \times S^1 = S^1$, where the metric on S^1 is g_0/ε^2. Let θ^D be the type (1,0) unitary connection of TD^2 and Θ^D its curvature form. Then the product metric of h and h^D define a Hermitian metric h^U on U, which coincides with h^V near W_1, and the direct sum of θ and θ^D define a unitary connection θ^U of TU, which coincides with θ^V near W_1. Let N denote the complex manifold with boundary $W_2 = W$ defined by gluing U to V along the boundary W_1. Then the metrics h^V, h^U and the connections θ^V, θ^U define a Hermitian metric h^N and a unitary connection θ^N of TN. We denote by Θ^U, Θ^N the curvature forms of θ^U, θ^N, respectively. Let $c_1(TM)$ be the first Chern class of TM and $[M]$ the fundamental cycle of M. Then it follows from (2.4) that

$$\int_N c_1^{m+1}(TN,\Theta^N) = \int_V c_1^{m+1}(TV,\Theta^V) + \int_U c_1^{m+1}(TU,\Theta^U)$$
$$= f(X) + (m+1)c_1^m(TM)[M]\int_{D^2} c_1(TD^2,\Theta^D) \equiv f(X) \pmod{\boldsymbol{Z}}. \tag{2.5}$$

On the other hand, let L^N be a virtual bundle over N defined by $L^N = \otimes^{m+1}(K^{-1}N - \tau)$. Then the metric h^N and the connection θ^N naturally define a metric and a unitary connection of L^N, whose restriction to $\partial N = W$ coincides with those of $L^W = L^N|_W$. Hence, using the metrics, the connections and the natural spin^c-structure of TN, we can define the L^N-valued spin^c-Dirac operator as in the previous section. Then since the restrictions of the metric h^N and the connection θ^N to $\partial N = W$ coincide with those of TW and h^N, θ^N are products near W, it follows from the Atiyah–Patodi–Singer Index Theorem (see [**2**, (4.2)] and [**1**, (4.3)]) that

$$\int_N \mathrm{ch}(L^N,\Theta^N)\,\mathrm{Td}(TN,\Theta^N) \equiv \xi_\varepsilon^W \pmod{\boldsymbol{Z}}, \tag{2.6}$$

where $\mathrm{ch}(L^N,\Theta^N)$ is the Chern character form of L^N and $\mathrm{Td}(TN,\Theta^N)$ is the Todd form of TN. Here, since

$$\mathrm{ch}(L^N,\Theta^N) = \{\mathrm{ch}(\wedge^{m+1}TN,\Theta^N) - 1\}^{m+1} = c_1^{m+1}(TN,\Theta^N)$$

and the leading term of $\mathrm{Td}(TN)$ is equal to 1, it follows from (2.5) and (2.6) that

$$f(X) \equiv \xi_\varepsilon^W \pmod{\boldsymbol{Z}}. \tag{2.7}$$

Moreover, since the Futaki invariant does not depend on the choice of ε, it follows from (2.2), (2.3) and (2.7) that

$$\mathrm{hol}_\zeta(\gamma) = (-1)^{\mathrm{Index}(D^L)}\mathrm{e}^{-2\pi \mathrm{i} f(X)}. \tag{2.8}$$

Here it follows from the Atiyah–Singer Index Theorem (see [**1**, (4.3)]) that

$$\mathrm{Index}(D^L) = \mathrm{ch}(L)\,\mathrm{Td}(TM)[M]$$
$$= (c_1(TM) + \text{higher-order terms})^{m+1}(1+\cdots)[M] = 0, \tag{2.9}$$

and hence it follows from (2.8) that

$$\mathrm{hol}_\zeta(\gamma) = \mathrm{e}^{-2\pi \mathrm{i} f(X)}.$$

This completes the proof of Theorem 2.1. □

The next corollary is an immediate consequence of Theorem 2.1 and the main theorem in [**4**].

Corollary 2.2. $\mathrm{hol}_\zeta(\gamma) = 1$ *for any loop* γ *in* B *if* M *admits an Einstein–Kähler metric.*

Remark 2.3. Let $\pi : EG \to BG$ be the universal G-bundle with the universal connection. Then, for any $g = \exp X \in G$, there exists a loop γ_g in BG such that a horizontal lift of γ_g connects a point $p \in \pi^{-1}(b)$ with the point $p \cdot g \in \pi^{-1}(b)$. Hence it follows from Theorem 2.1 that $\mathrm{hol}_\zeta(\gamma) = 1$ for any loop γ in BG if and only if $f(X) \in \boldsymbol{Z}$ for any $X \in \vartheta$, which is equivalent to the condition that $f(X) = 0$ for any $X \in \vartheta$. On the other hand, it is known (see [**6**, Theorems 5.1, 5.2]) that M does not admit an Einstein–Kähler metric unless the Lie algebra $h(M)$ coincides with the complexification of ϑ or ϑ itself. If $h(M)$ coincides with the complexification of ϑ or ϑ itself, $f(X) = 0$ for any $X \in \vartheta$ if and only if $f(X) = 0$ for any $X \in h(M)$.

Let ϑ_p be the $Ad(G)$-invariant dense subset of ϑ consisting of the elements X such that $\exp X$ is periodic and $\Omega_p(B)$ the set of loops in B whose horizontal lifts connect a point $p \in \pi^{-1}(b)$ with the point $p \cdot \exp(X) \in \pi^{-1}(b)$ for $X \in \vartheta_p$. Then the next proposition follows from [**7**, Theorem 1.4] and (2.9).

Proposition 2.4. *Assume that* γ *is an element of* $\Omega_p(B)$. *Then the following equality holds:*

$$\mathrm{hol}_\zeta(\gamma) = \exp \frac{2\pi \mathrm{i}}{p} \sum_{k=1}^{p-1} \frac{1}{\mathrm{e}^{-2\pi \mathrm{i} k/p} - 1} \,\mathrm{Index}(D^L, g^k),$$

where p *is the order of* $g := \exp X$ *and* $\mathrm{Index}(D^L, g^k)$ *is the Atiyah–Singer index of* D^L *evaluated at* g^k *(see [**1**]).*

Since $\mathrm{Index}(D^L, g^k)$ is computed by using the holomorphic Lefschetz theorem (4.6) in [**1**], the holonomy $\mathrm{hol}_\zeta(\gamma)$ is computed concretely for $\gamma \in \Omega_p(B)$.

Example 2.5. In this example we compute the holonomy for the complex manifold introduced in [**4**]. Let H_i denote the hyperplane bundle over $\boldsymbol{CP}^i$ and M the total space of the projective bundle $\boldsymbol{P}(E)$ of the vector bundle $E = \pi_1^* H_1 \oplus \pi_2^* H_2$ over $\boldsymbol{CP}^1 \times \boldsymbol{CP}^2$, where π_i is the ith factor projection. Then the factor group $P(\mathrm{GL}(2; \boldsymbol{C}) \times \mathrm{GL}(3; \boldsymbol{C}))$ of $\mathrm{GL}(2; \boldsymbol{C}) \times \mathrm{GL}(3; \boldsymbol{C})$ by the centre of $\mathrm{GL}(5; \boldsymbol{C})$ is isomorphic to the identity component of $H(M)$, hence $h(M)$ is the complexification of ϑ, and M does not admit an Einstein–Kähler metric (for details, see [**4**, § 3]). Now let $\pi : EG \to BG$ be the universal G-bundle with the universal connection. Then, for any $g \in G$ there exists a loop γ_g in BG such that a horizontal lift of γ_g connects a point $p \in \pi^{-1}(b)$ with the point $p \cdot g \in \pi^{-1}(b)$. Here let X be an element of ϑ_p represented by the diagonal matrix with diagonal entries $(2\pi \mathrm{i}/p, 2\pi \mathrm{i}/p, 0, 0, 0)$ and set $g := \exp X$, which is an element of $P(\mathrm{GL}(2; \boldsymbol{C}) \times \mathrm{GL}(3; \boldsymbol{C}))$ represented by the diagonal periodic matrix of order p with diagonal entries $(\alpha, \alpha, 1, 1, 1)$, where $\alpha := \exp(2\pi \mathrm{i}/p)$ is the primitive pth root of 1. Then the fixed-point set $\Omega(k) \subset M$ of the g^k-action is independent of k and coincides with the disjoint union of the two components N_1, N_2, which are isomorphic to the base space $\boldsymbol{CP}^1 \times \boldsymbol{CP}^2$ of E and whose normal bundles in M are isomorphic to $\pi_1^* H_1$, $\pi_2^* H_2$, respectively. Set $x = c_1(H_1)$ and $y = c_1(H_2)$, which are the positive generators of $H^2(\boldsymbol{CP}^1)$ and $H^2(\boldsymbol{CP}^2)$, respectively.

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Then we have

$$c_1(K_M^{-1}|_{N_i}) = c_1(TM|_{N_i}) = c_1(TN_i \oplus \pi_i^* H_i) = \begin{cases} 3x+3y & (i=1), \\ 2x+4y & (i=2). \end{cases}$$

Let $[N_i]$ denote the fundamental cycle of N_i. Since g^k acts on $\pi_1^* H_1$, $\pi_2^* H_2$ via multiplication by α^{-k}, α^k, respectively, it follows from [**1**, (4.6)] that

$$\begin{aligned} \operatorname{Index}(D^L, g^k) &= (\alpha^{-k} \mathrm{e}^{c_1(K_M^{-1}|_{N_1})} - 1)^5 (1 - \alpha^k \mathrm{e}^{-c_1(\pi_1^* H_1)})^{-1} \operatorname{Td}(TN_1)[N_1] \\ &\qquad + (\alpha^k \mathrm{e}^{c_1(K_M^{-1}|_{N_2})} - 1)^5 (1 - \alpha^{-k} \mathrm{e}^{-c_1(\pi_2^* H_2)})^{-1} \operatorname{Td}(TN_2)[N_2] \\ &= xy^2 \text{ coefficient of} \\ &\qquad (\alpha^{-k} \mathrm{e}^{3x+3y} - 1)^5 (1 - \alpha^k \mathrm{e}^{-x})^{-1} \left(\frac{x}{1-\mathrm{e}^{-x}} \right)^2 \left(\frac{y}{1-\mathrm{e}^{-y}} \right)^3 \\ &\qquad\quad + (\alpha^k \mathrm{e}^{2x+4y} - 1)^5 (1 - \alpha^{-k} \mathrm{e}^{-y})^{-1} \left(\frac{x}{1-\mathrm{e}^{-x}} \right)^2 \left(\frac{y}{1-\mathrm{e}^{-y}} \right)^3 \\ &= (1-\alpha^{-k})(2\alpha^{-k} - 245\alpha^{-2k} + 1699\alpha^{-3k} - 2176\alpha^{-4k}) \\ &\qquad + (1-\alpha^{-k})(2541\alpha^{5k} - 2034\alpha^{4k} + 306\alpha^{3k} - 3\alpha^{2k}). \end{aligned}$$

Hence we have

$$\begin{aligned} &\sum_{k=1}^{p-1} \frac{1}{\alpha^{-k} - 1} \operatorname{Index}(D^L, g^k) \\ &\qquad \equiv 2 - 245 + 1699 - 2176 + 2541 - 2034 + 306 - 3 = 90 \pmod{p}, \end{aligned}$$

because

$$\sum_{k=1}^{p-1} \alpha^{\mu k} \equiv -1 \pmod{p}$$

for any integer μ. Therefore it follows from Proposition 2.4 that

$$\operatorname{hol}_\zeta(\gamma_g) = \alpha^{90},$$

which is not equal to 1 unless p is a divisor of 90. Hence it follows from Corollary 2.2 that M does not admit an Einstein–Kähler metric.

References

1. M. F. Atiyah and I. M. Singer, The index of elliptic operators, III, *Ann. Math.* **87** (1968), 546–604.
2. M. F. Atiyah, V. K. Patodi and I. M. Singer, Spectral asymmetry and Riemannian geometry, I, *Math. Proc. Camb. Phil. Soc.* **77** (1975), 43–69.
3. J. M. Bismut and D. S. Freed, The analysis of elliptic families, II, *Commun. Math. Phys.* **107** (1986), 103–163.

4. A. Futaki, An obstruction to the existence of Einstein Kähler metrics, *Inventiones Math.* **73** (1983), 437–443.
5. A. Futaki and S. Morita, Invariant polynomials of the automophism group of a compact complex manifold, *J. Diff. Geom.* **21** (1985), 135–142.
6. S. Kobayashi, *Transformation groups in differential geometry*, Classics in Mathematics (Springer, 1995).
7. K. Tsuboi, On the determinant and the holonomy of equivariant elliptic operators, *Proc. Am. Math. Soc.* **123** (1995), 2275–2281.

https://doi.org/10.1017/S0013091500000389 Published online by Cambridge University Press