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An AF Algebra Associated with the Farey Tessellation

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Abstract. We associate with the Farey tessellation of the upper half-plane an AF algebra $\mathfrak{A}$ encoding the "cutting sequences" that define vertical geodesics. The Effros–Shen AF algebras arise as quotients of $\mathfrak{A}$. Using the path algebra model for AF algebras we construct, for each $\tau \in \left(0, \frac{1}{4}\right]$, projections (E_n) in $\mathfrak{A}$ such that $E_n E_{n\pm 1} E_n \leq \tau E_n$.

Introduction

The semigroup $\mathfrak{S}$ generated by the matrices $A = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ is isomorphic to $\mathbb{F}_2^+$, the free semigroup on two generators. This fact, intimately connected to the continued fraction algorithm, can be visualized by means of the *Farey tessellation* $\{g\mathbb{G} : g \in \mathfrak{S}\}$ of $\mathbb{H}$ depicted in Figure 1, where $\mathbb{G} = \left\{0 \leq \Re z \leq 1 : |z - \frac{1}{2}| \geq \frac{1}{2}\right\}$ [25].

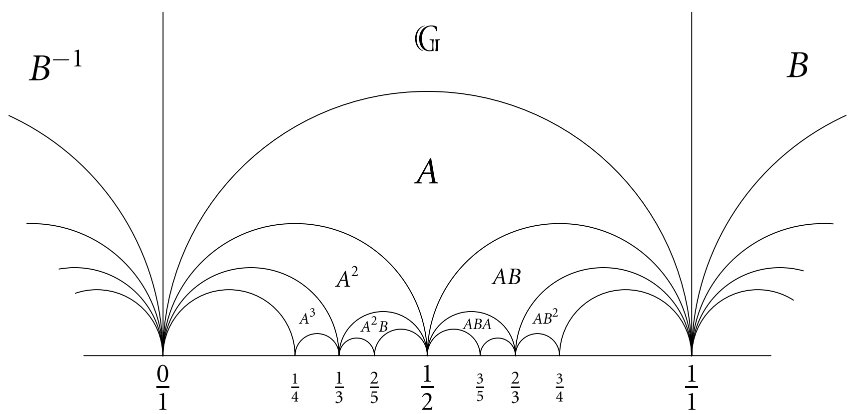

Figure 1: The Farey tessellation.

The strip $0 \leq \Re z \leq 1$ is tessellated precisely by the images of $\mathbb{G}$ under matrices from the set

$$\mathfrak{S}_* = \{I\} \cup \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) : 0 \leq a \leq c,\ 0 \leq b \leq d \right\}.$$

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By suspending the cusps in this tessellation (which correspond to rational numbers in $[0,1]$) with appropriate (infinite) multiplicities, one gets the diagram $\mathcal{G}$ from Figure 2 (see [19]). This diagram reflects both the elementary mediant construction, which produces from a pair $(p/q, p'/q')$ of rational numbers with $p'q - pq' = 1$ the new pairs $(\frac{p}{q}, \frac{p+p'}{q+q'})$ and $(\frac{p+p'}{q+q'}, \frac{p'}{q'})$ with the same property, and the "geometry" of the continued fraction algorithm. As in the case of the Pascal triangle, in $\mathcal{G}$ one writes the sum of the denominators of two neighbors from the same floor into the next floor of the diagram. One keeps, however, a copy of each denominator at the next floor. For this reason, such a diagram was called the *Pascal triangle with memory* in [18]. There is a remarkable one-to-one correspondence between the integer solutions of the equation $ad - bc = 1$ with $0 \leq a \leq c$, $0 \leq b \leq d$ and the rational labels of two neighbors at the same floor in $\mathcal{G}$ acquired by the mediant construction and by keeping each label at the next floor in the diagram.

The thrust of this paper is the remark that, by regarding $\mathcal{G}$ as a Bratteli diagram, one gets an AF algebra $\mathfrak{A} = \varinjlim \mathfrak{A}_n$ with interesting properties. This algebra is closely related with the Effros–Shen AF algebras [11, 21], which we show arising as primitive quotients of $\mathfrak{A}$. The primitive ideal space $\operatorname{Prim} \mathfrak{A}$ is identified with the disjoint union of the irrational numbers in $[0,1]$ and three copies of the rational ones, except for the endpoints 0 and 1, which are represented by only two copies.

In [3] it was shown that any separable abelian C^*-algebra $\mathfrak{Z}$ is the center $Z(\mathcal{A})$ of an AF algebra $\mathcal{A}$. The AF algebra $\mathfrak{A}$ can actually be retrieved from that abstract construction by embedding $\mathfrak{Z} = C[0,1]$ into the norm closure in $L^\infty[0,1]$ of the linear space of the characteristic functions of open sets $(\frac{k}{2^n}, \frac{k+1}{2^n})$ and singleton sets $\{\frac{\ell}{2^n}\}$, $n \geq 0$, $0 \leq k < 2^n$, $0 \leq \ell \leq 2^n$. In particular this shows that $Z(\mathfrak{A}) = C[0,1]$.

The connecting maps $K_0(\mathfrak{A}_n) \hookrightarrow K_0(\mathfrak{A}_{n+1})$ correspond to the polynomial relations $p_{n+1}(t) = (1+t+t^2)p_n(t^2)$. These polynomials are closely related to the Stern–Brocot sequence [6]. The origins of this remarkable sequence, which has attracted considerable interest over time, can be traced back to Eisenstein (see [5, 27], or the contemporary reference [26] for a thorough bibliography on this subject). In our framework the Stern–Brocot sequence $q(n,k)$, $n \geq 0$, $0 \leq k < 2^n$, simply appears as the sizes of the summands $\mathfrak{A}_n \cong \bigoplus_{k=0}^{2^n-1} \mathbb{M}_{q(n,k)} \oplus \mathbb{C}$, where $\mathbb{M}_r$ denotes the C^*-algebra of $r \times r$ matrices with complex entries.

The Bratteli diagram $\mathcal{G}$ has some apparent symmetries. In the last section we employ the AF path algebra model to express them, constructing sequences of projections in $\mathfrak{A}$ that satisfy certain braiding relations reminiscent of the Temperley–Lieb–Jones relations. In particular, for every $\tau \in (0, \frac{1}{4}]$, we construct projections E_n in $\mathfrak{A}$, $n \geq 0$, such that $E_n E_{n\pm 1} E_n \leq \tau E_n$ and $[E_n, E_m] = 0$ if $|n-m| \geq 2$. This suggests a possible connection with a class of statistical mechanics models with partition functions closely related to Riemann's zeta function, called Farey spin chains, which have been studied in recent years by Knauf, Kleban, and their collaborators (see [16–19, 22] and references therein).

1 The Pascal Triangle with Memory as a Bratteli Diagram

The *Pascal triangle with memory* is a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ defined as follows:

- The *vertex set* $\mathcal{V}$ is the disjoint union $\biguplus_{n\geq 0} \mathcal{V}_n$ of the sets $\mathcal{V}_n = \{(n,k) : 0 \leq k \leq 2^n\}$ of *vertices at floor n*;
- The set of *edges* is defined as $\mathcal{E} = \biguplus_{n\geq 0} \mathcal{E}_n$, where $\mathcal{E}_n$ is the set of edges connecting vertices at floor n with those at floor $n+1$ under the rule that (n,k) is connected with $(n+1,\ell)$ precisely when $|2k-\ell| \leq 1$. There are no edges connecting vertices from $\mathcal{V}_i$ and $\mathcal{V}_j$ when $|i-j| \geq 2$.

To each vertex (n,k) we attach the *label* $r(n,k) = \frac{p(n,k)}{q(n,k)}$, with non-negative integers $p(n,k)$, $q(n,k)$ defined recursively for $n \geq 0$ by

$$q(n,0) = q(n,2^n) = 1, \quad p(n,0) = 0, \quad p(n,2^n) = 1;$$
$$q(n+1,2k) = q(n,k), \quad p(n+1,2k) = p(n,k), \quad 0 \leq k \leq 2^n;$$
$$\left.\begin{aligned} q(n+1,2k+1) &= q(n,k) + q(n,k+1), \\ p(n+1,2k+1) &= p(n,k) + p(n,k+1), \end{aligned}\right. \qquad 0 \leq k < 2^n.$$

Note that $r(n,0) = 0 < r(n,1) = \frac{1}{n+1} < \cdots < r(n,2^n) = 1$ gives a partition of $[0,1]$, and

$$p(n,k+1)q(n,k) - p(n,k)q(n,k+1) = 1, \qquad n \geq 0,\ 0 \leq k < 2^n,$$

showing in particular that $p(n,k)$ and $q(n,k)$ are relatively prime.

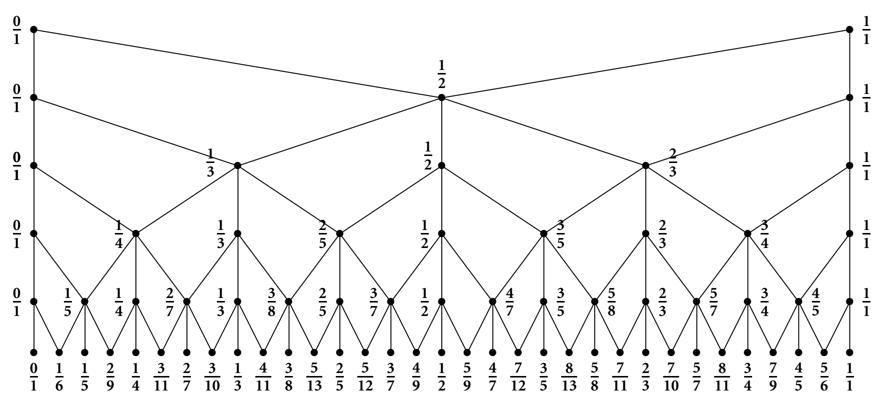

Figure 2: The Pascal triangle with memory, $\mathcal{G}$.

Conversely, for every pair $p/q < p'/q'$ of rational numbers with $p'q - pq' = 1$, $0 \leq p \leq q$ and $0 \leq p' \leq q'$, there exists a unique pair of integers (n,k) with $n \geq 0$, $0 \leq k < 2^n$, such that $r(n,k) = p/q$ and $r(n,k+1) = p'/q'$. This correspondence establishes a bijection between the vertices from $\mathcal{V} \setminus \{(n,2^n) : n \geq 0\}$ and the set

$$\Gamma^+ := \left\{ \begin{pmatrix} p' & p \\ q' & q \end{pmatrix} \in SL_2(\mathbb{Z}) : 0 \leq p \leq q,\ 0 \leq p' \leq q' \right\} \subset SL_2(\mathbb{Z}).$$

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Remark 1. The mapping $r(n,k) \mapsto k/2^n$, $0 \leq k \leq 2^n$, $n \geq 0$, extends by continuity to Minkowski's question mark map $?: [0,1] \to [0,1]$ defined on (reduced) continued fractions as

$$?([a_1,a_2,\dots]) = \sum_{k\geq 1} \frac{(-1)^{k-1}}{2^{(a_1+\cdots+a_k)-1}}.$$

The map $?$ is strictly increasing and singular, and establishes remarkable one-to-one correspondences between rational and dyadic numbers, and respectively between quadratic algebraic numbers and rational numbers in $[0,1]$ (see [7, 20, 24]).

In this paper we shall consider the AF algebra $\mathfrak{A}$ associated with the Bratteli diagram $D(\mathfrak{A}) = \mathcal{G}$ from Figure 2. For the connection between Bratteli diagrams, AF algebras, and their ideals, we refer to the classical reference [1]. We write $(n,k)\downarrow(n,'k')$ when $n' = n+1$ and there is at least one edge between the vertices (n,k) and (n',k') in the Bratteli diagram, and write $(n,k) \Downarrow (n',k')$ when $n < n'$ and there are vertices $(n,k_0 = k), (n+1,k_1),\dots,(n',k_{n'-n} = k')$ such that $(n+r,k_r) \downarrow (n+r+1,k_{r+1})$, $r = 0,\dots,n'-n-1$. In algebraic terms this is equivalent to $e_{(n,k)}e_{(n',k')} \neq 0$, where $e_{(n,k)}$ denotes the central projection in $\mathfrak{A}_n$ that corresponds to the vertex (n,k) of the diagram. The AF algebra $\mathfrak{A}$ is the inductive limit $\varinjlim \mathfrak{A}_n$, where

$$\mathfrak{A}_n = \bigoplus_{0\leq k\leq 2^n} \mathbb{M}_{q(n,k)}$$

and each embedding $\mathfrak{A}_n \hookrightarrow \mathfrak{A}_{n+1}$ is given by the Bratteli diagram from Figure 2.

Remark 2. Consider the set $\mathcal{V}_*$ of vertices of $\mathcal{G}$ of form (n,k) with $0 \leq k \leq 2^n$ and k odd, and the map $\Phi\colon \mathcal{V}_* \to \mathbb{N}$, $\Phi(n,k) = q(n,k)$. The inverse image $\Phi^{-1}(q)$ of q contains exactly $\varphi(q)$ elements, where φ denotes Euler's totient function; in particular q is prime if and only if $\#\Phi^{-1}(q) = q-1$. This remark shows [17] that the partition function associated with the corresponding Farey spin chain is $\sum_{n=1}^{\infty} \varphi(n)n^{-s}$, which is equal to $\zeta(s-1)/\zeta(s)$ when $\Re s > 2$.

Remark 3. (i) The integers $q(n,k)$ satisfy the equality $\sum_{0\leq k\leq 2^n} q(n,k) = 3^n + 1$.

(ii) Consider the Bratteli diagram obtained by deleting in $\mathcal{G}$ all vertices $(n,0)$ and denote the corresponding AF algebra by $\mathfrak{B} = \varinjlim \mathfrak{B}_n$. It is clear that $\mathfrak{B}$ is an ideal in $\mathfrak{A}$ and $\mathfrak{A}/\mathfrak{B} \cong \mathbb{C}$. Moreover,

$$\mathfrak{B}_n = \bigoplus_{1\leq k\leq 2^n} \mathbb{M}_{p(n,k)},$$

thus the ranks of the central summands of the building blocks of $\mathfrak{B}$ give the complete list of numerators $p(n,k)$. We also have

$$\sum_{0\leq k\leq 2^n} p(n,k) = \frac{3^n+1}{2}.$$

2 The Primitive Ideal Space of the AF Algebra $\mathfrak{A}$

We denote $\mathbb{I} = \{\theta \in (0,1) : \theta \notin \mathbb{Q}\}$ and $\mathbb{Q}_{(0,1)} = \mathbb{Q} \cap (0,1)$.

The C^*-algebra $\mathfrak{A}$ is not simple and has a rich (and potentially interesting) structure of ideals. We first relate $\mathfrak{A}$ with the AF algebra $\mathfrak{F}_\theta$ associated by Effros and Shen [11] to the continued fraction decomposition $\theta = [a_1, a_2, \dots]$ of $\theta \in \mathbb{I}$. The Bratteli diagram $D(\mathfrak{F}_\theta)$ of the simple C^*-algebra $\mathfrak{F}_\theta$ is given in Figure 3.

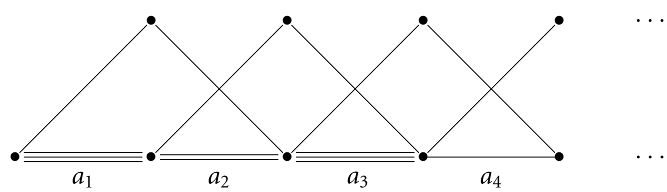

Figure 3: The Bratteli diagram $D(\mathfrak{F}_\theta)$.

The C^*-algebra of unitized compact operators $\widetilde{\mathbb{K}} = \mathbb{C}I + \mathbb{K}$ is an AF algebra and we have a short exact sequence $0 \to \mathbb{K} \to \widetilde{\mathbb{K}} \to \mathbb{C} \to 0$, made explicit by the Bratteli diagram in Figure 4, where the shaded subdiagram corresponds to the ideal $\mathbb{K}$. Replacing $\mathbb{C} \oplus \mathbb{C}$ by $\mathbb{M}_q \oplus \mathbb{M}_{q'}$ one gets an AF algebra $\mathfrak{A}_{(q,q')}$ which is an extension of $\mathbb{K}$ by $\mathbb{M}_q$.

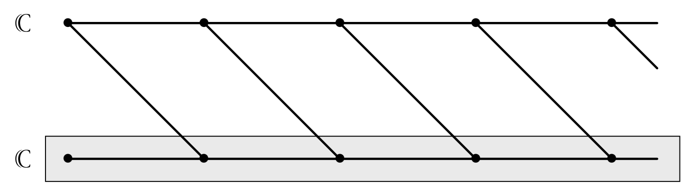

Figure 4: The Bratteli diagram of the C^*-algebra of unitized compact operators.

We first show that Effros–Shen algebras arise naturally as quotients of our AF algebra $\mathfrak{A}$ and that the corresponding ideals belong to the primitive ideal space Prim $\mathfrak{A}$. The *Farey map* $F\colon [0,1] \to [0,1]$ defined by

$$F(x) = \begin{cases} \dfrac{x}{1-x} & \text{if } x \in [0, \frac{1}{2}], \\[2ex] \dfrac{1-x}{x} & \text{if } x \in (\frac{1}{2}, 1] \end{cases}$$

(see [14]), acts on infinite (reduced) continued fractions as

$$F([a_1, a_2, a_3, \dots]) = [a_1 - 1, a_2, a_3, \dots].$$

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For each $y \in [0,1]$ the equation $F(x) = y$ has exactly two solutions $x \in [0,1]$ given by

$$x = F_1(y) = \frac{y}{1+y} \quad \text{and} \quad x = F_2(y) = \frac{1}{1+y} = 1 - F_1(y). \tag{2.1}$$

One has $F_1([a_1, a_2, \dots]) = [a_1+1, a_2, \dots]$ and $F_2([a_1, a_2, \dots]) = [1, a_1, a_2, \dots]$. Rational numbers are generated by the backwards orbit of F as follows:

$$\{F^{-n}(\{0\}) : n = 0, 1, 2, \dots\} = \mathbb{Q} \cap [0,1].$$

More precisely, for each $n \in \mathbb{N}$ one has

$$\begin{aligned} F^{-n}(\{0\}) &= \{r(n-1,k) : 0 \le k \le 2^{n-1}\} \\ &= \left\{ F_{i_1}^{\alpha_1} \cdots F_{i_k}^{\alpha_k}(0) : i_j \in \{1,2\},\ i_1 \neq \cdots \neq i_k,\ \alpha_1 + \cdots + \alpha_k = n \right\} \\ &= \left\{ [a_1, \dots, a_r] : a_1 + \cdots + a_r \le n \right\}. \end{aligned}$$

In the next statement, given relatively prime integers $0 < p < q$, $\overline{p}$ will denote the multiplicative inverse of p modulo q, *i.e.*, the unique integer $\overline{p} \in \{1, \dots, q-1\}$ with $p\overline{p} = 1 \pmod{q}$.

Proposition 4 (i) *For each $\theta \in \mathbb{I}$, there is $I_\theta \in \operatorname{Prim} \mathfrak{A}$ such that $\mathfrak{A}/I_\theta \cong \mathfrak{F}_\theta$.*

(ii) *Given $\frac{p}{q} \in \mathbb{Q}_{(0,1)}$ in lowest terms, there are $I_{\frac{p}{q}}, I^+_{\frac{p}{q}}, I^-_{\frac{p}{q}} \in \operatorname{Prim} \mathfrak{A}$ such that $\mathfrak{A}/I_{\frac{p}{q}} \cong \mathbb{M}_q$, $\mathfrak{A}/I^-_{\frac{p}{q}} \cong \mathfrak{A}_{(q,\overline{p})}$, and $\mathfrak{A}/I^+_{\frac{p}{q}} \cong \mathfrak{A}_{(q,q-\overline{p})}$.*

(iii) *There are $I_0, I_0^+, I_1, I_1^- \in \operatorname{Prim} \mathfrak{A}$ such that $\mathfrak{A}/I_0 \cong \mathfrak{A}/I_1 \cong \mathbb{C}$ and $\mathfrak{A}/I_0^+ \cong \mathfrak{A}/I_1^- \cong \widetilde{\mathbb{K}}$.*

Proof (i) Let $\theta \in \mathbb{I}$ with continued fraction $[a_1, a_2, \dots]$ and let $r_\ell = r_\ell(\theta) = p_\ell/q_\ell = [a_1, \dots, a_\ell]$ be its ℓ-th convergent, where $p_\ell = p_\ell(\theta)$ and $q_\ell = q_\ell(\theta)$ can be recursively defined by

$$\begin{gathered} p_{-1} = 1,\ q_{-1} = 0, \quad p_0 = 0,\ q_0 = 1; \\ \begin{pmatrix} p_\ell & q_\ell \\ p_{\ell-1} & q_{\ell-1} \end{pmatrix} = \begin{pmatrix} a_\ell & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_{\ell-1} & q_{\ell-1} \\ p_{\ell-2} & q_{\ell-2} \end{pmatrix}, \qquad \ell \ge 1. \end{gathered}$$

The relation $p_\ell q_{\ell-1} - p_{\ell-1} q_\ell = (-1)^{\ell-1}$ shows in particular that $\gcd(p_\ell, q_\ell) = 1$.

For each $a \in \mathbb{N} = \{1, 2, \dots\}$ consider the diagrams L_a and R_a from Figure 5. Also set $L_0 = R_0 = \varnothing$. Clearly L_{a+b} coincides with the concatenation $L_a \circ L_b$ of L_a followed by L_b, and we also have $R_{a+b} = R_a \circ R_b$. Using the obvious identifications between $L_a \circ R_b$, $R_a \circ L_b$, and $C_a \circ C_b$ (see Figure 6) and (2.1), we see that the AF algebras defined by the Bratteli diagrams $L_{a_1} \circ R_{a_2} \circ L_{a_3} \circ R_{a_4} \circ \dots$ and $R_{a_1} \circ L_{a_2} \circ R_{a_3} \circ L_{a_4} \circ \dots$ are isomorphic to $\mathfrak{F}_{[a_1+1, a_2, a_3, \dots]} \cong \mathfrak{F}_{F_1(\theta)} \cong \mathfrak{F}_{F_2(\theta)} \cong \mathfrak{F}_{[1, a_1, a_2, \dots]}$ (note that the AF algebra defined by $C_{a_1} \circ C_{a_2} \circ C_{a_3} \circ \cdots$ is isomorphic to $\mathfrak{F}_{[a_1+1, a_2, a_3, \dots]}$).

The Bratteli subdiagram $\mathcal{G}_\theta$ of $\mathcal{G}$ containing the vertices $(0,0)$ and $(0,1)$ and defined by $L_{a_1-1} \circ R_{a_2} \circ L_{a_3} \circ R_{a_4} \circ \cdots$ generates a copy of $\mathfrak{F}_\theta$.

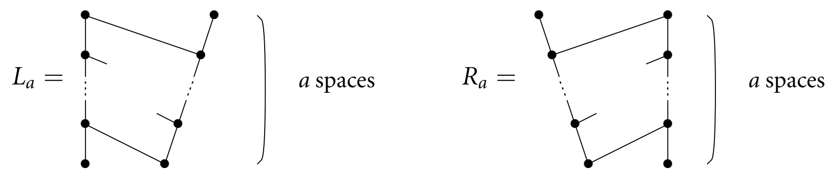

Figure 5: The diagrams L_a and R_a.

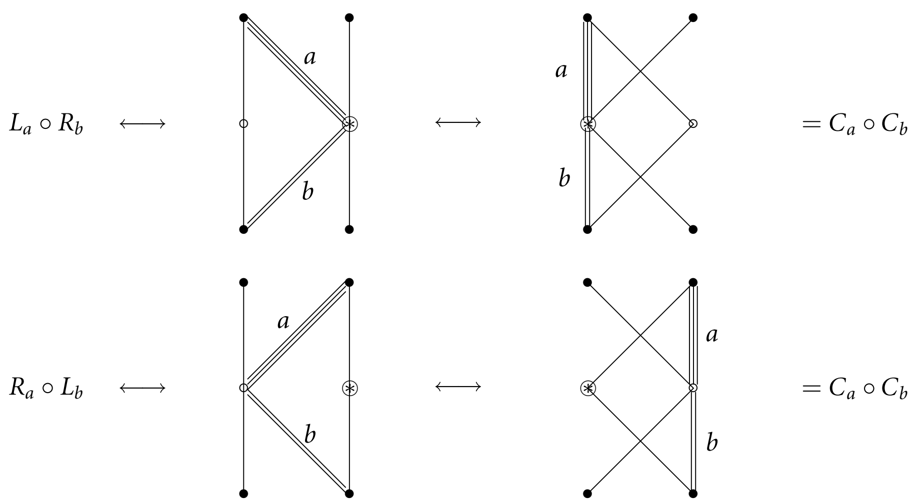

Figure 6: The identification between $L_a \circ R_b$, $R_a \circ L_b$, and $C_a \circ C_b$.

The complement $\mathcal{G} \setminus \mathcal{G}_\theta$ is a directed and hereditary Bratteli diagram as in [1, Lemma 3.2] (see also Figure 7). Thus there is an ideal I_θ in $\mathfrak{A}$ such that $D(I_\theta) = \mathcal{G} \setminus \mathcal{G}_\theta$, $D(\mathfrak{A}/I_\theta) = \mathcal{G}_\theta$, and $\mathfrak{A}/I_\theta \cong \mathfrak{F}_\theta$. Moreover I_θ is a primitive ideal [1, Theorem 3.8].

If $j_n = j_n(\theta)$ is the unique index for which $r(n, j_n) < \theta < r(n, j_n + 1)$ (see Figure 7), then

$$I_\theta \cap \mathfrak{A}_n = \bigoplus_{\substack{0 \le k \le 2^n \\ k \ne j_n, j_{n+1}}} \mathbb{M}_{q(n,k)}.$$

The vertices of $D(\mathfrak{A}/I_\theta)$ are explicitly related to the continued fraction decomposition of θ. For each $r \in \mathbb{Q}_{(0,1)}$, denote $ht(r) = \min\{n : \exists k,\ r(n,k) = r\}$. Let p_n/q_n be the continued fraction approximations of θ, and $h_n = ht(p_n/q_n)$. With this notation, the labels of the two vertices at floor m in $\mathcal{G}_\theta$ are

$$\frac{p_n}{q_n} \quad \text{and} \quad \frac{p_{n-1} + (m - h_n)p_n}{q_{n-1} + (m - h_n)q_n}$$

whenever $h_n \le m < h_{n+1}$.

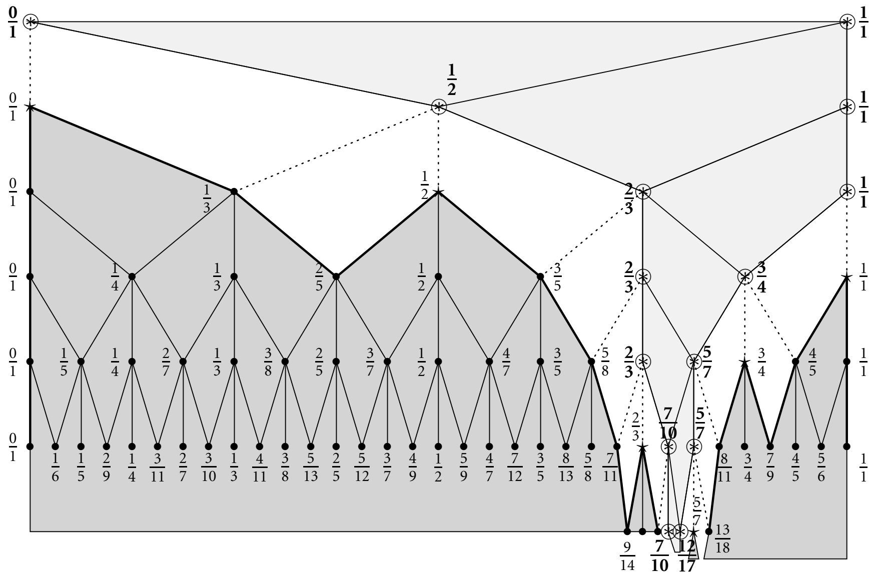

Figure 7: The diagrams $D(I_\theta) = \mathcal{G} \setminus \mathcal{G}_\theta$ (darker) and $D(\mathfrak{A}/I_\theta) = \mathcal{G}_\theta$ (lighter) when $\theta = [1, 2, 2, 1, 1, \dots]$.

(ii) For each $\theta = p/q \in \mathbb{Q}_{(0,1)}$ in lowest terms, consider the Bratteli subdiagram $\mathcal{G}_\theta$ of $\mathcal{G}$ defined by all vertices (n, j) with $r(n, j) = \theta$ and (m, i) with $(m, i) \Downarrow (n, j)$. The AF algebra associated with $\mathcal{G}_\theta$ is clearly isomorphic to $\mathbb{M}_q$. Again, the complement $\mathcal{G} \setminus \mathcal{G}_\theta$ is seen to be a directed and hereditary Bratteli diagram. Therefore there is a primitive ideal I_θ in $\mathfrak{A}$ such that $D(I_\theta) = \mathcal{G} \setminus \mathcal{G}_\theta$ and $\mathfrak{A}/I_\theta \simeq \mathbb{M}_q$.

Let $n_0 - 1 = n_0(\theta) - 1$ be the largest $n \in \mathbb{N}$ for which there exists $j = j_n(\theta)$ such that $r(n, j) < \theta < r(n, j + 1)$. For $n < n_0$ define j_n as above. By the choice of n_0 and the properties of the Pascal triangle with memory, for every $n \geq n_0$ there is $j_n = j_n(\theta)$ with $r(n, j_n) = \theta$. The ideal $I_{p/q}$ is generated by the direct summands $\mathbb{M}_{q(n_0, j_{n_0}-1)}$, $\mathbb{M}_{q(n_0, j_{n_0}+1)}$ and $\mathbb{M}_{q(n,c_n)}$, $n < n_0$, that is,

$$I_{\frac{p}{q}} \cap \mathfrak{A}_n = \begin{cases} \displaystyle\bigoplus_{\substack{0 \leq k \leq 2^n \\ k \neq j_n, j_{n+1}}} \mathbb{M}_{q(n,k)} & \text{if } n < n_0, \\ \displaystyle\bigoplus_{\substack{0 \leq k \leq 2^n \\ k \neq j_n}} \mathbb{M}_{q(n,k)} & \text{if } n \geq n_0. \end{cases}$$

The ideals $I^{\pm}_{\frac{p}{q}}$ defined by (see also Figures 9 and 10)

$$I^{+}_{\frac{p}{q}} \cap \mathfrak{A}_n = \bigoplus_{\substack{0 \leq k \leq 2^n \\ k \neq j_n, j_{n+1}}} \mathbb{M}_{q(n,k)},$$

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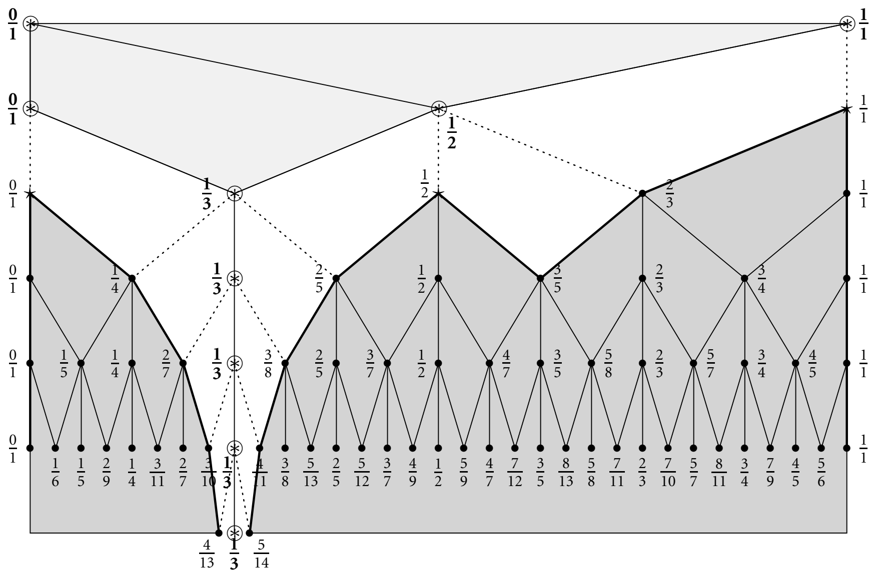

Figure 8: The diagrams $D(I_{\frac{1}{3}})$ (darker) and $D(\mathfrak{A}/I_{\frac{1}{3}})$ (lighter).

and respectively by

$$I^-_{\frac{p}{q}} \cap \mathfrak{A}_n = \begin{cases} \displaystyle\bigoplus_{\substack{0 \le k \le 2^n \\ k \ne j_n, j_{n+1}}} \mathbb{M}_{q(n,k)} & \text{if } n < n_0, \\ \displaystyle\bigoplus_{\substack{0 \le k \le 2^n \\ k \ne j_{n-1}, j_n}} \mathbb{M}_{q(n,k)} & \text{if } n \ge n_0, \end{cases}$$

are primitive, and we clearly have $\mathfrak{A}/I^-_{\frac{p}{q}} \cong \mathfrak{A}_{(q,\overline{p})}$ and $\mathfrak{A}/I^+_{\frac{p}{q}} \cong \mathfrak{A}_{(q,q-\overline{p})}$. (iii) is now obvious. ∎

Remark 5. A joint (and important) feature of all cases above is that

$$(n, j) \notin D(I_\theta) = \mathcal{G} \setminus \mathcal{G}_\theta \Longrightarrow r(n, j-1) < \theta < r(n, j+1).$$

Remark 6. In $GL_2(\mathbb{Z})$ consider the matrices

$$A \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad M(a) = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}.$$

The identification between $L_a \circ R_b$ and $C_a \circ C_b$ reflects the matrix equality

$$B^a A^b = M(a)M(b),$$

whereas the identification between $R_a \circ R_b$ and $C_a \circ C_b$ reflects the matrix equality

$$A^a B^b = JM(a)M(b)J.$$

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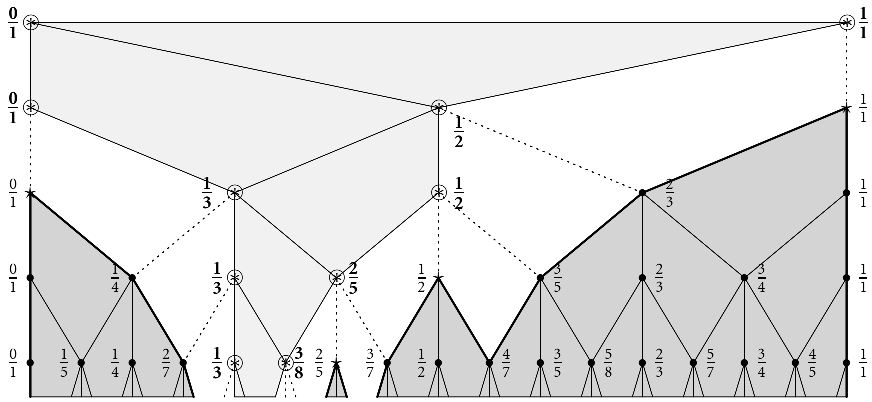

Figure 9: The diagrams $D(I^+_{\frac{1}{3}})$ (darker) and $D(\mathfrak{A}/I^+_{\frac{1}{3}})$ (lighter).

A combinatorial analysis based on Bratteli's correspondence between primitive ideals and subdiagrams of $\mathcal{G}$ shows that these are actually the only primitive ideals of $\mathfrak{A}$.

Proposition 7 Prim $\mathfrak{A} = \{I_\theta : \theta \in \mathbb{I}\} \cup \{I_\theta, I_\theta^\pm : \theta \in \mathbb{Q}_{(0,1)}\} \cup \{I_0, I_0^+, I_1, I_1^-\}$.

Proof Let $I \in$ Prim $\mathfrak{A}$. Consider the Bratteli diagrams $D = D(I)$ and $\widetilde{D} = D(\mathfrak{A}/I) = \mathcal{G} \setminus D$. If there is n_0 such that $(n_0, k) \in D$ for all $0 \leq k \leq 2^{n_0}$, then $I = \mathfrak{A}$. So for each n, the set $L_n = \{k : (n,k) \in \widetilde{D}\}$ is nonempty. Denote also $L_n^c = \{0, 1, \dots, 2^n\} \setminus L_n$.

We first notice that L_n must be a set of the form $\{a_n\}$ or $\{a_n, a_n + 1\}$. If not, there are $k, k' \in L_n$ such that $k' - k \geq 2$. Since I is a primitive ideal, a vertex (p, r) in $\mathcal{G}$ should exist such that $(n,k) \Downarrow (p,r)$ and $(n,k') \Downarrow (p,r)$. Since $k' - k > 2$, this is not possible due to the definition of $\mathcal{G}$.

To finish the proof it suffices to show that

$$L_{n+1} = \begin{cases} \{2a_n\} & \text{if } L_n = \{a_n\}, \\ \{2a_n, 2a_n + 1\},\ \{2a_n + 1, 2a_n + 2\}, & \\ \qquad\qquad\qquad \text{or } \{2a_n + 1\} & \text{if } L_n = \{a_n, a_n + 1\}, \end{cases} \tag{2.2}$$

that is, all links $(n, j) \downarrow (n+1, j')$ in $\widetilde{D}$ are exactly as indicated in Figure 11.

Indeed, if $L_n = \{a_n\}$, then $(n, a_n - 1), (n, a_n + 1)$ are vertices in the hereditary diagram D; thus we also have $(n + 1, 2a_n - 1), (n + 1, 2a_n + 1) \in D$. Because D is directed, $(n + 1, 2a_n) \in D$ would imply $(n, a_n) \in D$, which contradicts $a_n \in L_n$.

If $L_n = \{a_n, a_n + 1\}$, then $(n, a_n - 1), (n, a_n + 2) \in D$. Moreover, because D is hereditary the vertices $(n + 1, 2a_n - 1)$ and $(n + 1, 2a_n + 3)$ also belong to D. We now look at the consecutive vertices $(n + 1, 2a_n), (n + 1, 2a_n + 1), (n + 1, 2a_n + 2)$. From the first part, they cannot all belong to $\widetilde{D}$. If $(n + 1, 2a_n + 1) \in D$ and $(n + 1, 2a_n), (n + 1, 2a_n + 2) \in \widetilde{D}$, then L_{n+1} has a gap, thus contradicting the first

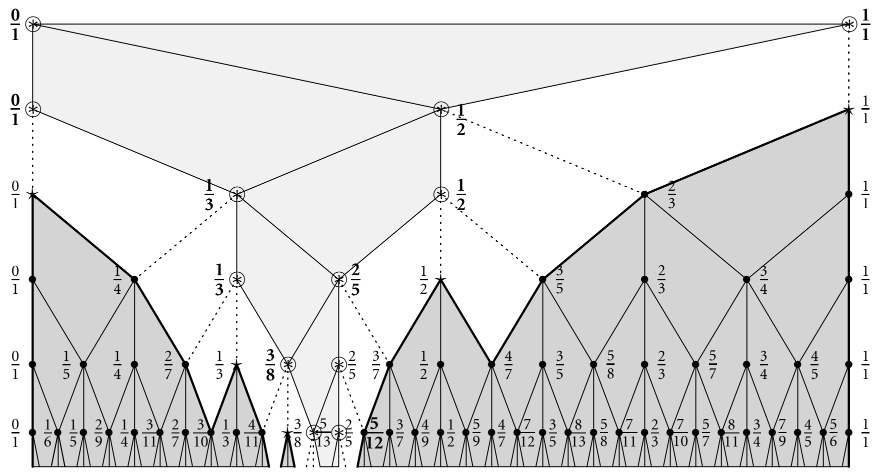

Figure 10: The diagrams $D(I^-_{\frac{2}{5}})$ (darker) and $D(\mathfrak{A}/I^-_{\frac{2}{5}})$ (lighter).

part. If $(n+1, 2a_n), (n+1, 2a_n+2) \in D$, it follows as a result of the fact that $(n+1, 2a_n-1) \in D$ and that D is directed that $(n+1, 2a_n+1) \in \widetilde{D}$. In a similar way one cannot have $(n+1, 2a_n+1), (n+1, 2a_n+2) \in D$. It remains that only the following cases can occur (see also Figure 11):

(i) $(n+1, 2a_n), (n+1, 2a_n+1) \in \widetilde{D}$ and $(n+1, 2a_n+2) \in D$, thus $L_{n+1} = \{2a_n, 2a_n+1\}$.

(ii) $(n+1, 2a_n) \in D$ and $(n+1, 2a_n+1), (n+1, 2a_n+2) \in \widetilde{D}$, thus $L_{n+1} = \{2a_n+1, 2a_n+2\}$.

(iii) $(n+1, 2a_n+1) \in \widetilde{D}$ and $(n+1, 2a_n), (n+1, 2a_n+2) \in D$, thus $L_{n+1} = \{2a_n+1\}$,

which concludes the proof of (2.2). ∎

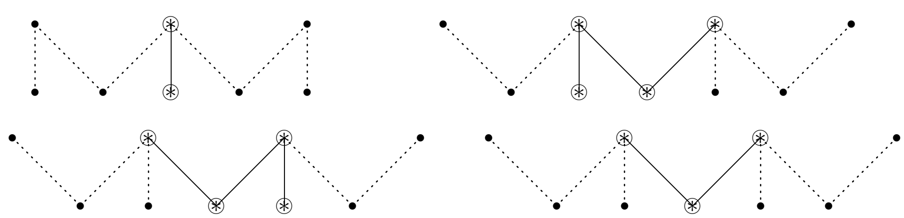

Figure 11: The possible links between two consecutive floors in $D(\mathfrak{A}/I)$.

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3 The Jacobson Topology on $\operatorname{Prim}\mathfrak{A}$

We first recall some basic things about the primitive ideal space of a C^*-algebra $\mathcal{A}$ following [8, 23]. For each set $S \subseteq \operatorname{Prim}\mathcal{A}$, consider the ideal $k(S) := \bigcap_{J\in S} J$ in $\mathcal{A}$, called the *kernel* of S. For each ideal I consider its *hull*, $h(I) := \{P \in \operatorname{Prim}\mathcal{A} : I \subseteq P\}$. The *closure* of a set $S \subseteq \operatorname{Prim}\mathcal{A}$ is defined as $\overline{S} := \{P \in \operatorname{Prim}\mathcal{A} : k(S) \subseteq P\}$. There is a unique topology on $\operatorname{Prim}\mathcal{A}$, called the *Jacobson* (or *hull-kernel*) *topology* such that its closed sets are exactly those with $S = \overline{S}$. The open sets in $\operatorname{Prim}\mathcal{A}$ are then precisely those of the form $\mathcal{O}_I := \{P \in \operatorname{Prim}\mathcal{A} : I \nsubseteq P\}$ for some ideal I in $\mathcal{A}$. The Jacobson topology is always T_0, *i.e.,* for any two distinct points in $\operatorname{Prim}\mathcal{A}$, one of them has a neighborhood which does not contain the other.

Moreover, the correspondence $S \mapsto k(S)$ establishes a one-to-one correspondence between the closed subsets S of $\operatorname{Prim}\mathcal{A}$ and the lattice of ideals in $\mathcal{A}$, with inverse given by $I \mapsto h(I)$. For any ideal I in $\mathcal{A}$, let p_I denote the quotient map $\mathcal{A} \to \mathcal{A}/I$. The mapping $P \mapsto P\cap I$ is a homeomorphism of the open set $\mathcal{O}_I$ onto $\operatorname{Prim} I$, whereas $Q \mapsto p_I^{-1}(Q)$ is a homeomorphism of $\operatorname{Prim}\mathcal{A}/I$ onto the closed set $h(I)$ of $\operatorname{Prim}\mathcal{A}$. A general study of the primitive ideal space of AF algebras was pursued in [2, 4, 9].

We collect some immediate properties of the primitive ideal space of $\mathfrak{A}$ in the following.

Remark 8. (i) For each $\theta \in \mathbb{I}$, $\overline{\{I_\theta\}} = \{I_\theta\}$.

(ii) For each $\theta \in \mathbb{Q}_{(0,1)}$, $I_\theta \nsubseteq I_\theta^+$, $I_\theta \nsubseteq I_\theta^-$, and $I_\theta = I_\theta^+ \cap I_\theta^-$. We also have $I_0 \nsubseteq I_0^+$ and $I_1 \nsubseteq I_1^-$. Therefore $\overline{\{I_\theta\}} = \{I_\theta, I_\theta^+, I_\theta^-\}$ whenever $\theta \in \mathbb{Q}_{(0,1)}$, $\overline{\{I_0\}} = \{I_0, I_0^+\}$ and $\overline{\{I_1\}} = \{I_1, I_1^-\}$, showing in particular that the Jacobson topology on $\operatorname{Prim}\mathfrak{A}$ is not Hausdorff. In spite of this we shall see that after removing the "singular points" $I_\theta^\pm$ from $\operatorname{Prim}\mathfrak{A}$ we retrieve the usual topology on $[0, 1]$.

For each set $E \subseteq [0, 1]$, consider the ideal

$$\mathfrak{I}(E) := \bigcap_{\theta\in E} I_\theta, \tag{3.1}$$

and denote by $\overline{E}$ the usual closure of E in $[0, 1]$.

Lemma 9 $\mathfrak{I}(E) = \mathfrak{I}(\overline{E})$ *for every set* $E \subseteq [0, 1]$.

Proof The inclusion $\mathfrak{I}(\overline{E}) \subseteq \mathfrak{I}(E)$ is obvious by (3.1). We prove $\mathfrak{I}(E) \subseteq I_x$ for all $x \in \overline{E}$. Suppose *ad absurdum* there is $x \in \overline{E}$ for which $\mathfrak{I}(E) \nsubseteq I_x$, *i.e.,* there is $(n, j) \in \mathcal{V}$ with $(n, j) \in D(\mathfrak{I}(E))$ and $(n, j) \notin D(I_x)$. The latter and Remark 5 yield

$$r(n, j - 1) < x < r(n, j + 1). \tag{3.2}$$

On the other hand, because $D(\mathfrak{I}(E))$ contains (n, j), every diagram $D(I_\theta)$, $\theta \in E$, must contain the whole "pyramid" starting at (n, j), see Figure 12. Thus

$$\forall\theta \in E, \forall k \geq 1, \quad \theta \in [0, r(n + k, 2^k j - 2^k + 1), 1] \cup [r(n + k, 2^k j + 2^k - 1), 1].$$

But

$$r(n+k, 2^k j + 2^k - 1) = \frac{kp(n, j+1) + p(n, j)}{kq(n, j+1) + q(n, j)} \xrightarrow{k} \frac{p(n, j+1)}{q(n, j+1)} = r(n, j+1),$$
$$r(n+k, 2^k j - 2^k + 1) = \frac{kp(n, j-1) + p(n, j)}{kq(n, j-1) + q(n, j)} \xrightarrow{k} \frac{p(n, j-1)}{q(n, j-1)} = r(n, j-1),$$

hence

$$E \subseteq [0, r(n, j-1)] \cup [r(n, j+1), 1],$$

which is in contradiction with (3.2). ■

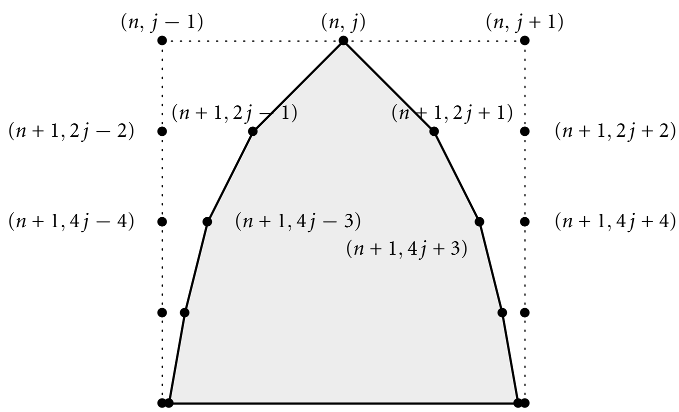

Figure 12: The ideal generated by (n, j).

Remark 10. We have $q(n, 2j) = q(n-1, j) < \min\{q(n, 2j-1), q(n, 2j+1)\}$, so if $r(n, 2j) = p/q$, then

$$r(n, 2j+1) - r(n, 2j-1) = \frac{1}{q(n, 2j-1)q(n, 2j)} + \frac{1}{q(n, 2j)q(n, 2j+1)} < \frac{2}{q^2}.$$

One can give a better estimate as follows. Let $\theta = p/q \in (0, 1)$ be a rational number in lowest terms and let $\bar{p} \in \{1, \dots, q-1\}$ denote the multiplicative inverse of p modulo q. Let $n_0 = n_0(\theta)$ be the smallest n such that $\theta = r(n, j_0)$ for some j_0. Then j_0 is odd and the labels $r' = p'/q'$ and respectively $r'' = p''/q''$ of the "left parent" $(n_0 - 1, \frac{j_0-1}{2})$ and respectively of the "right parent" $(n_0 - 1, \frac{j_0+1}{2})$ of the vertex (n_0, j_0), are given by $(q', p') = (\bar{p}, \frac{p\bar{p}-1}{q})$, and respectively by $(q'', p'') = \left(q - \bar{p}, p - \frac{p\bar{p}-1}{q}\right) = (q, p) - (q', p')$. Furthermore, we have

$$r(n_0 + k, 2^k j_0 - 1) = \frac{p + kp'}{q + kq'}, \quad r(n_0 + k, 2^k j_0 + 1) = \frac{p + kp''}{q + kq''},$$

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and

$$\max\{ r(n_0+k, 2^k j_0+1) - p/q, p/q - r(n_0+k, 2^k j_0-1)\} < \frac{1}{kq^2}.$$

Lemma 11 *For some $x \in [0,1]$ and $S \subseteq [0,1]$ suppose $\mathfrak{I}(S) \subseteq \mathfrak{I}_x$. Then $x \in \overline{S}$.*

Proof Obviously two cases may occur.

Case 1: $x \notin \mathbb{Q}$. Let (p_n/q_n) denote the sequence of continued fraction approximations of x. Taking stock on the definition of the ideal $\mathfrak{I}_x$, we get positive integers $k_1 < k_2 < \cdots$ and vertices $(k_n, j_n) \in D(\mathfrak{A})$ with the following properties:

(i) $r(k_n, j_n) = p_n/q_n$;
(ii) j_n is even;
(iii) $(k_n, j_n) \notin D(\mathfrak{I}_x)$.

Actually (iii) is a plain consequence of (i) and gives in turn, *cf.* Remark 5,

$$r(k_n, j_n - 1) < x < r(k_n, j_n + 1). \tag{3.3}$$

Case 2: $x \in \mathbb{Q}$. There is n_0 such that $(n, j_n) \notin D(\mathfrak{I}_x)$ and $r(n, j_n) = x$ for all $n \geq n_0$. In this case we take $k_n = n$.

Suppose that $\exists n \geq n_0, \forall \theta \in S, (k_n, j_n) \in D(\mathfrak{I}_\theta)$. Then $(k_n, j_n) \in D(\mathfrak{I}(S)) \setminus D(\mathfrak{I}_x)$, which contradicts the assumption of the lemma. Therefore we must have

$$\forall n, \exists \theta_n \in S, (k_n, j_n) \notin D(\mathfrak{I}_{\theta_n}),$$

which according to Remark 5 gives

$$r(k_n, j_n - 1) < \theta_n < r(k_n, j_n + 1). \tag{3.4}$$

From (3.3), (3.4) and Remark 10 we now infer

$$|x - \theta_n| < r(k_n, j_n + 1) - r(k_n, j_n - 1) < \frac{2}{q_n^2}, \quad \forall n \geq n_0,$$

and so $\operatorname{dist}(x, S) = 0$. This concludes the proof of the lemma. ∎

As a consequence, the Jacobson topology is Hausdorff when restricted to the subset $\operatorname{Prim}_0 \mathfrak{A} = \{I_\theta : \theta \in [0,1]\}$ of $\operatorname{Prim} \mathfrak{A}$. Moreover, we have the following.

Corollary 12 *Let (θ_n) be a sequence in $[0,1]$. The following are equivalent:*

(i) $\theta_n \to \theta$ *in* $[0,1]$.
(ii) $I_{\theta_n} \to I_\theta$ *in* $\operatorname{Prim} \mathfrak{A}$.

Proof (i) Suppose $\theta_n \to \theta$ in $[0,1]$ but $I_{\theta_n} \nrightarrow I_\theta$ in $\operatorname{Prim}\mathfrak{A}$. Then there is I ideal in $\mathfrak{A}$ such that $I \nsubseteq I_\theta$ and there is a subsequence (n_k) such that $I_{\theta_{n_k}} \notin \mathcal{O}_I$, so that $I \subseteq I_{\theta_{n_k}}$. By Lemma 9 this also yields $I \subseteq I_\theta$, which is a contradiction.

(ii) Suppose $I_{\theta_n} \to I_\theta$ in $\operatorname{Prim}\mathfrak{A}$ but $\theta_n \nrightarrow \theta$ in $[0,1]$. Then there is a subsequence (n_k) such that $\theta \notin \overline{\{\theta_{n_k}\}_k}$. By Lemma 11 we have $I := \bigcap_k I_{\theta_{n_k}} \nsubseteq I_\theta$, and so $I_\theta \in \mathcal{O}_I$. But on the other hand $I \subseteq I_{\theta_{n_k}}$, *i.e.*, $I_{\theta_{n_k}} \notin \mathcal{O}_I$ for all k, thus contradicting $I_{\theta_{n_k}} \to I_\theta$. ∎

4 A Description of the Dimension Group

By a classical result of Elliott ([12], see also [10]), AF algebras are classified up to isomorphism by their dimension groups. In this section we give a description of the dimension group $K_0(\mathfrak{C})$ of the codimension one ideal $\mathfrak{C} = I_1$ of $\mathfrak{A}$ obtained by erasing all vertices $(n, 2^n)$ from the Bratteli diagram. This is inspired by the generating function identity [6]

$$\sum_{n\geq 0} \theta_n X^n = \prod_{k\geq 0} (1 + X^{2^k} + X^{2^{k+1}}),$$

where $(\theta_n)_{n=0}^{\infty}$ is the Stern–Brocot sequence $q(0,0)$, $q(1,0)$, $q(1,1)$, $q(2,0)$, $q(2,1)$, $q(2,2)$, $q(2,3), \dots, q(n,0), \dots, q(n,2^n-1), q(n+1,0), \dots$.

For each integer $n \geq 0$, set

$$p_{(n,k)}(X) := \begin{cases} 1 & \text{if } k = 0, \\ X^k + X^{-k} & \text{if } 1 \leq k < 2^n, \end{cases}$$

and consider the abelian additive group

$$\mathcal{P}_n := \Big\{ \sum_{0\leq k<2^n} c_k p_{(n,k)} : c_k \in \mathbb{Z} \Big\}.$$

Set

$$\varrho(X) = X^{-1} + 1 + X, \quad \varrho_n(X) = \prod_{0\leq k<n} \varrho(X^{2^k}),$$

and define the injective group morphisms

$$\beta_m \colon \mathcal{P}_m \to \mathcal{P}_{m+1},\ (\beta_m(p))(X) = \varrho(X) p(X^2),$$

$$\beta_{m,n} \colon \mathcal{P}_m \to \mathcal{P}_n,\ (\beta_{m,n}(p))(X) = (\beta_{n-1} \cdots \beta_m(p))(X) = \varrho_{m-n}(X) p(X^{2^{n-m}}),\ m < n.$$

Note that

$$(4.1) \qquad (\beta_n(p_{(n,k)}))(X) = \varrho(X) p_{(n,k)}(X^2)$$
$$= \begin{cases} p_{(n+1,0)}(X) + p_{(n+1,1)}(X) & \text{if } k = 0, \\ p_{(n+1,2k-1)}(X) + p_{(n+1,2k)}(X) + p_{(n+1,2k+1)}(X) & \text{if } 1 \leq k < 2^n. \end{cases}$$

The group $K_0(\mathfrak{C}_n)$ identifies with the free abelian group $\mathbb{Z}^{2^n}$, generated by the Murray–von Neumann equivalence classes $[e_{(n,k)}]$ of minimal projections $e_{(n,k)}$ in the central summand $\mathfrak{C}_{(n,k)}$, $0 \leq k < 2^n$. We have $K_0(\mathfrak{C}) = \varinjlim (K_0(\mathfrak{C}_n), \alpha_n)$, the injective morphisms $\alpha_n \colon K_0(\mathfrak{C}_n) \to K_0(\mathfrak{C}_{n+1})$ being given by

$$\alpha_n([e_{(n,k)}]) = \begin{cases} [e_{(n+1,0)}] + [e_{(n+1,1)}] & \text{if } k = 0, \\ [e_{(n+1,2k-1)}] + [e_{(n+1,2k)}] + [e_{(n+1,2k+1)}] & \text{if } 1 \leq k < 2^n. \end{cases}$$

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The positive cone $K_0(\mathfrak{C}_n)^+$ consists of elements of form $\sum_{k=0}^{2^n-1} c_k[e_{(n,k)}]$, $c_k \in \mathbb{Z}_+$. The groups $K_0(\mathfrak{C}_n)$ and $\mathcal{P}_n$ are identified by the group isomorphism ϕ_n mapping $[e_{(n,k)}]$ onto $p_{(n,k)}$. Equalities (4.1) are reflected into the commutativity of the diagram

$$\begin{tikzcd} K_0(\mathfrak{C}_n) \arrow[r, "\phi_n"] \arrow[d, "\alpha_n"'] & \mathcal{P}_n \arrow[d, "\beta_n"] \\ K_0(\mathfrak{C}_{n+1}) \arrow[r, "\phi_{n+1}"'] & \mathcal{P}_{n+1} \end{tikzcd} \tag{4.2}$$

As a result, $K_0(\mathfrak{C})$ is isomorphic with the abelian group $\mathcal{P} = \varinjlim(\mathcal{P}_n, \beta_n)$ and can, therefore, be described as $(\cup_n \mathcal{P}_n)/_\sim = \mathbb{Z}[X + X^{-1}]/_\sim$, where $\sim$ is the equivalence relation given by equality on each $\mathcal{P}_n \times \mathcal{P}_n$, and for $p \in \mathcal{P}_m$, $q \in \mathcal{P}_n$, $m < n$, by

$$p \sim q \iff q(X) = (\beta_{m,n}(p))(X) = p(X^{2^{n-m}}) \prod_{0 \le k < n-m} (X^{-2^k} + 1 + X^{2^k}).$$

Let $[p]$ denote the equivalence class of $p \in \bigcup_n \mathcal{P}_n$. The addition on $\mathcal{P}$ is given by

$$[p] + [q] = [\beta_{m,n}(p) + q], \quad p \in \mathcal{P}_m,\ q \in \mathcal{P}_n,\ m \le n,$$

and does not depend on the choice of m or n. For example

$$\begin{aligned} [X^{-1} + X] + [X^{-3} + X^3] &= [(X^{-1} + 1 + X)(X^{-2} + X^2) + X^{-3} + X^3] \\ &= [2(X^{-3} + X^3) + (X^{-2} + X^2) + (X^{-1} + X)]. \end{aligned}$$

An element $[p]$, $p \in \mathcal{P}_n$, belongs to the positive cone $\mathcal{P}^+$ of the dimension group precisely when there is an integer $N > n$ such that $\beta_{n,N}(p)$ has nonnegative coefficients. The equality (where $c_{r+1} = 0$)

$$\begin{aligned} (X^{-1} + 1 + X) \sum_{0 \le k < 2^n} c_k(X^{2k} + X^{-2k}) = \sum_{0 \le k < 2^n} c_k(X^{2k} + X^{-2k}) \\ + \sum_{0 \le k < 2^n} (c_k + c_{k+1})(X^{2k+1} + X^{-2k-1}) \end{aligned}$$

shows that $p(X)$ has nonnegative coefficients if and only if $\varrho(X)p(X^2)$ has the same property. Therefore $[p] \in \mathcal{P}^+$ precisely when $p(X)$ has nonnegative coefficients.

Consider the positive integers $q'_{(n,k)}$, $n \ge 0$, $0 \le k < 2^n$, describing the sizes of central summands in

$$\mathfrak{C}_n = \bigoplus_{0 \le k < 2^n} \mathbb{M}_{q'_{(n,k)}}, \tag{4.3}$$

that is

$$\begin{aligned} q'_{(n,0)} &= q'_{(n,2^n-1)} = 1, \\ q'_{(n,2k)} &= q'_{(n-1,k)}, \\ q'_{(n,2k+1)} &= q'_{(n-1,k)} + q'_{(n-1,k+1)}, \quad 0 \le k < 2^n. \end{aligned}$$

For instance $q'(3,k)$, $0 \le k \le 7$, are given by $1,3,2,3,1,2,1,1$, and $q'(4,k)$, $0 \le k \le 15$, by $1,4,3,5,2,5,3,4,1,3,2,3,1,2,1,1$. From (4.3) we have

$$\sum_{0 \le k < 2^n} q'(n,k)[e_{(n,k)}] = [1] \quad \text{in } K_0(\mathfrak{C}).$$

This corresponds to

$$\sum_{0 \le k < 2^n} q'(n,k) p_{(n,k)}(X) = \varrho_n(X).$$

One can give a representation of $K_0(\mathfrak{C})$ where the injective maps β_n in (4.2) are replaced by inclusions $\iota_n(p) = p$. Define

$$\phi_{(n,k)}(X) = \frac{p_{(n,k)}(X^{1/2^n})}{\varrho_{(n,k)}(X^{1/2^n})} = \begin{cases} \dfrac{1}{\prod_{j=1}^n (X^{-1/2^j} + 1 + X^{1/2^j})} & \text{if } k = 0, \\ \dfrac{X^{k/2^n} + X^{-k/2^n}}{\prod_{j=1}^n (X^{-1/2^j} + 1 + X^{1/2^j})} & \text{if } 1 \le k < 2^n, \end{cases}$$

and consider the additive abelian group

$$\mathcal{R}_n := \Big\{ \sum_{0 \le k < 2^n} c_k \phi_{(n,k)} : c_k \in \mathbb{Z} \Big\}.$$

The equalities (4.1) become

$$\begin{aligned} &\phi_{(n+1,0)} + \phi_{(n+1,1)} = \phi_{(n,0)}, \\ &\phi_{(n+1,2k-1)} + \phi_{(n+1,2k)} + \phi_{(n+1,2k+1)} = \phi_{(n,k)}, \quad 1 \le k < 2^n, \end{aligned}$$

and show that $\mathcal{R}_n \subseteq \mathcal{R}_{n+1}$ and that the diagram

$$\begin{tikzcd} K_0(\mathfrak{C}_n) \arrow[r, "\psi_n"] \arrow[d, "\alpha_n"'] & \mathcal{R}_n \arrow[d, hook, "\iota_n"] \\ K_0(\mathfrak{C}_{n+1}) \arrow[r, "\psi_{n+1}"'] & \mathcal{R}_{n+1} \end{tikzcd}$$

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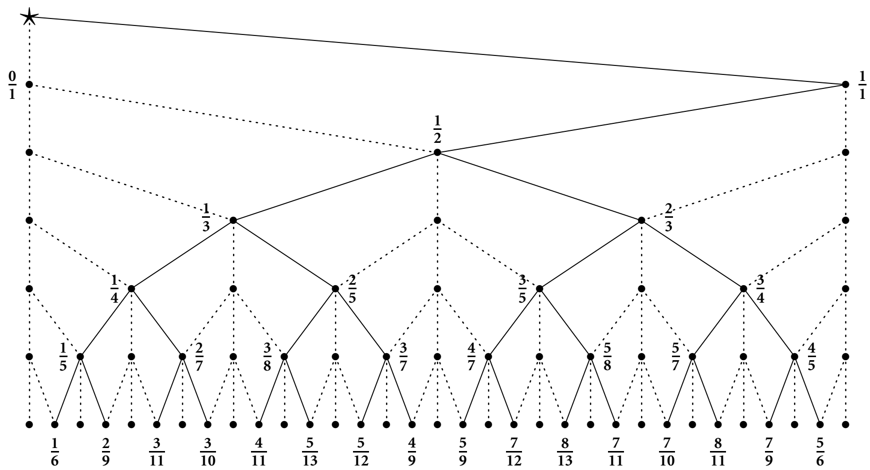

Figure 13: The diagram $\mathcal{T}$.

is commuting, where $\psi([e_{(n,k)}]) = \phi_{(n,k)}$. Therefore $K_0(\mathfrak{C}) = \mathcal{R} := \bigcup_n \mathcal{R}_n$. Taking $X = \mathrm{e}^Y$, we see that $K_0(\mathfrak{C})$ can be viewed as the $\mathbb{Z}$-linear span of $\widetilde{\phi}_{(n,k)}$, $n \geq 0$, $0 \leq k < 2^n$, where

$$\widetilde{\phi}_{(n,k)}(Y) = \begin{cases} \dfrac{1}{\prod_{j=1}^{n}(1 + 2\cosh(Y/2^j))} & \text{if } k = 0, \\ \dfrac{2\cosh(kY/2^n)}{\prod_{j=1}^{n}(1 + 2\cosh(Y/2^j))} & \text{if } 1 \leq k < 2^n. \end{cases}$$

One can certainly replace Y by iY and use cos instead of cosh.

5 Traces on $\mathfrak{A}$

We augment the diagram $\mathcal{G} = D(\mathfrak{A})$ into $\widetilde{\mathcal{G}}$, by adding a (-1)-st floor with only one vertex $\star = (-1, 0)$ connected to both $(0, 0)$ and $(0, 1)$. Traces τ on $\mathfrak{A}$ are in one-to-one correspondence [13, Section 3.6] with families $\alpha^\tau = (\alpha^\tau_{(n,k)})$ of numbers in $[0, 1]$, $n \geq -1$, $0 \leq k \leq 2^n$, such that

$$\begin{aligned} \alpha^\tau_\star &= 1, \\ \alpha^\tau_{(n,0)} &= \alpha^\tau_{(n+1,0)} + \alpha^\tau_{(n+1,1)} && \text{if } n \geq -1, \\ \alpha^\tau_{(n,2^n)} &= \alpha^\tau_{(n+1,2^{n+1})} + \alpha^\tau_{(n+1,2^{n+1}-1)} && \text{if } n \geq 0, \\ \alpha^\tau_{(n,k)} &= \alpha^\tau_{(n+1,2k-1)} + \alpha^\tau_{(n+1,2k)} + \alpha^\tau_{(n+1,2k+1)} && \text{if } n \geq 1,\ 0 < k < 2^n. \end{aligned}$$

An inspection of $\widetilde{\mathcal{G}}$ shows that such a family α^τ is uniquely determined by the numbers $\alpha^\tau_{(n,k)}$ with odd k. Let $\mathcal{T}$ denote the diagram obtained by removing the

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memory in $\widetilde{\mathcal{G}}$. Its set of vertices $V(\mathcal{T})$ consists of $\star$ and (n,k) with $n \geq 0$ and odd k. For $v = (n,k)$ define $Lv = (n+1, 2k-1)$ if $n \geq 0$, $0 < k \leq 2^n$, and $Rv = (n+1, 2k+1)$ if $n \geq -1$, $0 \leq k < 2^n$.

Given α_v^τ, $v = (n,k) \in V(\mathcal{T})$, define recursively for $r \geq 1$,

$$\begin{aligned} \alpha^\tau_{(n+r,0)} &= \alpha^\tau_{(n+r-1,0)} - \alpha^\tau_{(n+r,1)} && \text{if } n \geq -1, \\ \alpha^\tau_{(n+r,2^{n+r})} &= \alpha^\tau_{(n+r-1,2^{n+r-1})} - \alpha^\tau_{(n+r,2^{n+r}-1)} && \text{if } n \geq 0, \\ \alpha^\tau_{(n+r,2^r k)} &= \alpha^\tau_{(n+r-1,2^{r-1}k)} - \alpha^\tau_{(n+r,2^r k-1)} - \alpha^\tau_{(n+r,2^r k+1)} && \text{if } n \geq 1, \end{aligned}$$

or equivalently

$$\begin{aligned} \alpha^\tau_{(n,0)} &= \alpha^\tau_\star - \sum_{j=0}^n \alpha^\tau_{(j,1)} = \alpha^\tau_\star - \sum_{j=0}^n \alpha^\tau_{L^j R\star} && \text{if } n \geq 0, \\ \alpha^\tau_{(n,2^n)} &= \alpha^\tau_{(0,1)} - \sum_{j=1}^n \alpha^\tau_{(j,2^j-1)} = \alpha^\tau_{(0,1)} - \sum_{j=1}^n \alpha^\tau_{R^{j-1}L(0,1)} && \text{if } n \geq 1, \\ \alpha^\tau_{(n+r,2^r k)} &= \alpha^\tau_{(n,k)} - \sum_{j=1}^r (\alpha^\tau_{(n+j,2^j k-1)} + \alpha^\tau_{(n+j,2^j k+1)}) \\ &= \alpha^\tau_{(n,k)} - \sum_{j=1}^r \left(\alpha^\tau_{R^{j-1}L(n,k)} + \alpha^\tau_{L^{j-1}R(n,k)} \right) && \text{if } n \geq 2. \end{aligned} \tag{5.1}$$

There is an obvious order relation on $V(\mathcal{T})$ defined by $(n,k_n) \preceq (n',k'_n)$ if $n \leq n'$ and there is a chain of vertices $(n,k_n), \dots, (n',k'_n)$ such that $(n+i, k_{n+i})$ is connected to $(n+i+1, k_{n+i+1})$, *i.e.*, $k_{n+i+1} - 2k_{n+i} = \pm 1$. A function $f \colon V(\mathcal{T}) \to \mathbb{R}$ is monotonically decreasing if $f(v_1) \geq f(v_2)$ whenever $v_1 \preceq v_2$ in $V(\mathcal{T})$. For each vertex $v = (n,k) \in V(\mathcal{T})$, let

$$\mathcal{C}_v = \begin{cases} \{L^j R\star : j \geq 0\} & \text{if } v = \star, \\ \{R^{j-1}L(0,1) : j \geq 1\} & \text{if } v = (0,1), \\ \{R^{j-1}Lv : j \geq 1\} \cup \{L^{j-1}Rv : j \geq 1\} & \text{if } v \in V(\mathcal{T}) \setminus \{\star, (0,1)\}, \end{cases} \tag{5.2}$$

denote the set of vertices in $V(\mathcal{T})$ neighboring the vertical infinite segment originating at v. As a result of (5.1) and of non-negativity of α^τ we have the following.

Proposition 13 *There is a one-to-one correspondence between traces on $\mathfrak{A}$ and functions $\phi \colon V(\mathcal{T}) \to [0,1]$ such that $\phi(\star) = 1$ and*

$$\phi(v) \geq \sum_{w \in \mathcal{C}_v} \phi(w), \quad \forall v \in V(\mathcal{T}). \tag{5.3}$$

Note that a function satisfying (5.3) is necessarily monotonically decreasing.

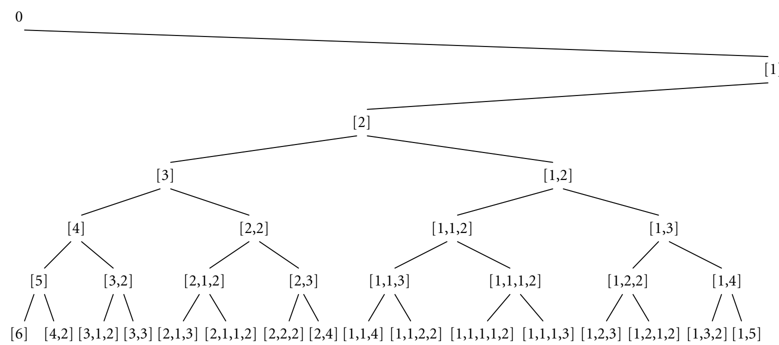

Figure 14: The diagram $\mathcal{T}$ in the continued fraction representation.

One can give a description of the set $\mathcal{C}_v$ using the one-to-one correspondence $v \mapsto r(v)$ between the sets $V(\mathcal{T})$ and $\mathbb{Q} \cap [0,1]$ (see Figure 14). Any number in $\mathbb{Q} \cap (0,1)$ can be uniquely represented as a (reduced) continued fraction $[a_1, \dots, a_t]$ with $a_t \geq 2$. It is not hard to notice and prove that for any $v \in V(\mathcal{T})$ with $r(v) = [a_1, \dots, a_t]$, $a_t \geq 2$, we have

$$r(Lv) = \begin{cases} [a_1, \dots, a_{t-1}, a_t - 1, 2] & \text{if } t \text{ even,} \\ [a_1, \dots, a_{t-1}, a_t + 1] & \text{if } t \text{ odd,} \end{cases} \tag{5.4}$$

$$r(Rv) = \begin{cases} [a_1, \dots, a_{t-1}, a_t + 1] & \text{if } t \text{ even,} \\ [a_1, \dots, a_{t-1}, a_t - 1, 2] & \text{if } t \text{ odd.} \end{cases}$$

As a result of (5.2) and (5.4) we have

$$\begin{aligned}&\{r(w) : w \in \mathcal{C}_v\} \\ &\qquad = \{[a_1, \dots, a_{t-1}, a_t - 1, 1, k] : k \geq 1\} \cup \{[a_1, \dots, a_{t-1}, a_t, k] : k \geq 1\},\end{aligned}$$

which shows in conjunction with Proposition 13 that there is a one-to-one correspondence between traces on $\mathfrak{A}$ and maps $\phi : \mathbb{Q} \cap [0,1] \to [0,1]$ which satisfy

$$1 = \phi(0) \geq \sum_{k=1}^{\infty} \phi\Big(\frac{1}{k}\Big), \quad \phi(1) \geq \sum_{k=1}^{\infty} \phi\Big(\frac{k}{k+1}\Big),$$

$$\begin{aligned}\phi([a_1, \dots, a_t]) \geq \sum_{k=1}^{\infty} \Big(&\phi([a_1, \dots, a_{t-1}, a_t - 1, 1, k]) \\ &+ \phi([a_1, \dots, a_{t-1}, a_t, k]) \Big), \quad a_t \geq 2.\end{aligned}$$

6 Generators, Relations, and Braiding

We shall use the path algebra model for AF algebras as in [15, §2.3.11] and [13, §2.9]. Here, however, a monotone increasing path ξ will be encoded by the sequence (ξ_n)

where ξ_n gives the "horizontal coordinate" of the vertex at floor n, instead of its edges. To use this model we again augment the diagram $\mathcal{G} = D(\mathfrak{A})$ into $\widetilde{\mathcal{G}}$.

Denote by Ω the (uncountable) set of monotone increasing paths starting at $\star$. Let $\Omega_{[r}$ denote the set of infinite monotone increasing paths starting on the r-th floor of $\widetilde{\mathcal{G}}$, $\Omega_{r]}$ the set of monotone increasing paths that connect $\star$ with a vertex on the r-th floor, and $\Omega_{[r,s]}$ the set of monotone increasing paths starting on the r-th floor and ending on the s-th floor. Let $\xi_{r]} \in \Omega_{r]}$, $\xi_{[r,s]} \in \Omega_{[r,s]}$, $\xi_{[s} \in \Omega_{[s}$ denote the natural truncations of a path $\xi \in \Omega$. By $\xi \circ \eta$ we denote the natural concatenation of two paths $\xi \in \Omega_{r]}$ and $\eta \in \Omega_{[r}$ with $\xi_r = \eta_r$. Consider the set R_r of pairs of paths $(\xi, \eta) \in \Omega_{r]} \times \Omega_{r]}$ with the same endpoint $\xi_r = \eta_r$. For each $(\xi, \eta) \in R_r$ the mapping

$$\Omega \ni \omega \mapsto T_{\xi,\eta}\omega = \delta(\eta, \omega_{r]})\xi \circ \omega_{[r} \in \Omega,$$

extends to a linear operator on the $\mathbb{C}$-linear space $\mathbb{C}\Omega$ with basis Ω, and also to a bounded operator $T_{\xi,\eta} \colon \ell^2(\Omega) \to \ell^2(\Omega)$ with $\|T_{\xi,\eta}\| = 1$. We have $\mathfrak{A} = \overline{\bigcup_{r\geq 1} \mathfrak{A}_r}$ where the linear span $\mathfrak{A}_r$ of the operators $T_{\xi,\eta}$, $(\xi, \eta) \in R_r$, forms a finite dimensional C^*-algebra as a result of

$$T^*_{\eta,\xi} = T_{\xi,\eta}, \quad T_{\xi,\eta}T_{\xi',\eta'} = \delta(\eta, \xi')T_{\xi,\eta'}, \quad \sum_{\xi\in\Omega_{r]}} T_{\xi,\xi} = 1.$$

Furthermore the inclusion $\mathfrak{A}_r \overset{\iota_r}{\hookrightarrow} \mathfrak{A}_{r+1}$ is given by

$$\iota_r(T_{\xi,\eta}) = \sum_{\substack{\lambda\in\Omega_{[r,r+1]} \\ \lambda_r=\xi_r(=\eta_r)}} T_{\xi\circ\lambda,\eta\circ\lambda}.$$

This model is employed to give a presentation by generators and relations of the C^*-algebra $\mathfrak{A}$ in the spirit of the presentation of the GICAR algebra from [13, Example 2.23]. We also construct two families of projections that satisfy commutation relations reminiscent of the Temperley–Lieb relations.

We consider the following elements in $\mathfrak{A}$:

(i) the projection e_n in $\mathfrak{A}_{n-1,n} \subseteq \mathfrak{A}_n$ onto the linear space of edges from N (north) to SW (south-west), $n \geq 1$.
(ii) the projection f_n in $\mathfrak{A}_{n-1,n} \subseteq \mathfrak{A}_n$ onto the linear span of edges from N to SE, $n \geq 0$.
(iii) the projection $g_n = 1 - e_n - f_n$ in $\mathfrak{A}_{n-1,n} \subseteq \mathfrak{A}_n$ onto the linear span of edges from N to S, $n \geq 0$.
(iv) the partial isometry $v_n \in \mathfrak{A}_{n-1,n+1} \subseteq \mathfrak{A}_{n+1}$ with initial support $v_n^* v_n = \widetilde{e}_n = g_n f_{n+1}$ and final support $v_n v_n^* = \widetilde{f}_n = f_n e_{n+1}$, which flips paths in the diamonds of shape N-S-SE-NE, $n \geq 0$.
(v) the partial isometry $w_n \in \mathfrak{A}_{n-1,n+2} \subseteq \mathfrak{A}_{n+1}$ with initial support $w_n^* w_n = \widetilde{e}'_n = g_n e_{n+1}$ and final support $w_n w_n^* = \widetilde{f}'_n = e_n f_{n+1}$, which flips paths in the diamonds of shape N-S-SW-NW, $n \geq 1$.

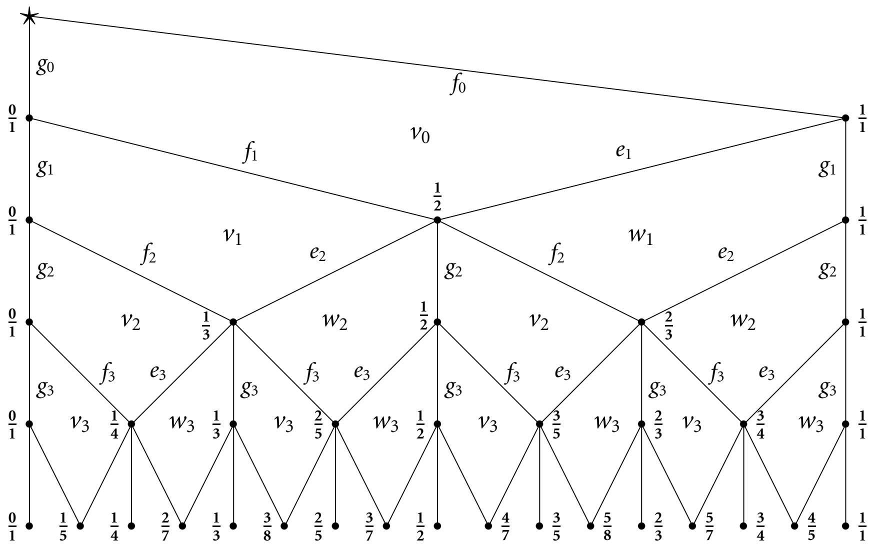

Figure 15: The generators of $\mathfrak{A}$.

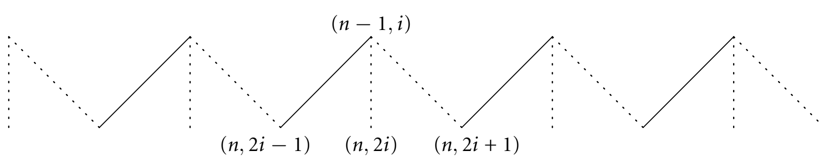

Figure 16: Support of projection e_n.

The AF-algebra $\mathfrak{A}$ is generated by the set $\mathfrak{G} = \{e_n\}_{n\geq 1} \cup \{f_n\}_{n\geq 0} \cup \{v_n\}_{n\geq 0} \cup \{w_n\}_{n\geq 1}$.

Straightforward commutation relations arise since elements defined by edges that reach up to floor $\leq r$ commute with elements defined by edges between the r-th and the s-th floors with $r < s$, as a result of $[\mathfrak{A}_r, \mathfrak{A}_r' \cap \mathfrak{A}_s] = 0$. For instance v_s commutes with e_r, f_r, g_r if $r \leq s-1$ or $r \geq s+2$, and $[v_s, v_r] = [v_s, v_r^*] = [v_s, w_r] = [v_s, w_r^*] = 0$ if $|r-s| \geq 2$. Besides, the elements of $\mathfrak{G}$ satisfy the following commutation relations:

(R1) $e_n^2 = e_n^* = e_n,\ f_n^2 = f_n^* = f_n,\ g_n^2 = g_n^* = g_n,\ e_n + f_n + g_n = 1$.
e_n, f_m, g_k mutually commute.

(R2) $(1-f_n)v_n = (1-e_{n+1})v_n = 0$, $v_n(1-g_n) = v_n(1-f_{n+1}) = 0$.
$(1-e_n)w_n = (1-f_{n+1})w_n = 0$, $w_n(1-g_n) = w_n(1-e_{n+1}) = 0$.

(R3) $v_n g_n = f_n v_n$, $v_n f_{n+1} = e_{n+1} v_n$, $w_n g_n = e_n w_n$, $w_n e_{n+1} = f_{n+1} w_n$.

(R4) $v_n^* v_n = g_n f_{n+1}$, $v_n v_n^* = f_n e_{n+1}$, $w_n^* w_n = g_n e_{n+1}$, $w_n w_n^* = e_n f_{n+1}$.

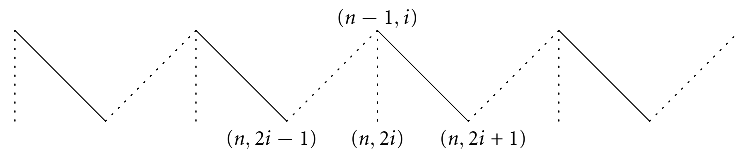

Figure 17: Support of projection f_n.

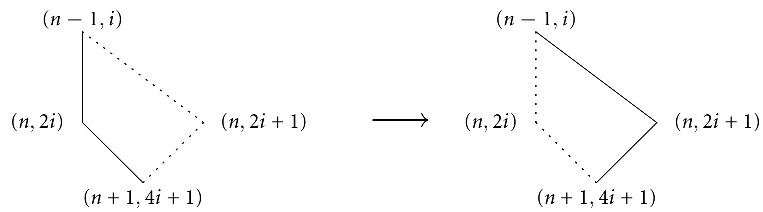

Figure 18: The partial isometry $v_n \colon g_n f_{n+1} \mapsto f_n e_{n+1}$.

As a result of (R1)–(R4) we also get

$$
\begin{aligned}
v_{n+1}v_n &= v_n^2 = v_{n\pm1}v_n^* = v_{n\pm1}^* v_n = 0,\\
w_{n+1}w_n &= w_n^2 = w_{n\pm1}w_n^* = w_{n\pm1}^* w_n = 0,\\
v_n w_n &= v_{n\pm1} w_n = w_n v_n = w_{n\pm1} v_n = 0,\\
v_n w_n^* &= v_{n\pm1} w_n^* = v_n^* w_n = v_n^* w_{n-1} = 0.
\end{aligned}
\tag{6.1}
$$

The only non-zero products ab with

$$a \in \{v_n, v_n^*, w_n, w_n^*\} \quad \text{and} \quad b \in \{v_{n+1}, v_{n+1}^*, w_{n+1}, w_{n+1}^*\}$$

are $v_n v_{n+1}$, $w_n w_{n+1}$, $w_n^* v_{n+1}$, and $v_n^* w_{n+1}$.

Let B_n denote Artin's braid group generated by $\sigma_1, \dots, \sigma_{n-1}$ with relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| > 1$ and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$. Relations (6.1) show in particular that the partial isometries v_{i-1}, respectively w_i, satisfy trivially the braid relations.

Taking $R_n(\lambda) := 1 + \lambda v_n$, the equalities $v_n^2 = 0$ and $v_n v_{n\pm1} v_n = 0$ yield the Yang–Baxter type relation $R_n(\lambda)R_{n+1}(\lambda+\mu)R_n(\mu) = R_{n+1}(\mu)R_n(\lambda+\mu)R_{n+1}(\lambda)$.

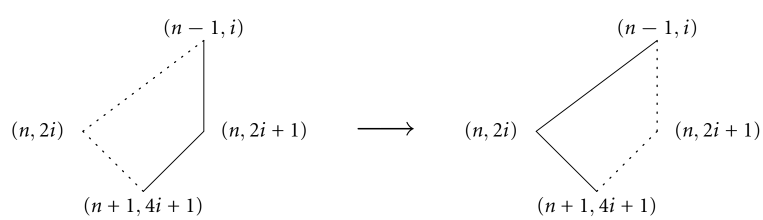

Figure 19: The partial isometry $w_n \colon g_n e_{n+1} \mapsto e_n f_{n+1}$.

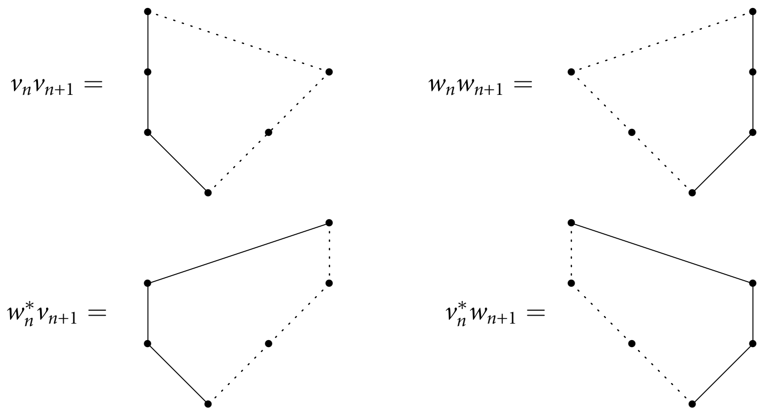

Figure 20: The partial isometries $v_n v_{n+1} \colon g_n g_{n+1} f_{n+2} \mapsto f_n e_{n+1} e_{n+2}$, $w_n w_{n+1} \colon g_n g_{n+1} e_{n+2} \mapsto e_n f_{n+1} f_{n+2}$, $w_n^* v_{n+1} \colon e_n g_{n+1} f_{n+2} \mapsto g_n e_{n+1} e_{n+2}$, $v_n^* w_{n+1} \colon f_n g_{n+1} e_{n+2} \mapsto g_n f_{n+1} f_{n+2}$.

By analogy with the construction of Temperley–Lieb–Jones projections in the GICAR algebra [13, 15] for each $\lambda > 0$ we put $\tau = \frac{\lambda}{(1+\lambda)^2} \in \big(0, \frac{1}{4}\big]$ and consider

$$E_n = \frac{1}{1+\lambda}(v_n^* v_n + \sqrt{\lambda} v_n + \sqrt{\lambda} v_n^* + \lambda v_n v_n^*) \in \mathfrak{A}, \quad n \geq 0,$$

and

$$F_n = \frac{1}{1+\lambda}(w_n^* w_n + \sqrt{\lambda} w_n + \sqrt{\lambda} w_n^* + \lambda w_n w_n^*) \in \mathfrak{A}, \quad n \geq 1.$$

Proposition 14 *The elements E_n and F_n define (self-adjoint) projections in the AF algebra $\mathfrak{A}$ satisfying the braiding relations*

$$E_n F_n = F_n E_n = 0, \tag{6.2}$$

$$[E_n, E_m] = [F_n, F_m] = [E_n, F_m] = 0 \quad \text{if } |n-m| \geq 2, \tag{6.3}$$

$$E_n E_{n+1} E_n = \tau E_n e_{n+2}, \quad E_{n+1} E_n E_{n+1} = \tau E_{n+1} g_n, \tag{6.4}$$

$$F_n F_{n+1} F_n = \tau F_n f_{n+2}, \quad F_{n+1} F_n F_{n+1} = \tau F_{n+1} g_n, \tag{6.5}$$

$$E_n F_{n+1} E_n = \lambda\tau E_n f_{n+2}, \quad F_n E_{n+1} F_n = \lambda\tau F_n e_{n+2}, \tag{6.6}$$

$$E_{n+1} F_n E_{n+1} = \lambda\tau E_{n+1} e_n, \quad F_{n+1} E_n F_{n+1} = \lambda\tau F_{n+1} f_n, \tag{6.7}$$

$$E_n E_{n+1} F_n = E_n F_{n+1} F_n = E_{n+1} E_n F_{n+1} = E_{n+1} F_n F_{n+1} = 0, \tag{6.8}$$

$$F_n E_{n+1} E_n = F_n F_{n+1} E_n = F_{n+1} E_n E_{n+1} = F_{n+1} F_n E_{n+1} = 0. \tag{6.9}$$

Proof The initial and final projections of the partial isometry v_n are orthogonal, thus E_n defines a projection in $\mathfrak{A}_n$ for every $\lambda \geq 0$. A similar property holds for F_n, which is seen to be orthogonal to E_n. The commutation relations (6.3) are obvious because

v_{n+2} and w_{n+2} commute with all elements in $\mathfrak{A}_{n+1}$, including E_n and F_n. By (6.1) we have $v_n^* E_{n+1} = v_n v_{n+1}^* = 0$, leading to

$$E_n E_{n+1} = \frac{\sqrt{\lambda}}{(1+\lambda)^2} (v_n^* v_n + \sqrt{\lambda}\, v_n)(v_{n+1} + \sqrt{\lambda}\, v_{n+1} v_{n+1}^*), \tag{6.10}$$

and also

$$E_{n+1} E_n = (E_n E_{n+1})^* = \frac{\sqrt{\lambda}}{(1+\lambda)^2} (v_{n+1}^* + \sqrt{\lambda}\, v_{n+1} v_{n+1}^*)(v_n^* v_n + \sqrt{\lambda}\, v_n^*). \tag{6.11}$$

From (6.10) and $v_{n+1} E_n = v_{n+1}^* v_n = 0$ we have

$$\begin{aligned} E_n E_{n+1} E_n &= \frac{\lambda}{(1+\lambda)^3} (v_n^* v_n + \sqrt{\lambda}\, v_n) v_{n+1} v_{n+1}^* (v_n^* v_n + \sqrt{\lambda}\, v_n^*) \\ &= \frac{\lambda}{(1+\lambda)^3} (\widetilde{e}_n + \sqrt{\lambda}\, v_n) \widetilde{f}_{n+1} (\widetilde{e}_n + \sqrt{\lambda}\, v_n^*). \end{aligned} \tag{6.12}$$

But $\widetilde{e}_n \widetilde{f}_{n+1} \widetilde{e}_n = \widetilde{e}_n \widetilde{f}_{n+1} = g_n f_{n+1} e_{n+1} = \widetilde{e}_n e_{n+2}$, $v_n \widetilde{f}_{n+1} \widetilde{e}_n = v_n \widetilde{e}_n e_{n+1} e_{n+2} = v_n e_{n+2}$ (and because $[e_{n+2}, v_n] = 0$ this also gives $\widetilde{e}_n \widetilde{f}_{n+2} v_n^* = v_n^* e_{n+2}$), and $v_n \widetilde{f}_{n+1} v_n^* = v_n f_{n+1} e_{n+2} v_n^* = v_n f_{n+1} v_n^* e_{n+2} = v_n g_n f_{n+1} v_n^* e_{n+2} = v_n v_n^* e_{n+2}$, which we insert in (6.12) to get $E_n E_{n+1} E_n = \tau E_n e_{n+2}$.

From (6.11) and $v_n^* E_{n+1} = v_n^* v_{n+1}^* = 0$ we find

$$E_{n+1} E_n E_{n+1} = \frac{\lambda}{(1+\lambda)^3} (v_{n+1}^* + \sqrt{\lambda}\, \widetilde{f}_{n+1}) \widetilde{e}_n (v_{n+1} + \sqrt{\lambda}\, \widetilde{f}_{n+1}). \tag{6.13}$$

As a result of $[g_n, v_{n+1}] = 0$ and $(1 - f_{n+1}) v_{n+1} = 0$ we have $v_{n+1}^* \widetilde{e}_n v_{n+1} = \widetilde{e}_{n+1} g_n$. It is also plain that $\widetilde{f}_{n+1} \widetilde{e}_n \widetilde{f}_{n+1} = \widetilde{f}_{n+1} \widetilde{e}_n = \widetilde{f}_{n+1} g_n$, $\widetilde{f}_{n+1} \widetilde{e}_n v_{n+1} = \widetilde{f}_{n+1} g_n v_{n+1} = \widetilde{f}_{n+1} v_{n+1} g_n = v_{n+1} g_n$, and $v_{n+1}^* \widetilde{e}_n \widetilde{f}_{n+1} = v_{n+1}^* \widetilde{f}_{n+1} g_n = v_{n+1}^* g_n$. Together with (6.13) these equalities yield

$$E_{n+1} E_n E_{n+1} = \tau E_{n+1} g_n.$$

Equalities (6.5)–(6.8) are checked in a similar way; (6.9) follows by taking adjoints in (6.8). ∎

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