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Pisot Numbers from $\{0, 1\}$-Polynomials

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Abstract. A *Pisot number* is a real algebraic integer greater than 1, all of whose conjugates lie strictly inside the open unit disk; a *Salem number* is a real algebraic integer greater than 1, all of whose conjugate roots are inside the closed unit disk, with at least one of them of modulus exactly 1. Pisot numbers have been studied extensively, and an algorithm to generate them is well known. Our main result characterises all Pisot numbers whose minimal polynomial is derived from a Newman polynomial — one with $\{0, 1\}$-coefficients — and shows that they form a strictly increasing sequence with limit $(1 + \sqrt{5})/2$. It has long been known that every Pisot number is a limit point, from both sides, of sequences of Salem numbers. We show that this remains true, from at least one side, for the restricted sets of Pisot and Salem numbers that are generated by Newman polynomials.

1 Introduction

A *Pisot* (or *Pisot–Vijayaraghavan*) *number* is a real algebraic integer $\alpha > 1$, all of whose conjugates lie inside the open unit disk. A real algebraic integer $\alpha > 1$ is a *Salem number* if all of its conjugate roots are inside the closed unit disk, and at least one of these conjugate roots has modulus exactly 1. The set of all Pisot numbers is usually denoted by $\mathcal{S}$, and the set of all Salem numbers is denoted by $\mathcal{T}$. Many results are known about the set $\mathcal{S}$. For example, $\mathcal{S}$ is known to be closed [9], and its minimum is known to be the largest root of $z^3 - z - 1$, which is approximately $1.3247179\cdots$ [10].

A *Newman polynomial* [3] is a polynomial with both constant term and leading coefficient equal to 1, and whose remaining coefficients are all either 0 or 1. In this paper we consider Newman polynomials that give rise to either Pisot numbers or Salem numbers. Since a Newman polynomial does not have positive real roots, it cannot have a Pisot or a Salem number as a root. This leads us to define the related set of monic polynomials

$$\mathcal{N}_d = \{\, (-1)^d h(-z) : h \in \mathbb{Z}[z] \text{ is a Newman polynomial of degree } d\}.$$

If $f(z) \in \mathcal{N}_d$ and $f(z) = \sum_{k=0}^{d} a_k z^k$, then $a_0 = (-1)^d$ and for $1 < k < d$, $a_k \in \{0, 1\}$ if k and d are of the same parity and $a_k \in \{0, -1\}$ if they are of opposite parity. We also set $\mathcal{N} = \bigcup_{d=1}^{\infty} \mathcal{N}_d$.

We can now define a *Newman Pisot number* to be a Pisot number whose minimal polynomial is in $\mathcal{N}$. We define a *Newman Salem number* slightly differently; a real number $\beta > 1$ is a *Newman Salem number* if it is a Salem number and it is the root of a polynomial in $\mathcal{N}$. We do not require the polynomial to be irreducible in this case.

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The following two theorems will be proved in Sections 3 and 4 of this paper, and in addition, some results concerning the largest element of the set of all roots of polynomials from $\mathcal{N}_d$ appear in Section 5.

Theorem 1.1 *The set of all Newman Pisot numbers forms a sequence $\{\nu_n\}$, for odd $n \geq 3$, where the minimal polynomial of ν_n is given by*

$$P(z) = z^n - z^{n-1} - z^{n-3} - \cdots - z^2 - 1.$$

The sequence $\{\nu_n\}$ is strictly increasing and has limit point $(1 + \sqrt{5})/2$.

Thus, for each odd $n \geq 3$, there exists precisely *one* polynomial in $\mathcal{N}_n$ that is a minimal polynomial for a Pisot number ν_n. When n is even, no such polynomials exist in $\mathcal{N}_n$.

The fact that every point of $\mathcal{S}$ is a limit of points of $\mathcal{T}$ from both sides has been known for many years [9]. We prove here that this remains partially true for the more restricted set of Pisot and Salem numbers arising from $\mathcal{N}$.

Theorem 1.2 *Every Newman Pisot number is a limit point, from below, of Newman Salem numbers.*

Based on the work of Dufresnoy and Pisot [4], Boyd constructed an algorithm that determines all Pisot numbers in a given interval $[a, b]$ of the real line [2]. A description of the algorithm can be found in [2, 7]; in this paper we very briefly outline the steps of the algorithm, using the same notation as in [7].

2 The Algorithm

For a given $\alpha \in \mathcal{S}$ with monic minimal polynomial $p(z)$ of degree d, we consider the set $\mathcal{C}$ of rational functions of the form $f(z) = r(z)/p^*(z)$, where $p^*(z) = z^d p(1/z)$ and $r(z)$ satisfies the two conditions $r(0) > 0$ and $|r(z)| \leqslant |p^*(z)|$ for $|z| = 1$. For any $f \in \mathcal{C}$, if $|z| < \alpha^{-1}$, then we can write $f(z) = u_0 + u_1 z + u_2 z^2 + \cdots$, where each u_i is an integer and $1 \leqslant u_0,\ u_0^2 - 1 \leqslant u_1$ and $w_n \leqslant u_n \leqslant w_n^+$ for $n \geqslant 2$. The values w_n and w_n^+ are determined by $u_0, \ldots, u_{n-1}$ and are finite except when $u_0 = 1$, in which case $w_2^+ = \infty$. For each integer $u_0 \geq 1$, we can view the sequences $\{u_j\}$ of integers as forming an infinite tree. The nodes of the tree at height n are finite subsequences $(u_0, u_1, \ldots, u_n)$. If $u_n = w_n$ or $u_n = w_n^+$, then such a node has no successors, and if $w_n < u_n < w_n^+$, then its successors are all the subsequences $(u_0, u_1, \ldots, u_n, u_{n+1})$, where $w_{n+1} \leq u_{n+1} \leq w_{n+1}^+$.

The bounds w_n and w_n^+ are calculated recursively, assuming that the values $u_0, u_1, \ldots, u_{n-1}$ are known. Set $D_n(z) = -z^n + d_1 z^{n-1} + \cdots + d_n$ and $E_n(z) = -z^n D_n(1/z)$, where we select $d_1, d_2, \ldots, d_n$ so that the first n coefficients of the Maclaurin series for $D_n(z)/E_n(z)$ are the given $u_0, u_1, \ldots, u_{n-1}$. Then w_n is simply the coefficient of z^n in this series. Similarly, let $D_n^+(z) = z^n + d_1^+ z^{n-1} + \cdots + d_n^+$ and $E_n^+(z) = z^n D_n^+(1/z)$, where we select $d_1^+, \ldots, d_n^+$ so that the first n coefficients of the series for $D_n^+(z)/E_n^+(z)$ are the given $u_0, u_1, \ldots, u_{n-1}$. Again, w_n^+ is the coefficient of z^n in this series. We can extend this construction to define $w_1 = u_0^2 - 1$ and $w_1^+ = 1 - u_0^2$. Thus, both w_n and w_n^+ are defined for $n \geq 1$, although the inequality $u_1 \leq w_1^+$ does not necessarily hold.

A node in the infinite tree of integer sequences $\{u_j\}$ that has no successors is called a *terminal node*. There are three possible ways for a node $(u_0, u_1, \dots, u_n)$ to be a terminal node:

(i) $w_n < u_n < w_n^+$, but there are no integers in $[w_{n+1}, w_{n+1}^+]$ and hence no candidates for u_{n+1},

(ii) $u_n = w_n$ (or w_n^+), but the polynomial D_n (or D_n^+) does not have integer coefficients, or

(iii) $u_n = w_n$ (or w_n^+), and D_n (or D_n^+) does have integer coefficients.

This third type of terminal node is in one-to-one correspondence with the points of $\mathcal{S}$ if we exclude the quadratics $(c + (c^2 - 4)^{1/2})/2$ for $c \geq 3$, which come from reciprocal quadratic polynomials of the form $1 - cz + z^2$. More specifically, if the terminal node $(u_0, u_1, \dots, u_n)$ is of the third type, then the polynomial $-D_n$(or D_n^+) is the minimal polynomial for a Pisot number.

We will make use of some of the many recursive relations between the polynomials defined above, and so we reproduce them here from [7, Section 2]. More details can be found in [1, Chapter 7]. We have

$$D_{n+1}(z) = (1+z)D_n(z) - \frac{u_n - w_n}{u_{n-1} - w_{n-1}} z D_{n-1}(z) \qquad \text{for } n \geq 2, \tag{2.1}$$

$$D_{n+1}^+(z) = (1+z)D_n^+(z) - \frac{w_n^+ - u_n}{w_{n-1}^+ - u_{n-1}} z D_{n-1}^+(z) \qquad \text{for } n \geq 4, \tag{2.2}$$

and also

$$D_{n+1}^+(z)E_n(z) - D_n(z)E_{n+1}^+(z) = (u_n - w_n)z^n(1+z), \tag{2.3}$$

$$D_{n+2}^+(z)E_n^+(z) - D_n^+(z)E_{n+2}^+(z) = (u_n - w_n^+)z^n(1-z^2). \tag{2.4}$$

The concise recursive relation

$$w_{n+1}^+ - w_{n+1} = \frac{4(w_n^+ - u_n)(u_n - w_n)}{w_n^+ - w_n} \tag{2.5}$$

will also be useful [1, 2].

Finally, a *path to infinity* in the tree described above corresponds to an infinite sequence $\{u_j\}$ which satisfies $w_j < u_j < w_j^+$ for all $j \geq 2$, and thus to limit points of the set $\mathcal{S}$. Such a limit point α' appears as the simple pole $1/\alpha'$ of the rational function $F(z) = \sum_{i=0}^{\infty} u_i z^i$.

3 The Set of Newman Pisot Numbers

Applying the algorithm of the previous section to the set $\mathcal{N}$, we can prove Theorem 1.1. The following two lemmas will be used to restrict appropriately the infinite tree of sequences $\{u_j\}$ in our proof. The first is a classical result due to Cauchy (see, [6]).

Lemma 3.1 *All the zeroes of the polynomial* $f(z) = a_0 + a_1 z + \cdots + a_n z^n$, $a_n \neq 0$, *lie in the circle*

$$|z| < 1 + \max |a_k/a_n|, \quad k = 0, 1, 2, \ldots, n-1.$$

The next lemma bounds the coefficients in the Maclaurin expansion of the ratio of polynomials that appears in the algorithm.

Lemma 3.2 *Suppose* $f(z) = \sum_{j=0}^{d} a_j z^j$ *is in* $\mathcal{N}_d$ *with* $f^*(z) = z^d f(1/z)$ *and let* $\epsilon = \pm 1$. *Let*

$$G(z) = \frac{\epsilon f(z)}{f^*(z)} = \sum_{i \geq 0} e_i z^i,$$

so that $e_i \in \mathbb{Z}$ *for* $i \geq 0$. *Suppose that* $1 \leq k \leq d$, *and that* $e_0 > 0$ *and* $e_i \geq 0$ *for* $1 \leq i \leq k-1$. *Then* $e_k \leq \sum_{i=0}^{(k-1)/2} e_{2i}$ *when* k *is odd, and* $e_k \leq 1 + \sum_{i=1}^{k/2} e_{2i-1}$ *when* k *is even.*

Proof of Lemma 3.2 Since $\epsilon f(z) = f^*(z) G(z)$, we match coefficients of z^k and find that $\epsilon a_k = \sum_{j=0}^{k} e_j a_{d-k+j}$, which we can rewrite as

$$\begin{aligned} -a_d e_k &= -\epsilon a_k + \sum_{j=0}^{k-1} e_j a_{d-k+j} \\ &= -\epsilon a_k + \sum_{\substack{j=0 \\ j \equiv k \bmod 2}}^{k-1} b_j e_j - \sum_{\substack{j=0 \\ j \not\equiv k \bmod 2}}^{k-1} b_j e_j, \end{aligned}$$

where $b_j = |a_{d-k+j}|$ is either 0 or 1. Thus

$$e_k = \epsilon a_k + \sum_{\substack{j=0 \\ j \not\equiv k \bmod 2}}^{k-1} b_j e_j - \sum_{\substack{j=0 \\ j \equiv k \bmod 2}}^{k-1} b_j e_j.$$

Since by assumption $e_0 > 0$, we have that $\epsilon = a_0$ and so ϵa_k takes value $(-1)^k$ or 0. Thus $e_k \leq e_0 + e_2 + \cdots + e_{k-1}$ when k is odd, and $e_k \leq 1 + e_1 + e_3 + \cdots + e_{k-1}$ when k is even. ∎

Proof of Theorem 1.1 By Lemma 3.1 we know that all the zeroes of a Newman polynomial must lie in the circle $|z| < 2$. Thus, all Newman Pisot numbers must lie in the real interval $[1, 2]$.

The algorithm described in Section 2 will be used to construct the set of Newman Pisot numbers. In what follows, we use the same notation from that section.

Let f_m denote the m-th *Fibonacci number*, and recall that $f_0 = 0$, $f_1 = 1$, and $f_m = f_{m-1} + f_{m-2}$ for $m \geq 2$. We claim that the subtree constructed by the following three formulas for $k \geq 1$, produces all Newman Pisot numbers:

$$D_k(z) = \begin{cases} 1 + z^2 + z^4 + \cdots + z^{k-1} - z^k & \text{if } k \text{ is odd,} \\ a_{0,k} + a_{1,k} z + \cdots + a_{k-1,k} z^{k-1} - z^k & \text{if } k \text{ is even,} \end{cases} \tag{3.1}$$

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where the coefficients $a_{j,k}$ are given by

$$a_{j,k} = \begin{cases} \frac{j+1}{k+2} & \text{if } j \text{ is odd,} \\ 1 - \frac{j}{k+2} & \text{if } j \text{ is even.} \end{cases}$$

Next, we have that

$$w_k = \begin{cases} f_{k+1} - 1 & \text{if } k \text{ is odd,} \\ f_{k+1} - \frac{k+4}{k+2} & \text{if } k \text{ is even.} \end{cases} \tag{3.2}$$

And finally, we have

$$w_k^+ = \begin{cases} f_{k+1} + \frac{(k+2)^2-5}{k^2-5} & \text{if } k \text{ is odd,} \\ f_{k+1} + \frac{k}{k-2} & \text{if } k \neq 2 \text{ is even.} \end{cases} \tag{3.3}$$

This would mean that $f_{k+1} - 2 < w_k < f_{k+1} - 1$ when k is even, and that $f_{k+1} + 1 < w_k^+ < f_{k+1} + 2$, for $k \geq 6$.

We will prove (3.1), (3.2), and (3.3) by induction on k, and we work out the first few cases ($k \leq 7$) as examples. Figure 1 illustrates the subtree being constructed up to height 7.

Since $1 \leq u_0$ and we wish to restrict $D_k(z)$ to the set $\mathcal{N}$, we have that $u_0 = 1$. Thus $D_1(z) = u_0 - z = 1 - z$ and $E_1(z) = 1 - z$ as well. So $D_1(z)/E_1(z) = 1$, meaning that $w_1 = 0$. Similarly, $D_1^+(z) = u_0 + z = 1 + z$ and thus $E_1^+(z) = 1 + z$ giving us $w_1^+ = 0$.

Next, suppose $D_2(z) = -z^2 + d_1 z + d_2$, so that $E_2(z) = -d_2 z^2 - d_1 z + 1$ and $D_2(z)/E_2(z) = 1 + u_1 z + w_2 z^2 + \cdots$. By clearing the denominator and comparing coefficients, we find that $d_2 = 1$, $d_1 = u_1/2$, and $w_2 = u_1^2/2$. Thus $D_2(z) = -z^2 + \frac{u_1}{2} z + 1$. Suppose the roots of $D_2(z)$ are τ_2 and $-\tau_2^{-1}$, where τ_2 is a Pisot number, so that $1 < \tau_2 < 2$. Then $0 < \tau_2 - \tau_2^{-1} < 3/2$ implies that $0 < u_1/2 < 3/2$, which implies that $u_1 = 1$ or $u_1 = 2$. By Lemma 3.2, the maximum possible value of u_1 is 1. So we select $u_1 = 1$, and thus $D_2(z) = -z^2 + \frac{1}{2} z + 1$ and also $w_2 = \frac{1}{2}$.

Now we cannot compute w_2^+ in this manner, since $u_0 = 1$ means that $w_2^+ = \infty$. But, using Lemma 3.2, we have that $u_2 \leq 1 + u_1 = 2$. So, $\frac{1}{2} < u_2 \leq 2$, meaning that $u_2 = 1$ or $u_2 = 2$. In either case, $w_2 < u_2 < w_2^+$. We consider the two possible values of u_2 in turn.

If $u_2 = 1$, then from (2.1) we have $D_3(z) = 1 + z - z^3$. So $E_3(z) = 1 - z^2 - z^3$ and $w_3 = 1$. Solving simultaneously using (2.3) and (2.4), we have that $D_3^+(z) = z^3 - 2z^2 - z + 1$ and $E_3^+(z) = z^3 - z^2 - 2z + 1$, with $w_3^+ = 3$. By Lemma 3.2, the maximum value of u_3 is $u_0 + u_2 = 2$. If $u_3 = w_3 = 1$, we have a terminal node and $1 + z - z^3$ corresponds to a Pisot number. But this is not a Newman Pisot number, since $z^3 - z - 1$ is not in $\mathcal{N}$. With the non-terminal value $u_3 = 2$, we have $D_4(z) = 1 + z^3 - z^4$, with $w_4 = 2$, and $D_4^+(z) = 1 - 2z^2 - z^3 + z^4$ with $w_4^+ = 4$. Again, neither polynomial corresponds to a Newman Pisot number, and so we select $u_4 = 3$.

Notice that both $w_3^+ - w_3 = 2$ and $w_4^+ - w_4 = 2$. Indeed, by (2.5) we can see that $w_n^+ - w_n = 2$ when $n \geq 4$, and that there is only one non-terminal choice of u_n in

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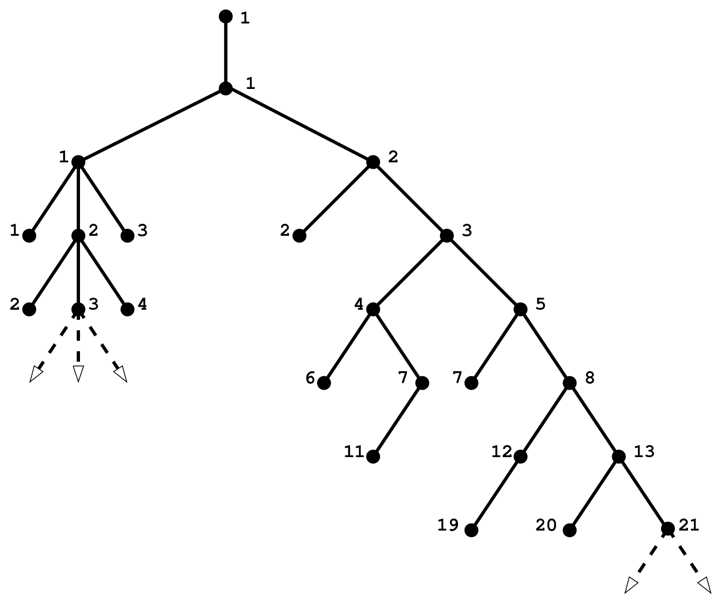

Figure 1: Nodes up to height 7 in the tree of integer sequences $\{u_j\}$.

each case. Thus the iterative formulas (2.1) and (2.2) become

$$D_{n+1}(z) = (1+z)D_n(z) - zD_{n-1}(z) \tag{3.4}$$

and

$$D^+_{n+1}(z) = (1+z)D^+_n(z) - zD^+_{n-1}(z) \tag{3.5}$$

for $n \geq 4$. With $n = 4$, we easily compute $D_5(z) = 1 - z^2 + z^3 + z^4 - z^5$ and $D^+_5(z) = 1 - z^2 - z^3 - z^4 + z^5$. Notice that the coefficients of z^2 and z^0 are of different signs in both cases, and so neither polynomial corresponds to a Newman Pisot number. The same situation occurs with $n = 5$; the polynomial $D_6(z) = 1 - z^2 + z^4 + z^5 - z^6$, and $D^+_6(z) = 1 - z^2 - z^4 - z^5 + z^6$. In fact, by (3.4) we see that if both $D_n(z)$ and $D_{n-1}(z)$ begin with the terms $1 - z^2$, then so does $D_{n+1}(z)$. The same argument, using (3.5), applies to $D^+_{n+1}(z)$. This means that none of the terminal nodes encountered here correspond to Newman Pisot numbers. Thus the choice of $u_2 = 1$ does not produce any Newman Pisot numbers. We next try $u_2 = 2$. This means (2.1) that $D_3(z) = 1 + z^2 - z^3$ and $E_3(z) = 1 - z - z^3$, implying that $w_3 = 2$. Also, by (2.3) and (2.4) we have $D^+_3(z) = 1 - 2z - 3z^2 + z^3$ and $E^+_3(z) = 1 - 3z - 2z^2 + z^3$, and so $w^+_3 = 8$. By Lemma 3.2, $w_3 = 2 \leq u_3 \leq u_0 + u_2 = 3$. If $u_3 = 2 = w_3$, we have a terminal node with $D_3(z)$ corresponding to a Newman Pisot number. If $u_3 = 3$, we have $w_3 < u_3 < w^+_3$, so we can continue.

By (2.1), $D_4(z) = 1 + \frac{1}{3}z + \frac{2}{3}z^2 + \frac{2}{3}z^3 - z^4$, so $E_4(z) = 1 - \frac{2}{3}z - \frac{2}{3}z^2 - \frac{1}{3}z^3 - z^4$, with $w_4 = \frac{11}{3}$. We can now use (2.5) to get $w^+_4 = \frac{11}{3} + \frac{20}{6} = 7$. By Lemma 3.2, $w_4 < 4 \leq u_4 \leq 5 < 7 = w^+_4$. Again, we consider both values of u_4 in turn. If $u_4 = 4$,

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then (2.1) gives $D_5(z) = 1 + z + z^2 + z^3 - z^5$ and so $E_5(z) = 1 - z^2 - z^3 - z^4 - z^5$, with $w_5 = 6$ and $w_5^+ = 7\frac{1}{5}$. This gives us two possible choices for u_5, namely 6 or 7. However, with $u_5 = 6 = w_5$ we have a terminal node with $D_5(z)$ corresponding to a Pisot number, but not to a Newman Pisot number. So we select the value $u_5 = 7$ and continue to get $D_6(z) = 1 - z + z^2 - z^4 + 2z^5 - z^6$. This gives $w_6 = 11$ and $w_6^+ = 11\frac{2}{3}$. Thus the only choice for u_6 here is the terminal value 11, which once again does not yield a Newman Pisot number. Therefore, the choice of $u_4 = 4$ leads to terminal nodes with no Newman Pisot numbers, and so we select $u_4 = 5$.

With this value, we have $D_5(z) = 1 + z^2 + z^4 - z^5$ and $E_5(z) = 1 - z - z^3 - z^5$, with $w_5 = 7$. From (2.5) we have $w_5^+ = 10\frac{1}{5}$. Lemma 3.2 gives $w_5 = 7 \le u_5 \le u_0 + u_2 + u_4 = 8$; if $u_5 = 7 = w_5$, we have a terminal node with $D_5(z)$ corresponding to a Newman Pisot number, and if $u_5 = 8$, we have $w_5 < u_5 < w_5^+$, so we continue.

Using (2.1),

$$\begin{aligned} D_6 &= 1 + \frac{1}{4}z + \frac{3}{4}z^2 + \frac{1}{2}z^3 + \frac{1}{2}z^4 + \frac{3}{4}z^5 - z^6, \\ E_6 &= 1 - \frac{3}{4}z - \frac{1}{2}z^2 - \frac{1}{2}z^3 - \frac{3}{4}z^4 - \frac{1}{4}z^5 - z^6, \\ w_6 &= 11\frac{3}{4}. \end{aligned}$$

By (2.5), $w_6^+ = 14\frac{1}{2}$. Lemma 3.2 gives $w_6 < 12 \le u_6 \le 13 < w_6^+$. The choice of $u_6 = 12$ gives $D_7(z) = 1+z+z^2+z^3+z^4+z^5-z^7$ and so $E_7(z) = 1-z^2-z^3-z^4-z^5-z^6-z^7$, with $w_7 = 19$ and $w_7^+ = 19\frac{10}{11}$. So $u_7 = 19 = w_7$, and we have a terminal node with $D_7(z)$ corresponding to a Pisot number, but not to a Newman Pisot number. Selecting $u_6 = 13$ gives $D_7(z) = 1+z^2+z^4+z^6-z^7$ and $E_7(z) = 1-z-z^3-z^5-z^7$, with $w_7 = 20$ and $w_7^+ = 22\frac{8}{11}$. Lemma 3.2 gives $w_7 = 20 \le u_7 \le 21$; if $u_7 = 20 = w_7$, we have a terminal node with $D_7(z)$ corresponding to a Newman Pisot number, and if $u_7 = 21$, we have $w_7 < u_7 < w_7^+$.

Suppose then that we have reached a node $(u_0, \dots, u_k)$ of height $k \ge 6$ in the search tree, and that equations (3.1), (3.2), and (3.3) hold for all $0 \le j \le k$. In particular, this means that $u_j = f_{j+1}$ for all $0 \le j \le k-1$ and that $w_k \le u_k \le w_k^+$.

We first suppose that k is even, so that $w_k < f_{k+1} - 1 \le u_k \le f_{k+1} + 1 < w_k^+$. By Lemma 3.2, the maximum possible value of u_k is $1 + \sum_{i=1}^{k/2} u_{2i-1} = 1 + \sum_{i=1}^{k/2} f_{2i} = f_{k+1}$. We first consider the case of $u_k = f_{k+1} - 1$. By (2.1), (3.1), and (3.2), we have that

$$\begin{aligned} (3.6)\quad D_{k+1}(z) &= (1+z)\Big(\sum_{i=0}^{k-1} a_{i,k} z^i - z^k\Big) - \frac{2z}{k+2}(1 + z^2 + \cdots + z^{k-2} - z^{k-1}) \\ &= 1 + z + z^2 + \cdots + z^{k-1} - z^{k+1}. \end{aligned}$$

Since w_{k+1} is the coefficient of z^{k+1} in the Maclaurin expansion of $D_{k+1}(z)/E_{k+1}(z)$, we have $w_{k+1} = -1 + \sum_{i=0}^{k-1} u_i = -1 + \sum_{i=1}^{k} f_i = f_{k+2} - 2$. By (2.5) we then have $w_{k+1}^+ = f_{k+2} - 2k(k-2)/(k^2 + 2k - 4)$. Now since $k \ge 6$, we have that $1 < 2k(k-2)/(k^2 + 2k - 4) < 2$, and so the only integer in the interval $[w_{k+1}, w_{k+1}^+]$

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is $f_{k+2} - 2$. But the choice of $u_{k+1} = w_{k+1} = f_{k+2} - 2$ leads to a terminal node with the polynomial $D_{k+1}(z)$ corresponding to a Pisot number, although not to a Newman Pisot number.

Thus we must have that $u_k = f_{k+1}$. In this case, we have that

$$\begin{aligned} D_{k+1}(z) &= (1+z)\Big(\sum_{i=0}^{k-1} a_{i,k} z^i - z^k\Big) - \frac{k+4}{k+2}(z + z^3 + z^5 + \cdots + z^{k-1} - z^k) \\ &= 1 + z^2 + z^4 + \cdots + z^k - z^{k+1}, \end{aligned}$$

which is of the required form in (3.1). We then compute w_{k+1} from the Maclaurin expansion of $D_{k+1}(z)/E_{k+1}(z)$, and find that $w_{k+1} = -1 + \sum_{i=0}^{k/2} f_{2i+1} = f_{k+2} - 1$. We can use (2.5) to get $w_{k+1}^+ = f_{k+2} - 1 + 2k(k+4)/(k^2+2k-4) = f_{k+2} + \frac{(k+3)^2-5}{(k+1)^2-5}$. Both are of the required form in (3.2) and (3.3).

Next suppose that k is odd, so that $w_k = f_{k+1} - 1 \le u_k \le f_{k+1} + 1 < w_k^+$. As before, we use Lemma 3.2 to find the maximum possible value of u_k to be $\sum_{i=0}^{(k-1)/2} u_{2i} = \sum_{i=0}^{(k-1)/2} f_{2i+1} = f_{k+1}$. If $u_k = w_k = f_{k+1} - 1$, then we have a *terminal node* of the third type, where $D_k(z) = 1 + z^2 + z^4 + \cdots + z^{k-1} - z^k$ corresponds to a Pisot number. Otherwise, $u_k = f_{k+1}$ and we have that

$$\begin{aligned} D_{k+1}(z) &= (1+z)(1 + z^2 + \cdots + z^{k-1} - z^k) - \frac{k+1}{k+3}\Big(\sum_{i=0}^{k-2} a_{i,k-1} z^{i+1} - z^k\Big) \\ &= -z^{k+1} + \sum_{\substack{j=0 \\ j \text{ odd}}}^{k} \frac{j+1}{k+3} z^j + \sum_{\substack{j=0 \\ j \text{ even}}}^{k} \frac{k+3-j}{k+3} z^j \\ &= \sum_{i=0}^{k} a_{i,k+1} z^i - z^{k+1}, \end{aligned}$$

which is of the required form in (3.1). We then compute w_{k+1} from the Maclaurin expansion of $D_{k+1}(z)/E_{k+1}(z)$ and find that $w_{k+1} = -1 + \sum_{i=0}^{k} a_{i,k+1} f_{i+1}$. That is,

$$\begin{aligned} w_{k+1} &= -1 + \sum_{\substack{j=0 \\ j \text{ odd}}}^{k} \frac{j+1}{k+3} f_{j+1} + \sum_{\substack{j=0 \\ j \text{ even}}}^{k} \frac{k+3-j}{k+3} f_{j+1} \\ &= -1 + \sum_{\substack{j=2 \\ j \text{ even}}}^{k+1} \frac{j}{k+3} f_j + \sum_{\substack{j=1 \\ j \text{ odd}}}^{k} f_j - \sum_{\substack{j=2 \\ j \text{ odd}}}^{k} \frac{j-1}{k+3} f_j \\ &= -1 + f_{k+1} + \frac{f_{k+1} - 1}{k+3} + \frac{1}{k+3} \sum_{j=2}^{k+1} (-1)^j j f_j \end{aligned}$$

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$$= -1 + f_{k+1} + \frac{f_{k+1}-1}{k+3} + f_k - \frac{f_{k+1}+1}{k+3}$$
$$= f_{k+2} - \frac{k+5}{k+3}.$$

Here we have used the fact that for odd k we have both $f_1 + f_3 + f_5 + \cdots + f_k = f_{k+1}$ and $\sum_{j=2}^{k+1}(-1)^j j f_j = (k+3)f_k - f_{k+1} - 1$. These are easily proved using induction, for example.

Finally we can use (2.5) to see that

$$w_{k+1}^{+} = f_{k+2} - \frac{k+5}{k+3} + \frac{4((k+2)^2-5)}{(k+2)^2-5+k^2-5} = f_{k+2} + \frac{k+1}{k-1}.$$

Both are of the required form in (3.2) and (3.3).

Thus whenever n is odd, for $u_n = w_n = f_{n+1} - 1$ we have a Newman Pisot number ν_n which is a root of $D_n(z) = 1 + z^2 + z^4 + \cdots + z^{n-1} - z^n$. There are no others in $\mathcal{N}_n$, and in particular there are none of even degree. Further, the *path to infinity* in this search tree corresponds to the limit point of the ν_n (see Section 2). This path is given by the sequence of u_n such that $w_n < u_n < w_n^*$ for all n, with $u_n = f_{n+1}$. This corresponds to the function $f(z) = 1 + z + 2z^2 + \cdots + f_{n+1}z^n + \cdots = \frac{1}{1-z-z^2}$ which has only one pole inside the unit circle at $a = (\sqrt{5}-1)/2$. Thus $\nu' = 1/a = (1+\sqrt{5})/2$ is the only limit point of Newman Pisot numbers. Finally, noting the change of sign between $D_n(\nu_{n-2})$ and $D_n(2)$ for odd $n \geq 5$ it is easy to see that the sequence $\{\nu_n\}$ is strictly increasing. ■

4 Salem Numbers from Newman Polynomials

We now prove Theorem 1.2, using a construction that is analogous to the one from [7, §4]. For convenience we change our notation slightly from previous sections.

Proof of Theorem 1.2 For even $m \geq 4$, let

$$f_m(z) = z^{m-1} - \sum_{k=0}^{(m-2)/2} z^{2k},$$

and let ν_m be the Newman Pisot number with $f_m(z)$ as its minimal polynomial. For each $n \geq 1$, define the sequence of polynomials

$$A_{m,n}(z) = z^{2mn} + f_m^*(z) \sum_{k=0}^{2n-1} z^{mk}.$$

Notice that $A_{m,n}(z)$ is reciprocal, and is in $\mathcal{N}$. We will show that for each m, the polynomials $A_{m,n}(z)$ give a sequence of Salem numbers that approach ν_m from below. Write $A_{m,n}(z) = P_{m,n}(z) - zQ_{m,n}(z)$, where

$$P_{m,n}(z) = \sum_{k=0}^{2n} z^{mk} \quad \text{and} \quad Q_{m,n}(z) = \sum_{k=0}^{mn-1} z^{2k}.$$

Define

$$\begin{aligned}a_{m,n}(t) &= e(-mnt)A_{m,n}(e(t)),\\ p_{m,n}(t) &= e(-mnt)P_{m,n}(e(t)),\\ q_{m,n}(t) &= e(-(mn-1)t)Q_{m,n}(e(t)),\end{aligned}$$

where $e(t) = e^{2\pi i t}$. Then $a_{m,n}(t) = p_{m,n}(t) - q_{m,n}(t)$, where both $p_{m,n}(t)$ and $q_{m,n}(t)$ are real-valued, periodic functions with period 1. Further, $p_{m,n}(t)$ has mn simple zeroes in the interval $(0, 1/2)$, at the points

$$S_p = \left\{ \frac{k}{2mn+m} \;\middle|\; 1 \le k < \frac{2mn+m}{2} \text{ and } (2n+1) \nmid k \right\},$$

and $q_{m,n}(t)$ has $mn - 1$ simple zeroes in the same interval, at the points

$$S_q = \left\{ \frac{j}{2mn} \;\middle|\; 1 \le j \le mn-1 \right\}.$$

Now the inequalities

$$\frac{k}{2mn+m} < \frac{k-j_k}{2mn} \le \frac{k+1}{2mn+m} < \frac{k+1-j_k}{2mn}, \tag{4.1}$$

where $j_k = \lfloor k/(2n+1) \rfloor$, hold for $1 \le k < (2mn+m)/2$ and $(2n+1) \nmid k$. As k varies in this range, j_k varies from 0 to $(m-2)/2$. The central equality in (4.1) is attained if and only if $k \equiv 2n \pmod{2n+1}$, in which case

$$\frac{k-j_k}{2mn} = \frac{j_k+1}{m} = \frac{k+1}{2mn+m},$$

but then this last fraction is not in S_p. For these values of k then, we have

$$\frac{k}{2mn+m} < \frac{k-j_k}{2mn} < \frac{k+2}{2mn+m} < \frac{k+1-j_k}{2mn},$$

where $k + 1 - j_k = k + 2 - j_{k+2}$. Thus, between every two consecutive zeroes of $p_{m,n}(t)$ in $(0, 1/2)$ there is exactly one zero of $q_{m,n}(t)$. This means that $a_{m,n}(t)$ has at least $mn - 1$ zeroes in $(0, 1/2)$, and so the polynomial $A_{m,n}(z)$ has at least $2mn - 2$ zeroes on the unit circle.

Since $A_{m,n}(0) = 1$ and $A_{m,n}(1) = 1 - n(m-2) < 0$, it follows that $A_{m,n}(z)$ has a real root in the interval $(0, 1)$, and, since $A_{m,n}(z)$ is reciprocal, one real root $\alpha_{m,n}$ in $(1, \infty)$ as well. This accounts for all $2mn$ roots of $A_{m,n}(z)$ and we conclude that $\alpha_{m,n}$ is a Newman Salem number.

We next show that for each m, the sequence $\{\alpha_{m,n}\}_{n=1}^{\infty}$ converges to ν_m from below. Since $A_{m,n}(z) = A^*_{m,n}(z) = 1 + z f_m(z) \sum_{k=0}^{2n-1} z^{mk}$, it follows that $A_{m,n}(\nu_m) = 1$. Recalling that $A_{m,n}(1) < 0$, we conclude that $\alpha_{m,n} < \nu_m$ for all n. Using the reciprocity of $A_{m,n+1}(z)$, we have that $A_{m,n+1}(z) = A_{m,n}(z) + z^{2mn+1} f_m(z)(z^m + 1)$, which

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means that $\operatorname{sgn}(A_{m,n+1}(\alpha_{m,n})) = \operatorname{sgn}(f_m(\alpha_{m,n}))$. But $\alpha_{m,n} < \nu_m$, and so $f_m(\alpha_{m,n}) < 0$, by which we conclude that $\alpha_{m,n+1} > \alpha_{m,n}$ for all n. The sequence $\{\alpha_{m,n}\}_{n=1}^{\infty}$, being strictly increasing and bounded above, must be convergent.

Finally, by writing

$$A_{m,n}(z) = z^{2mn}\Big[1 + \frac{f_m^*(z)}{z^m - 1}\Big] + \frac{f_m^*(z)}{1 - z^m}$$

for $z \in (-1, 1)$, we see that as $n \to \infty$, $A_{m,n}(z)$ converges uniformly to $f_m^*(z)/(1-z^m)$ on compact subsets of $(-1, 1)$. This means that $\{\alpha_{m,n}\}_{n=1}^{\infty}$ converges to ν_m, and thus the polynomials $A_{m,n}(z)$ yield the required sequence of Newman Salem numbers. ∎

5 Bounds on Roots of Newman Polynomials

Suppose that $\mathcal{F}$ is a given class of polynomials of degree d, and consider the set $R = \{\xi \in \mathbb{C} : f(\xi) = 0 \text{ for some } f \in \mathcal{F}\}$. What can be said about the element of R of maximum modulus? The set of polynomials of degree d with all coefficients from the set $\{-1, +1\}$ is commonly denoted $\mathcal{L}_d$, and was studied in [7]. In that paper, it was proved that the polynomial $z^d - z^{d-1} - \cdots - z - 1$ is the only member of $\mathcal{L}_d$ that is a minimal polynomial for a Pisot number. Following the notation of that paper, we let γ_d denote this Pisot number. If $\mathcal{F} = \mathcal{L}_d$, then the following result due to Cauchy [6, Chapter 7] easily shows that γ_d is the element of R of maximum modulus.

Theorem 5.1 *All the zeroes of the polynomial $f(z) = \sum_{i=0}^{d} a_i z^i$, $a_d \neq 0$, lie in the circle $|z| \leq r$, where r is the positive root of the equation*

$$|a_0| + |a_1|z + \cdots + |a_{d-1}|z^{d-1} - |a_d|z^d = 0.$$

The next theorem considers the case $\mathcal{F} = \mathcal{N}_d$ for odd d.

Theorem 5.2 *Suppose that $f(z)$ is a Newman polynomial of odd degree $d > 2$. Then all the zeroes of $f(z)$ lie in the circle $|z| \leq \nu_d$, where ν_d is the positive root of the equation*

$$z^d - z^{d-1} - z^{d-3} - \cdots - z^2 - 1 = 0.$$

Proof of Theorem 5.2 We first observe that

$$z^d(z^2 - z - 1) + 1 = (1 - z^2)(1 + z^2 + z^4 + \cdots + z^{d-1} - z^d)$$

for odd $d \geq 3$. Thus the roots of $z^d(z^2 - z - 1) + 1$ are simply those of

$$1 + z^2 + z^4 + \cdots + z^{d-1} - z^d,$$

together with $+1$ and -1.

Set $f(z) = z^d + a_{d-1}z^{d-1} + \cdots + a_1 z + 1$, where each a_i, $1 \leq i \leq d-1$, is either 0 or 1. Consider the polynomial

$$\begin{aligned} h(z) &= (z - a_{d-1})f(z) \\ &= z^{d+1} + (a_{d-2} - a_{d-1}^2)z^{d-1} \\ &\qquad + (a_{d-3} - a_{d-1}a_{d-2})z^{d-2} + \cdots + (1 - a_{d-1}a_1)z - a_{d-1}. \end{aligned}$$

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If $|z| > 1$, then

$$\begin{aligned}
|h(z)| &\geq |z|^{d+1} - \left(|a_{d-2} - a_{d-1}^2||z|^{d-1} + \cdots + |1 - a_{d-1}a_1||z| + |a_{d-1}|\right) \\
&\geq |z|^{d+1}\left(1 - \frac{1}{|z|^2} - \frac{1}{|z|^3} - \cdots - \frac{1}{|z|^{d+1}}\right) \\
&= |z|^{d+1}\left(1 - \frac{|z|^d - 1}{|z|^{d+2} - |z|^{d+1}}\right) \\
&= \frac{|z|^d\left(|z|^2 - |z| - 1\right) + 1}{|z| - 1} \\
&> 0,
\end{aligned}$$

for $|z| > \nu_d$. Since $|z - a_{d-1}| > 0$ when $|z| > 1$, we conclude that $|f(z)| > 0$ for $|z| > \nu_d$ and the required result follows. ■

Thus when $\mathcal{F} = \mathcal{N}_d$ for odd d, we see that ν_d is the element of R of maximum modulus. In both of the cases above, this maximal element is also a Pisot number.

Remark 1 The inequalities in the above proof do not require d to be odd. When d is even, $z^d(z^2 - z - 1) + 1 = (1 - z)(1 + z + z^2 + \cdots + z^{d-1} - z^{d+1})$; notice that the latter factor is one of the types of polynomials (3.6) that was produced by the algorithm in the proof of Theorem 1.1. If r_d denotes the root, greater than 1, of the equation $z^d(z^2 - z - 1) + 1 = 0$, then we could conclude that all zeroes of the Newman polynomial $f(z)$ lie in the circle $|z| \leq r_d$. However, the bound is only sharp for odd d.

Remark 2 In the above proof, if we replace the sum

$$1 - \frac{1}{|z|^2} - \frac{1}{|z|^3} - \cdots - \frac{1}{|z|^{d+1}}$$

by the infinite series

$$1 - \frac{1}{|z|^2} - \frac{1}{|z|^3} - \cdots - \frac{1}{|z|^{d+1}} - \frac{1}{|z|^{d+2}} - \cdots,$$

which is equal to

$$\frac{|z|^2 - |z| - 1}{|z| - 1},$$

we can conclude the following. If $f(z)$ is a Newman polynomial and ξ any root of $f(z)$, then $|\xi| < (1 + \sqrt{5})/2$. This bound also appears in [8], as well as independently in [5, 11]; the first of these is a more thorough investigation of the geometric properties of the set R with $\mathcal{F} = \mathcal{N}$.

This second remark leads to Theorem 1.1 once again. From [1, Chapter 7.2], we know that the only Pisot numbers strictly less than $(1 + \sqrt{5})/2$, aside from the root

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$\theta'' = 1.56175\cdots$ of $z^6 - 2z^5 + z^4 - z^2 + z - 1$, are roots of the following three types of polynomials:

$$P_{2n}(z) = 1 + z + z^2 + \cdots + z^{2n-1} - z^{2n+1},$$
$$P_{2n+1}(z) = 1 + z^2 + z^4 + \cdots + z^{2n} - z^{2n+1}$$
$$F_n(z) = 1 - z^2 + z^n(1 + z - z^2),$$

each for $n \geq 1$. Of the three types of polynomials above, only P_{2n+1} can be derived from a Newman polynomial. Thus, Theorem 1.1 would also follow from Remark 2 and [1, Chapter 7.2].

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