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ON EXPLICIT BOUNDS IN LANDAU'S THEOREM. II

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1. Quite some years ago a number of mathematicians were interested in obtaining explicit expressions for the bounds in Schottky's and Landau's theorems, specifically numerically evaluable bounds of a particular form. The best bounds of this type in Schottky's theorem were given by the author [**3**]. For Landau's theorem the chosen form is as follows. Let $F(Z)$ be regular in $|Z| < 1$, omit the values 0 and 1 and have Taylor expansion about $Z = 0$

$$F(Z) = a_0 + a_1 Z + \ldots .$$

Then

$$|a_1| \leqq 2|a_0|\{|\log|a_0|| + K\}.$$

Using the same method employed for Schottky's theorem the author showed that one can take $K = 5.94$. By a slight modification of the author's method Lai [**6**] gave the further value $K = 4.76$. On the other hand it is known [**7**] that one cannot have K less than $(1/4\pi^2)\Gamma(\frac{1}{4})^4$ which is approximately 4.37. In this paper we will prove that this particular value is indeed the best value for K.

The proof begins as before with the remark that for given a_0 the maximal value of $|a_1|$ is attained for the function $F_0(Z)$ mapping $|Z| < 1$ onto the universal covering surface of the W-sphere punctured at $0, 1, \infty$. This defines a function which we will call $\mu(W)$. The first step is to show that

$$\mu(W) \leqq 2|W|\{|\log|W|| + K^*\}$$

with K^* equal to one-half the maximum of $\mu(W)$ on $|W| = 1$. This uses the basic remark that $(\mu(W))^{-1}|dW|$ is the Poincaré metric for the punctured sphere and employs a technique of [**1**, p. 13]. Finally it is shown that the maximum of $\mu(W)$ on $|W| = 1$ is attained at $W = -1$. This is done by the methods of the Topological Theory of Functions.

2. LEMMA 1. *Let $2K^*$ denote the maximum of $\mu(W)$ on $|W| = 1$. Then*

$$(1) \quad \mu(W) \leqq 2|W|\{|\log|W|| + K^*\}$$

for W in D, the sphere punctured at $0, 1, \infty$.

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We denote $(\mu(W))^{-1}$ by $\lambda(W)$, $[2r(\log r + K^*)]^{-1}$ by $\rho(W)$ for $W = re^{i\theta}$, $r \geqq 1$, θ real. As remarked in [**3**], to prove (1) it is enough to show $\lambda(W) \geqq \rho(W)$ for $|W| \geqq 1$. From the inequality on p. 425 of [**4**]

$$|a_1| \leqq 2|a_0|\{\log|a_0| + M(t)\}$$

it follows that $\lambda(W) > \rho(W)$ for $|W|$ sufficiently large. Further as W tends to 1 from $|W| \geqq 1$

$$\lim \lambda(W) = \infty, \lim \rho(W) = (2K^*)^{-1}.$$

Thus if we had $\lambda(W) < \rho(W)$ at a point in $|W| > 1$, $\log \lambda(W) - \log \rho(W)$ would have a point of minimum W_0 in this set at which we would have

$$\log \lambda(W_0) < \log \rho(W_0).$$

Since, as remarked above, $\lambda(W)|dW|$ is the Poincaré metric for D (in the usual notation, in Ahlfors' notation it would be $2\lambda(W)|dW|$) we have

$$\Delta \log \lambda(W) = 4(\lambda(W))^2.$$

Moreover, by direct calculation

$$\Delta \log \rho(W) = 4r^2(\rho(W))^2.$$

At the minimum point W_0 we would have

$$\Delta(\log \lambda(W) - \log \rho(W)) \geqq 0$$

thus

$$4\lambda^2(W_0) - 4|W_0|^2\rho^2(W_0) \geqq 0$$

and

$$\lambda(W_0) > \rho(W_0)$$

a contradiction.

3. LEMMA 2. *The maximum of* $\mu(W)$ *on* $|W| = 1$ *is* $(1/2\pi^2)\Gamma(\frac{1}{4})^4$.

This is the value of $\mu(W)$ for $W = -1$ [**7**]. It might be possible to obtain this result from an explicit representation of the function $F_0(Z)$ but the most familiar ones do not seem particularly suited to such an application. We will proceed instead as follows. We denote $\nu(W) = |W|^{-1}\mu(W)$ and study the level sets of ν by the methods of the Topological Theory of Functions. We observe first that ν is symmetric both in the real axis and in the unit circle. It tends to zero as we approach $W = 1$ and to infinity as we approach $W = 0$ and $W = \infty$. At a non-critical point of ν the level sets have the structure of a regular curve family. A priori the critical points of ν need not be isolated but the curve family

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structure at them is either that of a regular curve family (non-isolated) or that of a saddle point or circle domain (isolated). Since $\log \nu$ is superharmonic, ν can have no points of minimum in D. Again, since $\Delta \log \nu < 0$, all saddle points are simple (i.e., the limiting end point of four level arcs). Thus, while a priori the function ν need not be pseudoharmonic, if we delete from D the isolated maximum points, in the residual domain the level sets of ν form a harmonique curve family $\mathscr{F}$ [**5**]. Since they are level sets recurrence of elements of $\mathscr{F}$ is ruled out.

LEMMA 3. *In a suitable deleted neighbourhood of* $W = 0, 1$ *or* ∞, *the elements of* $\mathscr{F}$ *have the structure of a circle domain.*

Consideration of the explicit asymptotic behavior of the Poincaré metric at the points $0, 1, \infty$ shows that they cannot be accumulation points of critical points of ν. Consider then, for example, the case $W = 0$ and a deleted disc neighborhood of this point not containing $W = 1$, $W = \infty$ or an isolated point of maximum for ν. We can apply the analysis of harmonique curve families in doubly-connected domains given in [**2**]. No element of $\mathscr{F}$ can have a limiting end point at 0 and since the elements of $\mathscr{F}$ are level sets of a C^2 function there can be no asymptotes. The result is then immediate. The cases of $W = 1$ and $W = \infty$ are just the same.

It follows similarly that every element of $\mathscr{F}$ is either a Jordan curve or an open arc joining two (not necessarily distinct) saddle points. Moreover there are only a finite number of saddle points for $\mathscr{F}$. The elements of $\mathscr{F}$ with limiting end points at them divide the sphere into a finite number of domains. From the index theory [**5**] for a harmonique curve family it follows that each such domain is simply- or doubly-connected. A simply-connected domain contains either 0, 1 or ∞ or an isolated point of maximum for ν. The elements of $\mathscr{F}$ in a double-connected domain have the structure of a ring domain.

Consider the simply-connected domain E containing $W = 1$. Consider also a saddle point P_1 for ν at which the value of ν is ν_1. In two opposite sectors determined by elements of $\mathscr{F}$ with limiting end points at P_1 we will have $\nu < \nu_1$. If either of these sectors did not lie in E it would lie either in a simply-connected domain, in which there would be a point of minimum for ν, which is impossible, or in a doubly-connected domain, say G_1. On the opposite boundary component of G_1 we would have a saddle point P_2 where ν would have a value ν_2 with $\nu_2 < \nu_1$. In G_1, $\nu > \nu_2$. In some sector at P_2 we would have $\nu < \nu_2$ and this sector could not lie in E or G_1. In a finite number of steps we would obtain a contradiction.

Thus every saddle point for ν lies on the boundary of E and opposite sectors lie in pairs in E. Further since there are three exception points for ν, by the symmetry of ν and the index theory, $W = -1$ must be a saddle

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point. Moreover opposite pairs of sectors there will contain respectively arcs on $|W| = 1$ and $\mathscr{I}W = 0$. The latter cannot lie in E since this domain is simply-connected and has the same symmetries as ν. Thus the former arcs lie in E and again for the same reasons the open arcs on $|W| = 1$ determined by $W = \pm 1$ lie in their entirety in E. Thus on these arcs ν decreases steadily from its values at $W = -1$ to its limiting value (zero) at $W = 1$. This completes the proof.

THEOREM. *If* $F(Z)$ *is regular for* $|Z| < 1$, *does not take the values* 0 *and* 1 *and has Taylor expansion about* $Z = 0$

$$F(Z) = a_0 + a_1 Z + \ldots$$

then

$$|a_1| \leqq 2|a_0|\{|\log|a_0|| + K^*\}$$

with $K^* = (1/4\pi^2)\Gamma(\frac{1}{4})^4$. *This result is best possible.*

Remark. The proof of Lemma 3 shows that in a neighborhood of an isolated boundary point the level sets of a representative function for the Poincaré metric have the structure of a circle domain.

REFERENCES

1. Lars V. Ahlfors, *Conformal invariants* (McGraw-Hill, 1973).

2. James A. Jenkins and Marston Morse, *Curve families F* locally the level curves of a pseudoharmonic function*, Acta Mathematica *91* (1954), 1–42.

3. James A. Jenkins, *On explicit bounds in Schottky's theorem*, Can. J. Math. *7* (1955), 76–82.

4. ——— *On explicit bounds in Landau's theorem*, Can. J. Math. *8* (1956), 423–425.

5. ——— *A topological Three Pole Theorem*, Indiana University Mathematics Journal *21* (1972), 1013–1018.

6. W. Lai, *Über den Satz von Landau*, Science Record *4* (1960), 339–342.

7. Review of [**3**], Mathematical Reviews *16* (1955), 579.

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