A NOTE ON CERTAIN INTEGRAL EQUATIONS OF ABEL-TYPE

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Let $f(x)$ be a function of a real variable x, such that $f(x)$ is monotone increasing for $a \leqq x \leqq b$. Integral equations of the kind

$$\int_a^x \{f(x)-f(t)\}^{-\alpha} h(t)dt = g(x), \quad a<x<b, \quad 0<\alpha<1 \qquad \text{...........(1)}$$

$$\int_x^b \{f(t)-f(x)\}^{-\alpha} h(t)dt = g(x), \quad a<x<b, \quad 0<\alpha<1 \qquad \text{...........(2)}$$

frequently occur in problems of mathematical physics, where $g(x)$ is known and $h(t)$ is to be determined. The solution, however, does not appear to be well known. The purpose of this note is to give an elementary solution of these integral equations.

To solve (1), we consider

$$\int_a^x f'(u)\{f(x)-f(u)\}^{\alpha-1} g(u)du. \qquad \text{..........................(3)}$$

Substituting the value of $g(u)$ from (1) in (3) and interchanging the order of integrations we get

$$\int_a^x f'(u)\{f(x)-f(u)\}^{\alpha-1} g(u)du$$

$$= \int_a^x h(t)dt \int_t^x f'(u)\{f(u)-f(t)\}^{-\alpha}\{f(x)-f(u)\}^{\alpha-1}du. \qquad \text{......(4)}$$

However, it is easily shown that

$$\int_t^x f'(u)\{f(u)-f(t)\}^{-\alpha}\{f(x)-f(u)\}^{\alpha-1}du = \pi \operatorname{cosec} \pi\alpha \qquad \text{............(5)}$$

Hence

$$h(t) = \frac{\sin \alpha\pi}{\pi}\frac{d}{dt}\int_a^t f'(u)\{f(t)-f(u)\}^{\alpha-1} g(u)du. \qquad \text{..................(6)}$$

In a similar manner it may be shown that

$$\int_t^b f'(u)\{f(u)-f(t)\}^{\alpha-1} g(u)du = \pi \operatorname{cosec} \pi\alpha \int_t^b h(x)dx \qquad \text{...........(7)}$$

and hence that the solution of (2) is

$$h(t) = -\frac{\sin \pi\alpha}{\pi}\frac{d}{dt}\int_t^b f'(u)\{f(u)-f(t)\}^{\alpha-1}g(u)du. \qquad \text{(8)}$$

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