Bull. Aust. Math. Soc. **82** (2010), 254–264
doi:10.1017/S000497271000002X

TURÁN TYPE INEQUALITIES FOR MODIFIED BESSEL FUNCTIONS

ÁRPÁD BARICZ

(Received 5 December 2009)

Dedicated to my daughter Boróka

Abstract

In this paper our aim is to deduce some sharp Turán type inequalities for modified Bessel functions of the first and second kinds. Our proofs are based on explicit formulas for the Turánians of the modified Bessel functions of the first and second kinds and on a formula which is related to the infinite divisibility of the Student t-distribution.

2000 *Mathematics subject classification*: primary 33C10; secondary 39B62.

Keywords and phrases: modified Bessel functions, Bessel functions, Turán type inequalities.

1. Introduction

Since the publication in 1948 by Szegő [29] of the famous Turán inequality for classical Legendre polynomials [31], many authors have deduced analogous results for classical (orthogonal) polynomials and special functions. In the last 60 years it has been shown by several researchers that the most important special functions satisfy a Turán inequality; see, for example, the most recent papers on this topic written in the last two years [2, 3, 5–11, 15] and the references therein. The Turán type inequalities now have an extensive literature and some of the results have been applied successfully in problems which arise in information theory, economic theory and biophysics. For more details the interested reader is referred to the papers [9, 11, 20, 26].

Motivated by these applications, the Turán type inequalities have recently come under the spotlight once again and it has been shown that, for example, the classical Gauss and Kummer hypergeometric functions, as well the generalized hypergeometric functions, satisfy naturally some Turán type inequalities [5, 7, 11]. We note that since the spherical functions on a sphere (that is, the Legendre polynomials) approach the spherical functions on the plane (the Bessel functions) as the radius approaches infinity, it was natural to find analogous inequalities for Bessel functions. In the

Research supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences.

1950s important contributions to the subject were the papers [25, 27, 28, 30], where it was shown, among other things, that the Bessel and modified Bessel functions of the first kind satisfy naturally some Turán type inequalities. Some of the results related to Turán type inequalities concerning Bessel and modified Bessel functions of the first kind were rediscovered later in the papers [4, 6, 17, 19] by using different approaches. Recently, the author [9] used the corresponding Turán type inequalities for the modified Bessel functions of the first and second kinds in order to present a simpler proof of a monotonicity problem, which arises in biophysics. In this paper our aim is to complete the results from [9] by proving that the corresponding Turán type inequalities are valid for a wider range of the parameter.

The paper is organized as follows. In Section 2 we reconsider the Turán type inequalities for the modified Bessel functions of the first kind and show that these inequalities are sharp and hold for a wider range of the parameter. This gives in fact a partial affirmative answer to a conjecture of Lorch [19]. In Section 3 we reconsider the Turán type inequalities for the modified Bessel functions of the second kind by proving a recent conjecture of the author [9] and by showing that these inequalities are sharp too. As far as we know the sharpness of the Turán type inequalities for modified Bessel functions has not been discussed in the literature. The key tools in our proofs are some explicit formulas for the Turánians of the modified Bessel functions of the first and second kinds and an integral representation of Grosswald [13] and Ismail [14], which was the chief tool in the proof of the infinite divisibility of the Student t-distribution. Finally, in Section 4 we improve the main results of [9, 23, 24] and present an open problem and a conjecture, which may be of interest for further research.

2. Turán type inequalities for modified Bessel functions of the first kind

In this section our aim is to point out that the region of validity of the Turán-type inequalities for modified Bessel functions of the first kind can be extended by using the ideas of Mukherjee and Nanjundiah [21]; see also the paper by Skovgaard [25] for further details.

THEOREM 2.1. *Let I_ν be the modified Bessel functions of the first kind. Then the following Turán-type inequalities hold for all $\nu > -1$ and $u \in \mathbb{R}$:*

$$0 \le I_\nu^2(u) - I_{\nu-1}(u)I_{\nu+1}(u) \le I_\nu^2(u)/(\nu+1). \tag{2.1}$$

In each of these inequalities equality holds if and only if $u = 0$. Moreover, these inequalities are sharp in the sense that the constants $\alpha_\nu = 0$ and $\beta_\nu = 1/(\nu+1)$ are the best possible such that the inequalities

$$\alpha_\nu I_\nu^2(u) \le I_\nu^2(u) - I_{\nu-1}(u)I_{\nu+1}(u) \le \beta_\nu I_\nu^2(u)$$

hold for all $u \in \mathbb{R}$ and $\nu > -1$.

We note that inequality (2.1) for $\nu \ge 0$ is well known. As far as we know it was deduced first in 1951 by Thiruvenkatachar and Nanjundiah [30]. The left-hand side of (2.1) was also deduced by Amos [4] in 1974, and later by Joshi and Bissu [17] in 1991. Note that, as was shown in [10, 19], the function $\nu \mapsto I_{\nu+a}(u)/I_\nu(u)$ is decreasing for each fixed $a \in (0, 2]$ and $u > 0$, where $\nu > -1$ and $\nu \geq -(a+1)/2$. Consequently, the function $\nu \mapsto I_\nu(u)$ is log-concave on $(-1, \infty)$, as the author pointed out in [10]. However, this implies just that the left-hand side of (2.1) holds for all $\nu > 0$. All the same, in 1994 Lorch [19] proved that the left-hand side of (2.1) holds for all $\nu > -1/2$. Moreover, Lorch [19] conjectured that the generalized Turánian $I_\nu^2(u) - I_{\nu-a}(u)I_{\nu+a}(u)$ is positive for all $u > 0$, $a \in (0, 1]$ and $\nu \in (-1, -1/2]$. We note that our result stated in Theorem 2.1 gives an affirmative answer to Lorch's conjecture for the case $a = 1$.

Note that the right-hand side of (2.1) for $\nu \geq 0$ was also proved in [17] and recently in a more general context by the author [6]. Finally, we note that the new proof presented below for the left-hand side of (2.1) is actually a slight modification of the proof stated in [17] by using the ideas of Mukherjee and Nanjundiah [21]. In [17] the authors had already deduced the explicit formula (2.2), but they overlooked the well-known fact that for $\nu > -1$ all the zeros of the Bessel function of the first kind J_ν are real and there is no need to assume in (2.2) that $\nu \geq 0$. The proof, stated below, of the right-hand side of (2.1) is somewhat different from the proof given in [17] and has the advantage that there is no restriction on the parameter ν being positive.

Proof of Theorem 2.1. Since the expressions in (2.1) are even in u, in what follows we suppose without loss of generality that $u \geq 0$. Let us consider the Turánian $\Delta_\nu(u) = I_\nu^2(u) - I_{\nu-1}(u)I_{\nu+1}(u)$, which in view of the recurrence relations $I_{\nu-1}(u) = (\nu/u)I_\nu(u) + I_\nu'(u)$ and $I_{\nu+1}(u) = -(\nu/u)I_\nu(u) + I_\nu'(u)$, can be rewritten as $\Delta_\nu(u) = (1 + \nu^2/u^2)I_\nu^2(u) - [I_\nu'(u)]^2$. On the other hand, it is known that I_ν is in fact a particular solution of the second-order differential equation $u^2 v''(u) + uv'(u) - (u^2 + \nu^2)v(u) = 0$, and consequently we obtain that

$$I_\nu''(u) = (1 + \nu^2/u^2)I_\nu(u) - (1/u)I_\nu'(u).$$

These relations together imply that [17]

$$\Delta_\nu(u) = \frac{1}{u} I_\nu^2(u) \left[\frac{u I_\nu'(u)}{I_\nu(u)}\right]'.$$

Now, by again using the recurrence formula

$$I_\nu'(u)/I_\nu(u) = \nu/u + I_{\nu+1}(u)/I_\nu(u)$$

and the Mittag–Leffler expansion

$$\frac{I_{\nu+1}(u)}{I_\nu(u)} = \sum_{n\geq 1} \frac{2u}{u^2 + j_{\nu,n}^2},$$

where $j_{\nu,n}$ is the nth positive zero of the Bessel function J_ν, we obtain that

$$\left[\frac{u I_\nu'(u)}{I_\nu(u)}\right]' = \sum_{n\geq 1} \frac{4u j_{\nu,n}^2}{(u^2 + j_{\nu,n}^2)^2}$$

https://doi.org/10.1017/S000497271000002X Published online by Cambridge University Press

and consequently [17]

$$\Delta_\nu(u) = 4I_\nu^2(u) \sum_{n\geq 1} \frac{j_{\nu,n}^2}{(u^2 + j_{\nu,n}^2)^2}. \tag{2.2}$$

Since for $\nu > -1$ all the zeros of J_ν are real, this in turn implies that the left-hand side of (2.1) holds. Now consider the function $\varphi_\nu : [0, \infty) \to \mathbb{R}$, defined by

$$\varphi_\nu(u) = \frac{\Delta_\nu(u)}{I_\nu^2(u)} = 1 - \frac{I_{\nu-1}(u)I_{\nu+1}(u)}{I_\nu^2(u)} = \sum_{n\geq 1} \frac{4j_{\nu,n}^2}{(u^2 + j_{\nu,n}^2)^2}.$$

Clearly,

$$\varphi_\nu'(u) = -\sum_{n\geq 1} \frac{16u j_{\nu,n}^2}{(u^2 + j_{\nu,n}^2)^3} \leq 0$$

for all $u \geq 0$ and $\nu > -1$, which implies that φ_ν is decreasing for all $\nu > -1$. On the other hand, by using the Rayleigh formula

$$\sum_{n\geq 1} \frac{1}{j_{\nu,n}^2} = \frac{1}{4(\nu + 1)} \tag{2.3}$$

and the asymptotic formula [1, p. 377]

$$I_\nu(u) \sim \frac{e^u}{\sqrt{2\pi u}} \left[1 - \frac{4\nu^2 - 1}{1!(8u)} + \frac{(4\nu^2 - 1)(4\nu^2 - 9)}{2!(8u)^2} - \cdots\right],$$

which holds for large values of u and for fixed $\nu > -1$, we obtain

$$\varphi_\nu(0) = \sum_{n\geq 1} \frac{4}{j_{\nu,n}^2} = \frac{1}{\nu + 1} \quad \text{and} \quad \lim_{u\to\infty} \varphi_\nu(u) = 0.$$

Thus we have proved that for all $u \geq 0$ and $\nu > -1$,

$$\lim_{u\to\infty} \varphi_\nu(u) = 0 \leq \frac{\Delta_\nu(u)}{I_\nu^2(u)} \leq \frac{1}{\nu + 1} = \varphi_\nu(0),$$

that is, the Turán-type inequalities in (2.1) hold. Moreover, from this discussion we conclude that the results are sharp. More precisely, the constants $\alpha_\nu = 0$ and $\beta_\nu = 1/(\nu + 1)$ are the best possible such that the inequalities

$$\alpha_\nu I_\nu^2(u) \leq I_\nu^2(u) - I_{\nu-1}(u)I_{\nu+1}(u) \leq \beta_\nu I_\nu^2(u)$$

hold for all $u \geq 0$ and $\nu > -1$. □

3. Turán type inequalities for modified Bessel functions of the second kind

Let us consider the modified Bessel function of the second kind K_ν, which is sometimes called the MacDonald or Hankel function. It is known that K_ν satisfies the Turán type inequality

$$K_\nu^2(u) - K_{\nu-1}(u)K_{\nu+1}(u) < 0 \tag{3.1}$$

for all $u > 0$ and $\nu \in \mathbb{R}$. This was proved in 1978 by Ismail and Muldoon [16]. For $\nu > 1/2$ inequality (3.1) was also deduced by Laforgia and Natalini [18] in 2006. Ismail and Muldoon [16], by using the Nicholson formula concerning the product of two modified Bessel functions of different order, proved that [16, Lemma 2.2] the function $\nu \mapsto K_{\nu+a}(u)/K_\nu(u)$ is increasing on $\mathbb{R}$ for each fixed $u > 0$ and $a > 0$. As Muldoon [22] pointed out, this implies that $\nu \mapsto K_\nu(u)$ is log-convex on $\mathbb{R}$ for each fixed $u > 0$. Recently, by using the classical Hölder–Rogers inequality, the author [10] pointed out that the function $\nu \mapsto K_\nu(u)$ is in fact strictly log-convex on $\mathbb{R}$ for each fixed $u > 0$.

Now, consider the Turán type inequality [9]

$$(2\nu - 1)K_\nu^2(u) - (\nu - 1)K_{\nu-1}(u)K_{\nu+1}(u) > 0, \tag{3.2}$$

which holds for all positive integers $\nu = n \geq 1$ and for $u > 0$. We note that in [9] it was conjectured that (3.2) holds for all real $\nu \geq 0$ and $u > 0$. Now suppose that $\nu = 0$. In this case (3.2) becomes $K_0^2(u) - K_{-1}(u)K_1(u) < 0$, which in view of (3.1) is true for all $u > 0$. Similarly, when $\nu = 1$ the inequality (3.2) clearly holds. When $\nu \in (0, 1)$, we can easily see that (3.1) implies (3.2). To see this, just rewrite (3.2) as

$$K_\nu^2(u) - K_{\nu-1}(u)K_{\nu+1}(u) < \frac{\nu}{1-\nu}K_\nu^2(u).$$

Thus, the conjecture stated in [9] for the case $\nu \in [0, 1]$ follows immediately from (3.1). In this section our aim is to prove the remaining part when $\nu > 1$ by improving the inequality (3.2). Moreover, we give an alternative proof for (3.1), and we show that in a sense inequality (3.1) is sharp. The new proof given below for (3.1) actually clarifies the connection between (3.1) and the improved version of (3.2) when $\nu > 1$, that is, in this case the sharp version of (3.2) is the counterpart of (3.1).

THEOREM 3.1. *Let K_ν be the modified Bessel function of the second kind. Then the following Turán type inequalities hold for all $\nu > 1$ and $u > 0$:*

$$K_\nu^2(u)/(1-\nu) < K_\nu^2(u) - K_{\nu-1}(u)K_{\nu+1}(u) < 0. \tag{3.3}$$

Moreover, the right-hand side of (3.3) holds for all $\nu \in \mathbb{R}$. These inequalities are sharp in the sense that the constants $\alpha_\nu = 1/(1-\nu)$ and $\beta_\nu = 0$ are the best possible such that the inequalities

$$\alpha_\nu K_\nu^2(u) < K_\nu^2(u) - K_{\nu-1}(u)K_{\nu+1}(u) < \beta_\nu K_\nu^2(u)$$

hold for all $u > 0$, $\nu > 1$ and $\nu \in \mathbb{R}$, respectively.

https://doi.org/10.1017/S000497271000002X Published online by Cambridge University Press

Observe that when $\nu > 1$, inequality (3.2) becomes

$$K_\nu^2(u) - K_{\nu-1}(u)K_{\nu+1}(u) > \frac{\nu}{1-\nu}K_\nu^2(u).$$

This is an immediate consequence of the left-hand side of (3.3). Thus our main result not only gives an affirmative answer to the conjecture stated in [9], but even improves the conjectured result. Another interesting fact here is the following: we believe, but are unable to prove, that the following conjecture is true.

CONJECTURE 3.2. The function

$$\nu \mapsto \frac{2^{\nu-1}\Gamma(\nu)u^{-\nu}}{K_\nu(u)}$$

is strictly log-convex on $(0, \infty)$ for each fixed $u > 0$.

If our present conjecture were correct, then this would lead to a generalization of the left-hand side of (3.3).

PROOF OF THEOREM 3.1. Our strategy here is as in the proof of Theorem 2.1. Let us first consider the corresponding Turánian,

$$\Delta_\nu(u) = K_\nu^2(u) - K_{\nu-1}(u)K_{\nu+1}(u).$$

Observe that, since $K_\nu(u) = K_{-\nu}(u)$, the Turánian $\Delta_\nu(u)$ is even in ν. Thus, it is enough to show inequality (3.3) for $\nu \geq 0$. In this spirit, in what follows, we assume without loss of generality that $\nu \geq 0$. Now, by using the recurrence relations $K_{\nu-1}(u) = -(\nu/u)K_\nu(u) - K_\nu'(u)$ and $K_{\nu+1}(u) = (\nu/u)K_\nu(u) - K_\nu'(u)$, clearly $\Delta_\nu(u) = (1 + \nu^2/u^2)K_\nu^2(u) - [K_\nu'(u)]^2$. On the other hand, recall that K_ν is a particular solution of the second-order differential equation

$$u^2v''(u) + uv'(u) - (u^2 + \nu^2)v(u) = 0,$$

and this in turn implies that

$$K_\nu''(u) = (1 + \nu^2/u^2)K_\nu(u) - (1/u)K_\nu'(u).$$

Consequently,

$$\Delta_\nu(u) = \frac{1}{u}K_\nu^2(u)\left[\frac{uK_\nu'(u)}{K_\nu(u)}\right]',$$

which in view of the recurrence relation

$$uK_\nu'(u)/K_\nu(u) = -\nu - uK_{\nu-1}(u)/K_\nu(u),$$

can be rewritten as

$$\Delta_\nu(u) = -\frac{1}{u}K_\nu^2(u)\left\{\frac{K_{\nu-1}(u)}{K_\nu(u)} + u\left[\frac{K_{\nu-1}(u)}{K_\nu(u)}\right]'\right\}. \tag{3.4}$$

https://doi.org/10.1017/S000497271000002X Published online by Cambridge University Press

Now consider the integral representation

$$\frac{K_{\nu-1}(\sqrt{u})}{\sqrt{u}K_\nu(\sqrt{u})} = \frac{4}{\pi^2}\int_0^\infty \frac{\gamma(t)\,dt}{u+t^2} \quad \text{where } \gamma(t) = \frac{t^{-1}}{J_\nu^2(t)+Y_\nu^2(t)} \tag{3.5}$$

and $u>0$, $\nu\geq 0$. Here J_ν and Y_ν stand for the Bessel functions of the first and second kind. The above integral representation was given implicitly by Grosswald [13] and written in the form (3.5) by Ismail [14]. Let $\phi_\nu:(0,\infty)\to\mathbb{R}$ be defined by

$$\phi_\nu(u) = \frac{\Delta_\nu(u)}{K_\nu^2(u)} = 1 - \frac{K_{\nu-1}(u)K_{\nu+1}(u)}{K_\nu^2(u)}.$$

Then by using relations (3.4) and (3.5),

$$\begin{aligned}\phi_\nu(u) &= -\frac{1}{u}\bigg[\frac{4}{\pi^2}\int_0^\infty \frac{u\gamma(t)\,dt}{u^2+t^2} + \frac{4}{\pi^2}\int_0^\infty \frac{u(t^2-u^2)\gamma(t)\,dt}{(u^2+t^2)^2}\bigg]\\ &= -\frac{4}{\pi^2}\int_0^\infty \frac{(u^2+t^2+1)\gamma(t)\,dt}{(u^2+t^2)^2} < 0,\end{aligned}$$

and consequently,

$$\phi_\nu'(u) = \frac{8}{\pi^2}\int_0^\infty \frac{u(u^2+t^2+2)\gamma(t)\,dt}{(u^2+t^2)^3} > 0$$

for all $u>0$ and $\nu\geq 0$. Thus, we have proved that ϕ_ν maps $(0,\infty)$ into $(-\infty,0)$ and is strictly increasing. These in turn imply that for all $u>0$ and $\nu\geq 0$,

$$\lim_{u\to 0}\phi_\nu(u) < \phi_\nu(u) < \lim_{u\to\infty}\phi_\nu(u).$$

By using the asymptotic expansion [1, p. 378]

$$K_\nu(u) \sim \sqrt{\frac{\pi}{2u}}e^{-u}\bigg[1+\frac{4\nu^2-1}{8u}+\frac{(4\nu^2-1)(4\nu^2-9)}{2!(8u)^2}+\cdots\bigg],$$

which holds for large values of u and for fixed $\nu\geq 0$, we obtain

$$\beta_\nu = \lim_{u\to\infty}\phi_\nu(u) = 0,$$

which shows that in (3.3) the constant $\beta_\nu = 0$ is the best possible. With this the proof of the right-hand side of (3.3) is done. Now let us focus on the sharpness of the left-hand side of (3.3). Suppose that $\nu>1$. Recall that when $\nu>0$ is fixed and u tends to zero, the asymptotic relation [1, p. 375] $2K_\nu(2u)\sim u^{-\nu}\Gamma(\nu)$ holds. Using this relation, we easily obtain

$$\alpha_\nu = \lim_{u\to 0}\phi_\nu(u) = \frac{1}{1-\nu}$$

for all $\nu>1$, which shows that in this case the constant α_ν is the best possible. □

https://doi.org/10.1017/S000497271000002X Published online by Cambridge University Press

4. Concluding remarks and open problems

4.1. Concluding remarks. In this section we present an immediate consequence of the main results of the previous sections in order to improve the main result of [9].

COROLLARY 4.1. *If* $\nu > -1$ *and* $u > 0$, *then*

$$uI_\nu'(u)/I_\nu(u) < \sqrt{u^2+\nu^2}. \tag{4.1}$$

Moreover, $u \mapsto I_\nu(u)K_\nu(u)$ *is strictly decreasing on* $(0, \infty)$ *for all* $\nu > -1$.

For the sake of completeness recall that (4.1) for $\nu > 0$ was first proved by Gronwall [12] in 1932, motivated by a problem in wave mechanics; in 1950 it also appeared in Phillips and Malin's paper [24] for positive integers $\nu \geq 1$. Moreover, in [9] we proved that (4.1) holds for all $\nu \geq -1/2$ real. We note that the monotonicity of $u \mapsto I_\nu(u)K_\nu(u)$ was also studied for integer $\nu \geq 1$ by Phillips and Malin [24]. Recently, in 2007, it was reconsidered by Penfold *et al.* [23] for $\nu \geq 0$ real, motivated by a problem which arises in biophysics.

In order to extend the above result to the case where $\nu \geq -1/2$ and shorten the proof given in [23], the author [9] recently pointed out that in fact (4.1) is equivalent to the left-hand side of (2.1). Thus, in view of Theorem 2.1 with this the proof of (4.1) is complete. Now consider the monotonicity of $u \mapsto I_\nu(u)K_\nu(u)$. Recall that in [9] it was shown that (3.1) is equivalent to

$$uK_\nu'(u)/K_\nu(u) < -\sqrt{u^2+\nu^2}, \tag{4.2}$$

where $u > 0$ and $\nu \in \mathbb{R}$. Consequently, by using (4.1) and (4.2), it follows that

$$u[\log(I_\nu(u)K_\nu(u))]' < 0,$$

that is, the function $u \mapsto I_\nu(u)K_\nu(u)$ is strictly decreasing on $(0, \infty)$ for all $\nu > -1$, as we required.

There is another proof which follows from the results of the previous sections. From the proofs of Theorems 2.1 and 3.1 it is clear that the function $u \mapsto uI_\nu'(u)/I_\nu(u)$ is strictly increasing on $(0, \infty)$ for all $\nu > -1$ and the function $u \mapsto uK_\nu'(u)/K_\nu(u)$ is strictly decreasing on $(0, \infty)$ for all $\nu \in \mathbb{R}$. Thus, by using the Wronskian recurrence relation

$$\frac{uI_\nu'(u)}{I_\nu(u)} - \frac{uK_\nu'(u)}{K_\nu(u)} = \frac{1}{I_\nu(u)K_\nu(u)},$$

we conclude that $u \mapsto I_\nu(u)K_\nu(u)$ is strictly decreasing on $(0, \infty)$ for all $\nu > -1$.

Now recall that Phillips and Malin [24] showed that for each $u > 0$ and positive integers $\nu > 1$ the inequality

$$uK_\nu'(u)/K_\nu(u) > -\sqrt{u^2\nu/(\nu-1)+\nu^2} \tag{4.3}$$

holds. In [9] we pointed out that (4.3) is equivalent to (3.2). Since the left-hand side of (3.3) implies (3.2), we conclude that (4.3) holds for all $\nu > 1$ real.

https://doi.org/10.1017/S000497271000002X Published online by Cambridge University Press

4.2. On a conjecture and an open problem. Consider the Bessel function of the first kind J_ν and consider the Turánian $\Delta_\nu(u) = J_\nu^2(u) - J_{\nu-1}(u)J_{\nu+1}(u)$. In 1950 and 1951 Szász [27, 28], by using recursions, proved that for all $u \in \mathbb{R}$ and $\nu > 0$ the Turán type inequality

$$J_\nu^2(u) - J_{\nu-1}(u)J_{\nu+1}(u) > J_\nu^2(u)/(\nu+1) \tag{4.4}$$

holds. In 1951 Thiruvenkatachar and Nanjundiah [30] proved the weaker result that, for all $u \in \mathbb{R}$ and $\nu > 0$, the Turán type inequality $\Delta_\nu(u) > 0$ holds. We note that Szász's proof [28] from 1951 was reproduced in 1991 by Joshi and Bissu [17]. Skovgaard [25] showed in 1954 that in fact $\Delta_\nu(u) > 0$ holds for all $\nu > -1$. The key tool in Skovgaard's proof was the identity [25]

$$\Delta_\nu(u) = 4J_\nu^2(u) \sum_{n\geq 1} \frac{j_{\nu,n}^2}{(u^2 - j_{\nu,n}^2)^2},$$

which is analogous to (2.2). Now consider the function $\Phi : (0, \infty)\backslash\Xi \to (0, \infty)$, defined by

$$\Phi_\nu(u) = \frac{\Delta_\nu(u)}{J_\nu^2(u)} = 1 - \frac{J_{\nu-1}(u)J_{\nu+1}(u)}{J_\nu^2(u)} = \sum_{n\geq 1} \frac{4j_{\nu,n}^2}{(u^2 - j_{\nu,n}^2)^2},$$

where $\Xi = \{j_{\nu,n} \mid n \geq 1\}$ is the set of the zeros of the Bessel function of the first kind of order ν. We easily obtain

$$\Phi_\nu'(u) = -\sum_{n\geq 1} \frac{16uj_{\nu,n}^2}{(u^2 - j_{\nu,n}^2)^3} > 0 \quad \forall \nu > -1,\, u \in (0,\, j_{\nu,1}),$$

$$\Phi_\nu''(u) = \sum_{n\geq 1} \frac{16j_{\nu,n}^2(5u^2 + j_{\nu,n}^2)}{(u^2 - j_{\nu,n}^2)^4} > 0 \quad \forall \nu > -1,\, u \in (0, \infty)\backslash\Xi.$$

Thus, the function Φ_ν is strictly increasing and convex on $(0, j_{\nu,1})$ and is strictly convex on $(j_{\nu,n}, j_{\nu,n+1})$ for each $n \geq 1$. Consequently, we have that $\Phi_\nu(u) > \Phi_\nu(0^+)$ for all $\nu > -1$ and $u \in (0, j_{\nu,1})$. On the other hand, by again using the Rayleigh formula (2.3), we obtain $\Phi_\nu(0^+) = 1/(\nu+1)$, which shows that (4.4) holds for all $\nu > -1$ and $u \in (0, j_{\nu,1})$, and the constant $1/(\nu+1)$ is the best possible. Moreover, since Φ_ν is strictly convex on $(j_{\nu,n}, j_{\nu,n+1})$ for each $n \geq 0$, we conjecture that Szász's result (4.4) can be improved significantly as follows.

CONJECTURE 4.2. Let $\nu > 0$. Then the equation $\Phi_\nu'(u) = 0$ has infinitely many roots. Denoting these roots by $\alpha_{\nu,n}$, where $\alpha_{\nu,n} \in (j_{\nu,n}, j_{\nu,n+1})$ for all $n \geq 1$, the Turán type inequality (4.4) can be improved as follows:

$$J_\nu^2(u) - J_{\nu-1}(u)J_{\nu+1}(u) > \beta_{\nu,n}J_\nu^2(u),$$

where $u \in (j_{\nu,n}, j_{\nu,n+1})$, $\beta_{\nu,n} = \Phi_\nu(\alpha_{\nu,n})$ for all $n \geq 0$ and the sequence $\{\beta_{\nu,n}\}_{n\geq 0}$ is strictly increasing, where $\alpha_{\nu,0} = j_{\nu,0} = 0$ and $\beta_{\nu,0} = \Phi(\alpha_{\nu,0}^+) = 1/(\nu+1)$.

Now consider the Bessel function of the second kind Y_ν, which is sometimes called the von Neumann function. By using the recurrence formulas

$$Y_{n+1/2}(u) = (-1)^{n+1} J_{-n-1/2}(u) \quad \text{and} \quad Y_{-n-1/2}(u) = (-1)^n J_{n+1/2}(u),$$

from (4.4) we obtain the following inequality, corresponding to (3.3):

$$Y_\nu^2(u) - Y_{\nu-1}(u) Y_{\nu+1}(u) > Y_\nu^2(u)/(1-\nu), \tag{4.5}$$

where $u > 0$, $\nu = \pm(n + 1/2) < 0$ and n is an integer. This suggests the following open problem.

OPEN PROBLEM. Is it true that the Turán type inequality (4.5) holds for all $\nu \in \mathbb{R}$? For those values of ν for which (4.5) holds, is the constant $1/(1-\nu)$ the best possible? Moreover, can (4.5) be improved in the spirit of the above conjecture?

Furthermore, we note that by using the asymptotic relation [1, p. 360] $\pi Y_\nu(2u) \sim -u^{-\nu}\Gamma(\nu)$, which holds when $\nu > 0$ is fixed and u tends to zero, the expression $1 - Y_{\nu-1}(u) Y_{\nu+1}(u)/Y_\nu^2(u)$ tends to $1/(1-\nu)$ if $\nu > 1$ is fixed and u tends to zero. This shows that if (4.5) holds for $\nu > 1$ then the constant $1/(1-\nu)$ is the best possible. As far as we know the Turán type inequalities for the Bessel function of the second kind have not been discussed in the literature.

References

[1] M. Abramowitz and I. A. Stegun (eds), *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables* (Dover Publications, New York, 1965).

[2] H. Alzer and G. Felder, 'A Turán-type inequality for the gamma function', *J. Math. Anal. Appl.* **350** (2009), 276–282.

[3] H. Alzer, S. Gerhold, M. Kauers and A. Lupaş, 'On Turán's inequality for Legendre polynomials', *Expo. Math.* **25** (2007), 181–186.

[4] D. E. Amos, 'Computation of modified Bessel functions and their ratios', *Math. Comp.* **28** (1974), 239–251.

[5] Á. Baricz, 'Turán type inequalities for generalized complete elliptic integrals', *Math. Z.* **256**(4) (2007), 895–911.

[6] Á. Baricz, 'Functional inequalities involving Bessel and modified Bessel functions of the first kind', *Expo. Math.* **26**(3) (2008), 279–293.

[7] Á. Baricz, 'Turán type inequalities for hypergeometric functions', *Proc. Amer. Math. Soc.* **136**(9) (2008), 3223–3229.

[8] Á. Baricz, 'Mills' ratio: monotonicity patterns and functional inequalities', *J. Math. Anal. Appl.* **340**(2) (2008), 1362–1370.

[9] Á. Baricz, 'On a product of modified Bessel functions', *Proc. Amer. Math. Soc.* **137**(1) (2009), 189–193.

[10] Á. Baricz, 'Turán type inequalities for some probability density functions', *Studia Sci. Math. Hungar.*, in press.

[11] R. W. Barnard, M. B. Gordy and K. C. Richards, 'A note on Turán type and mean inequalities for the Kummer function', *J. Math. Anal. Appl.* **349**(1) (2009), 259–263.

[12] T. H. Gronwall, 'An inequality for the Bessel functions of the first kind with imaginary argument', *Ann. of Math.* (2) **33**(2) (1932), 275–278.

https://doi.org/10.1017/S000497271000002X Published online by Cambridge University Press

[13] E. Grosswald, ‘The Student t-distribution of any degree of freedom is infinitely divisible’, *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **36**(2) (1976), 103–109.

[14] M. E. H. Ismail, ‘Bessel functions and the infinite divisibility of the Student t-distribution’, *Ann. Probab.* **5**(4) (1977), 582–585.

[15] M. E. H. Ismail and A. Laforgia, ‘Monotonicity properties of determinants of special functions’, *Constr. Approx.* **26** (2007), 1–9.

[16] M. E. H. Ismail and M. E. Muldoon, ‘Monotonicity of the zeros of a cross-product of Bessel functions’, *SIAM J. Math. Anal.* **9**(4) (1978), 759–767.

[17] C. M. Joshi and S. K. Bissu, ‘Some inequalities of Bessel and modified Bessel functions’, *J. Aust. Math. Soc. Ser. A* **50** (1991), 333–342.

[18] A. Laforgia and P. Natalini, ‘On some Turán-type inequalities’, *J. Inequal. Appl.* **2006** (2006), 6, Article 29828.

[19] L. Lorch, ‘Monotonicity of the zeros of a cross product of Bessel functions’, *Methods Appl. Anal.* **1**(1) (1994), 75–80.

[20] R. J. McEliece, B. Reznick and J. B. Shearer, ‘A Turán inequality arising in information theory’, *SIAM J. Math. Anal.* **12**(6) (1981), 931–934.

[21] B. N. Mukherjee and T. S. Nanjundiah, ‘On an inequality relating to Laguerre and Hermite polynomials’, *Math. Student* **19** (1951), 47–48.

[22] M. E. Muldoon, ‘Convexity properties of special functions and their zeros’, in: *Recent Progress in Inequalities*, Math. Appl., 430 (ed. G. V. Milovanovic) (Kluwer Academic Publishers, Dordrecht, 1998), pp. 309–323.

[23] R. Penfold, J.-M. Vanden-Broeck and S. Grandison, ‘Monotonicity of some modified Bessel function products’, *Integral Transforms Spec. Funct.* **18**(2) (2007), 139–144.

[24] R. S. Phillips and H. Malin, ‘Bessel function approximations’, *Amer. J. Math.* **72** (1950), 407–418.

[25] H. Skovgaard, ‘On inequalities of the Turán type’, *Math. Scand.* **2** (1954), 65–73.

[26] Y. Sun and Á. Baricz, ‘Inequalities for the generalized Marcum Q-function’, *Appl. Math. Comput.* **203**(1) (2008), 134–141.

[27] O. Szász, ‘Inequalities concerning ultraspherical polynomials and Bessel functions’, *Proc. Amer. Math. Soc.* **1** (1950), 256–267.

[28] O. Szász, ‘Identities and inequalities concerning orthogonal polynomials and Bessel functions’, *J. Anal. Math.* **1** (1951), 116–134.

[29] G. Szegő, ‘On an inequality of P. Turán concerning Legendre polynomials’, *Bull. Amer. Math. Soc.* **54** (1948), 401–405.

[30] V. K. Thiruvenkatachar and T. S. Nanjundiah, ‘Inequalities concerning Bessel functions and orthogonal polynomials’, *Proc. Indian Nat. Acad. Part A* **33** (1951), 373–384.

[31] P. Turán, ‘On the zeros of the polynomials of Legendre’, *Časopis Pest. Mat. Fys.* **75** (1950), 113–122.

ÁRPÁD BARICZ, Department of Economics, Babeş-Bolyai University,
Cluj-Napoca 400591, Romania
e-mail: bariczocsi@yahoo.com

https://doi.org/10.1017/S000497271000002X Published online by Cambridge University Press