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FACTORIZATION IN PRÜFER DOMAINS

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Abstract. We construct a norm on the nonzero elements of a Prüfer domain and extend this concept to the set of ideals of a Prüfer domain. These norms are used to study factorization properties Prüfer of domains.

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1. Introduction and motivation. This paper is motivated by three areas of study, namely, elemental factorization, ideal factorization, and Prüfer domains of finite character. For the interested reader, the following are recent works in these three areas respectively: **[1–3; 6, 13, 15]** and [**4**, **5**, **8**]. In this paper, we attempt to combine some of these classical approaches in a new theatre of operations, namely, Prüfer domains of finite character.

We first recall that in a domain D, an element a is said to be an irreducible, or an atom, if $a = bc$ implies that either b or c is a unit. And a domain D is said to be atomic if for every nonzero, nonunit $b \in D$ there exists atoms $a_1, a_2, \ldots, a_k$ such that $b = a_1a_2 \ldots a_k$. This paper explores various aspects of classical factorization in the realm of domains that are not necessarily Noetherian (or even atomic). In particular, certain factorization properties in Prüfer domains of finite character are characterized by adapting some standard tools to a more general setting.

Another aspect of classic factorization theory is the study of ideal decomposition, in particular, if D is a Dedekind domain, then every nonzero proper ideal of D factors uniquely into a product of maximal ideals. Classically, a tool that is utilized extensively in the study of factorization (both at the elemental level as well as factorization at the ideal level) is the Dedekind–Hasse norm. To this end, we construct a norm-like map on the set of ideals of a Prüfer domain. Our norm map will serve two purposes, first it will be used to characterize invertible ideals and second it will give an aesthetically pleasing viewpoint for ideal decomposition.

For ease of reading, we first acquaint the reader with some more definitions pertinent to this study. An atomic domain D is said to be finite factorization domain (FFD) if every nonzero nonunit has only finitely many nonassociate divisors. An atomic domain D is said to be a bounded factorization domain (BFD) if for all nonzero nonunits $b \in D$ there exists $\pi(b) \in \mathbb{N}$, such that whenever $b = a_1a_2 \cdots a_k$ is a factorization of b, we have $k \leq \pi(b)$. Last, a domain is said to satisfy the ascending chain condition on principal ideals (ACCP), if every increasing chain of principal ideals stabilizes. It was shown in [**3**] that FFD $\Longrightarrow$ BFD $\Longrightarrow$ ACCP $\Longrightarrow$ atomic, and none of the arrows can be reversed.

Let D be an integral domain. We will denote the set of maximal ideals of D by $\mathrm{Max}(D)$ and the set of prime ideals by $\mathrm{Spec}(D)$. A Noetherian domain D is said to be Dedekind if for all maximal ideals M, the localization D_M is a Noetherian valuation domain. One natural generalization of this concept is to drop the initial Noetherian requirement. A domain D is said to be almost Dedekind if D_M is a Noetherian valuation domain for all maximal ideals M. Generalizing further, we say a domain D is Prüfer if D_M is a valuation domain for all $M \in \mathrm{Max}(D)$ (equivalently D_P is a valuation domain for all primes $P \in \mathrm{Spec}(D)$, see [**9**]). Thus, all Dedekind domains are almost Dedekind and all almost Dedekind domains are Prüfer. A (Prüfer) domain D is said to be of finite character if for all nonzero ideals $I \subseteq D$, I is contained in at most finitely many maximal ideals.

A natural question arising from [**3**] is can we characterize Prüfer domains in which one or all of the arrows of FFD $\Longrightarrow$ BFD $\Longrightarrow$ ACCP $\Longrightarrow$ atomic can be reversed? We will show if D is an atomic Prüfer domains of finite character, then D is a BFD. This result shows that in the realm of Prüfer domains of finite character, the concepts atomic, ACCP, and BFD are all equivalent (and this is far from true for general integral domains). Additionally, we provide a characterization of FFD in the class of Prüfer domains of finite character.

In [**12**], the second-named author constructed a norm-like map to study atomicity in almost Dedekind domains. We generalize this norm to Prüfer domains and use it to glean information about atomicity in Prüfer domains with nonzero Jacobson radical. In [**13**], ideal factorization of a similar spirit was studied in almost Dedekind domains. In this paper, we consider some similar ideal factorization questions, and to this end, we construct a norm on the set of ideals of one-dimensional Prüfer domains. This norm is developed to avoid some of the hazards raised in the realm of general one-dimensional Prüfer domains, in particular, the possibility of the existence of idempotent maximal ideals. The norm is then used to classify the set of invertible ideals for Prüfer domains of finite character.

2. Norms on Prüfer domains. Most of our focus in this paper will be on Prüfer domain of dimension one. But the first few results of this section hold for general Prüfer domains. When the "one-dimensional" assumption needs to be employed will be made clear in the sequel.

Here, we wish to generalize the norm from [**12**] to a Prüfer domain D. For $P \in \mathrm{Spec}(D)$, we let $\nu_P : D_P \setminus \{0\} \to G_P$ denote the local valuation map into the value group G_P (written additively).

DEFINITION 2.1. Let D be a Prüfer domain. For nonzero $b \in D$, we define

$$N(b) = (\nu_M(b))_{M \in \mathrm{Max}(D)} \in \prod_{M \in \mathrm{Max}(D)} G_M$$

and

$$\bar{N}(b) = (\nu_P(b))_{P \in \mathrm{Spec}(D)} \in \prod_{P \in \mathrm{Spec}(D)} G_P.$$

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From the properties of valuations, we see that $N(ab) = N(a) + N(b)$ and $\bar{N}(ab) = \bar{N}(a) + \bar{N}(b)$, where addition is defined componentwise.

We define the images of these two norms, called normsets, as follows:

$$\text{Norm}(D) = \{N(b) : b \in D \setminus \{0\}\}$$

and

$$\overline{\text{Norm}(D)} = \{\bar{N}(b) : b \in D \setminus \{0\}\}.$$

We observe that both Norm(D) and $\overline{\text{Norm}}(D)$ form additive monoids with the identity element being the zero net. It is also worth noting that if $U(D)$ denotes the set of units of D, then both N and $\bar{N}$ are homomorphisms from D^* into their respective normsets with kernel $U(D)$ we get the following theorem.

THEOREM 2.2. *$D^*/U(D) \cong Norm(D)$ and $D^*/U(D) \cong \overline{Norm}(D)$. Moreover, we see $Norm(D) \cong \overline{Norm}(D)$.*

The previous shows it suffices to only consider the set of norms determined by the maximal ideals. Factorization in Prüfer domains (from this perspective) is completely determined by localizations at maximal ideals, regardless of the dimension of the Prüfer domain.

DEFINITION 2.3. We say $N(a) \leq N(b)$ if for all $M \in \text{Max}(D)$ we have $\nu_M(a) \leq \nu_M(b)$. We say $N(a) < N(b)$ if $N(a) \leq N(b)$ and there exists an $M \in \text{Max}(D)$ with $\nu_M(a) < \nu_M(b)$.

We pause for the following elementary, but useful observation.

PROPOSITION 2.4. *Let D be a Prüfer domain with $a, b \in D^*$, then $a|b$ if and only if $N(a) \leq N(b)$. Furthermore, a is a proper divisor of b if and only if $N(a) < N(b)$.*

Proof. We prove only the first statement as the second is very similar. Suppose first that $a|b$ and we write $b = da$ for some $d \in D^*$. Since $N(b) = N(d) + N(a)$ and $d \in D^*$, we have that $N(a) \leq N(b)$. Conversely, if $N(a) \leq N(b)$ then $\nu_M(a) \leq \nu_M(b)$ for all $M \in \text{Max}(D)$. Hence, for any $M \in \text{Max}(D)$, $\frac{b}{a} \in D_M$. So, $\frac{b}{a} \in \bigcap_{M \in \text{Max}(M)} D_M = D$ and so $a|b$. □

We also have the following theorem.

THEOREM 2.5. *Let X be any one of the conditions (a) unique factorization, (b) half-factorial, (c) bounded factorization, (d) finite factorization, (e) ACCP, (f) atomic. A Prüfer domain D has factorization property X if and only if Norm(D) has factorization property X.*

Proof. D has factorization property X if and only if D^* has factorization property X if and only if $D^*/U(D)$ has factorization property X if and only if Norm(D) has factorization property X. □

Let D be a one-dimensional Prüfer domain and let $H = \{I \subseteq D | I \text{ is a nonzero ideal of } D\}$. We wish to extend this notion to define a norm, $\hat{N}$ on H. We want the property that if $I = (a)$, then the $\hat{N}(I) = N(a)$. If D is almost Dedekind, the value of an ideal at a localization of a maximal ideal M must be $\nu_M(I) = k$, where $I \subseteq M^k$ and $I \not\subseteq M^{k+1}$. If D is not almost Dedekind, then a problem

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could potentially arise when a maximal ideal M is idempotent, since $M^k = M$ for all $k \in \mathbb{N}$.

To get around this difficulty, we use valuation ideals. Let M be a maximal ideal and let $\gamma \in G_M$, we define

$$^{\gamma}M = \{b \in D : \nu_M(b) \geq \gamma\},$$

and

$$\overline{^{\gamma}M} = \{b \in D : \nu_M(b) > \gamma\}.$$

Now $^{\gamma}M$ and $\overline{^{\gamma}M}$ are ideals for all $\gamma \in G_M$. We wish to use these types of ideal to define the value of an ideal at a particular maximal ideal. The problem is that the difference between these two types of ideals is very subtle, so we need to extend our value group to a set that has the ability to distinguish between these two types of ideals while preserving the property that $\hat{N}(IJ) = \hat{N}(I) + \hat{N}(J)$.

Recall a value group is said to be Archimedean if given $a, b \in G$ with $a < b$ there exists an $n \in \mathbb{N}$ such that $na > b$. Hölder's theorem states that every Archimedean value group is isomorphic to some subgroup of the additive group of real numbers with the standard order (see **[7]** p. 45.) A Prüfer domain is locally Archimedean (every local value group is Archimedean) if and only if it is one-dimensional (see **[14]**, Corollary 1.4).

Restricting to one-dimensional Prüfer domains will allow us to use the completeness of the real numbers; however, in order to distinguish between ^{g}M and $\overline{^{g}M}$, we will need to extend the real numbers to the surreal numbers $\mathbb{S}$. A thorough treatment of the surreal numbers may be found in **[10]**.

Let ω denote the cardinality of the natural numbers. We set $\epsilon = \frac{1}{w} \in \mathbb{S}$. It is known that ϵ has the property that for $r \in \mathbb{R}$ and $n \in \mathbb{N}$ we have $r + n\epsilon < t$ for all real numbers $t > r$ (see **[10]**). For the remainder of the paper, ϵ will be this fixed surreal number.

Now for $M \in \text{Max}(D)$ and ideal I, we define

$$T_M(I) = \{\nu_M(b)\}_{b \in I \setminus \{0\}}$$

and we set

$$s_M(I) = \inf\ T_M(I).$$

Definition 2.6. Let D be a one-dimensional Prüfer domain and let I be an ideal of D. We define the value of I at the maximal ideal M as

$$\nu_M(I) = \begin{cases} s_M(I) & : s_M(I) \in T_M(I) \\ s_M(I) + \epsilon & : s_M(I) \notin T_M(I) \end{cases}.$$

Note if $\nu_M(I) = s_M(I) + \epsilon$, we have $\gamma > s_M(I) + \epsilon$ for all $\gamma \in T_M(I)$. For $s, t \in \mathbb{S}$, we say $s \sim_M t$ if and only if $^{t}M = {}^{s}M$. Note that $m\epsilon \sim_M n\epsilon$ for all $m, n \in \mathbb{N}$, and if $G_M \cong \mathbb{N}$, then $\epsilon \sim_M 1$. Now for $M \in \text{Max}(D)$ and nonzero ideal I of D, we will assign a value to I from $\mathcal{I}_{G_M} = \mathbb{S}/\sim_M$. We will now use ideals of the form

$$^{\gamma}M = \{b \in D : \nu_M(b) \geq \gamma\},$$

where $\gamma \in \mathcal{I}_{G_M}$ to construct our norm.

LEMMA 2.7. *Let I and J be ideals of a one-dimensional Prüfer domain. Then, $\nu_M(IJ) = \nu_M(I) + \nu_M(J)$ for all $M \in Max(D)$.*

Proof. If t is the infimum of a set S and $t \notin S$, then there exists a countable sequence that converges to t. If $t \in S$, we will take the sequence to be constant.

Now, we have sequences $\{a_k\} \subseteq I$ and $\{b_k\} \subseteq J$ with $\lim_{k\to\infty} \nu_M(a_k) = s_M(I)$ and $\lim_{k\to\infty} \nu_M(b_k) = s_M(J)$. Hence,

$$s_M(IJ) = \lim_{k\to\infty} \nu_M(a_k b_k) = \lim_{k\to\infty} (\nu_M(a_k) + \nu_M(b_k)) = s_M(I) + s_M(J).$$

The definition of the product of ideals makes an element of smaller value impossible, since $IJ = \{\sum r_i x_i y_i : r_i \in D, x_i \in I, y_i \in J\}$ and $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$ for any valuation ν.

Now if both $s_M(I)$ and $s_M(J)$ are in $T_M(I)$ and $T_M(J)$, respectively, it is clear that $s_M(IJ) \in T_M(IJ)$, hence $\nu_M(IJ) = s_M(IJ) = s_M(I) + s_M(J) = \nu_M(I) + \nu_M(J)$.

Suppose $s_M(I) \in T_M(I)$ and $s_M(J) \notin T_M(J)$. Then, $s_M(IJ) \notin T_M(IJ)$. Thus, $\nu_M(IJ) = s_M(IJ) + \epsilon = s_M(I) + s_M(J) + \epsilon = \nu_M(I) + \nu_M(J)$.

Now suppose $s_M(I) \notin T_M(I)$ and $s_M(J) \notin T_M(J)$. Then, we must have $s_M(IJ) \notin T_M(IJ)$. Thus, we have $\nu_M(IJ) = s_M(IJ) + \epsilon = (s_M(I) + s_M(J) + \epsilon) \sim_M (s_M(I) + s_M(J) + 2\epsilon) = \nu_M(I) + \nu_M(J)$. □

Now, we define a norm on the set of nonzero ideals of a one-dimensional Prüfer domain.

DEFINITION 2.8. Let D be a one-dimensional Prüfer domain, and let I be an ideal of D. The norm of I is defined to be

$$\hat{N}(I) = (\nu_M(I))_{M \in \mathrm{Max}(D)} \in \prod_{M \in \mathrm{Max}(D)} \mathcal{I}_{G_M}.$$

From the lemma, we immediately get the desired theorem.

THEOREM 2.9. *Let D be a one-dimensional Prüfer domain, and let I and J be ideals of D. Then, $\hat{N}(IJ) = \hat{N}(I) + \hat{N}(J)$ where addition is done componentwise.*

The norm on the set of ideals also satisfies several nice properties. We partially order the image of $\hat{N}$ in the same manner as we ordered the image of N.

THEOREM 2.10. *Let D be a one-dimensional Prüfer domain, and let I and J be an ideals of D. Then,*

(*i*) $\hat{N}(I) = \hat{N}(J)$ *if and only if* $I = J$
(*ii*) $\hat{N}(I) < \hat{N}(J)$ *if and only if* $J \subset I$.

Proof. Suppose $\hat{N}(I) = \hat{N}(J)$ and a is in I, then $\nu_M(a) \geq \nu_M(I)$ for all M, hence $\nu_M(a) \geq \nu_M(J)$ for all M. Thus, $a \in J$. The proof of the other containment is similar.

The proof of (ii) is almost identical and is omitted. □

3. One-dimensional Prüfer domains with $\mathcal{J} \neq (0)$. In this section, we investigate factorization properties of Prüfer domains with nonzero Jacobson radicals. We denote the Jacobson radical of D by $\mathcal{J}$, and let $\max(a) = \{M : a \in M\}$. It will be assumed that all Prüfer domains are one-dimensional.

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DEFINITION 3.1. Let D be a one-dimensional Prüfer domain. We say D is uniformly bounded if for all nonzero nonunit $b \in D$, there exists $\delta, \rho > 0 \in \mathbb{R}$ such that for all $M \in \mathrm{Max}(D)$, we have $\delta < \nu_M(b) < \rho$.

THEOREM 3.2. *Let D be a uniformly bounded one-dimensional Prüfer domain. If D is atomic, then $\mathcal{J} = (0)$, or D is a semilocal PID.*

Proof. First, suppose D has only finitely many maximal ideals, say $\{M_1, M_2, \ldots, M_l\}$. Then, clearly $\mathcal{J} \neq (0)$. Now for all $i = 1, \ldots, l$, there exists $b \in D$ with $\max(b) = \{M_i\}$. Thus, we conclude that for all i there exists an atom α_i with $\max(\alpha_i) = \{M_i\}$.

Now suppose some maximal ideal is idempotent, say M_1. Set $\nu_{M_1}(\alpha_1) = r$ for some $r \in G_{M_1}$. Since G_{M_1} is nondiscrete there must exist $b_1 \in D$ with $\nu_{M_1}(b_1) = t$ where $t < r$ in G_{M_1}. Now, we find $b_2, b_3, \ldots, b_l$ where $b_i \in M_i$ and $b_i \notin M_1$. Now $N(\alpha_1 + b_1 b_2 \ldots b_l) = (t, 0, 0, \ldots, 0)$. Hence, $\alpha_1 + b_1 b_2 \ldots b_l$ divides α_1 which is impossible as α_1 is an atom and not associated to $\alpha_1 + b_1 b_2 \ldots b_l$ by construction. Thus, M_1 is not idempotent. We conclude that $\mathrm{Max}(D)$ contains no idempotents, and hence D must be almost Dedekind. But an almost Dedekind domain with only finitely many maximal ideals must be Dedekind, hence a semilocal PID (see **[9]**).

Now, we suppose that $|\mathrm{Max}(D)| = \infty$.

For $b \in \mathcal{J} \setminus \{0\}$, we set

$$b_k = \min\{k : b = \alpha_1 \alpha_2 \cdots \alpha_k \text{ is an atomic factorization of } b\}.$$

We set $k = \min_{b \in \mathcal{J} \setminus \{0\}} \{b_k\}$.

Suppose $k = 1$, then there exists an atom $b \in \mathcal{J}$. Now, we find $a \in D$ with $a \notin \mathcal{J}$. Since D is one-dimensional and hence Archimedean and also uniformly bounded, we can find $n \in \mathbb{N}$ such that $\nu_M(a^n) > \nu_M(b)$ for all $M \in \max(a)$. Now we see that

$$\nu_M(a^n + b) = \begin{cases} \nu_M(b) & : M \in \max(a) \\ 0 & : M \notin \max(a) \end{cases}$$

hence, $a^n + b$ strictly divides b and b is not an atom.

Now suppose $k > 1$. We find $b \in \mathcal{J}$ with $b = \alpha_1 \alpha_2 \cdots \alpha_k$ an atomic factorization. Now it must be the case that one of these atoms is contained in infinitely many maximal ideals, say α_1. We set $b' = \alpha_2 \ldots \alpha_k$. Now as k is minimal, it must be the case that $b' \notin \mathcal{J}$. Further, there must exist $P \in \max(\alpha_1)$ with $P \notin \max(b')$ and $Q \in \max(b')$ with $Q \notin \max(\alpha_1)$. We find $c \in D$ with $c \in Q$ and $c \notin P$.

Now by assumption, we can find $n \in \mathbb{N}$ such that $\nu_M((cb')^n) > \nu_M(\alpha_1)$ for all $M \in \max(cb')$. Now we have

$$\nu_M(\alpha_1 + (cb')^n) = \begin{cases} \nu_M(\alpha_1) & : M \in \max(\alpha_1) \cap \max(cb') \\ 0 & : M \notin \max(\alpha_1) \cap \max(cb') \end{cases}.$$

But then we have $\alpha_1 + (cb')^n$ strictly dividing α_1. Thus, no element of $\mathcal{J}$ can be factored into a finite product of atoms. We conclude that D is not atomic.

Thus, if D is atomic, we must have $J = (0)$. □

4. One-dimensional Prüfer domains of finite character. A Prüfer domain is said to be of finite character if for all $b \in D$, $\max(b)$ is finite.

DEFINITION 4.1. Let D be a one-dimensional Prüfer domain of finite character. For nonzero $b \in D$, we define the length of b to be $\|b\| = \sum_{M \in \mathrm{Max}(D)} \nu_M(b)$.

It is clear that $\|ab\| = \|a\| + \|b\|$. Further, if $a|b$, then $\nu_M(a) \leq \nu_M(b)$ for all $M \in \mathrm{Max}(D)$ thus $\|a\| \leq \|b\|$. The converse is clearly not true, as we can have $\|a\| \leq \|b\|$ with $\max(a) \cap \max(b) = \emptyset$.

We denote the set of nonunit divisors of b by $Z(b)$. That is $Z(b) = \{a : a|b, a \notin U(D)\}$.

DEFINITION 4.2. Let D be a one-dimensional Prüfer domain of finite character. For $b \in D$, we define $S_b = \{\|d\| : d \in Z(b)\}$.

THEOREM 4.3. *Let D be a one-dimensional Prüfer domain of finite character. The following are equivalent:*

(*i*) *For all* $b \in D$, $\inf S_b > 0$
(*ii*) *D is a BFD.*
(*iii*) *D satisfies ACCP.*
(*iv*) *D is atomic.*

Proof. Let $b \in D$ and assume $\inf S_b = t > 0$. Now for all $d \in Z(b)$, we have $\|d\| \geq t$. Thus, $\pi(b) = \lceil \frac{\|(b)\|}{t} \rceil$ is a bound on the length of all possible factorizations of b. Thus, D is a BFD.

Clearly (*ii*) implies (*iii*) and (*iii*) implies (*iv*).

Now we assume that D is atomic and let $b \in D$ with $|\max(b)| = k$. Now, set $H = \{A \in \mathcal{P}(\max(b)) :$ there exists an atom a with $\max(a) = A\}$, where $\mathcal{P}(\max(b))$ is the power set of $\max(b)$. Since H is finite, we will rewrite $H = \{A_1, A_2, \ldots A_l\}$. Now we find atoms $a_1, a_2, \ldots, a_l$ with $\max(a_i) = A_i$.

Now set $t_i = \min\{\nu_M(a_i) : M \in \max(a_i)\}$ and set $T = \min\{t_1, t_2, \ldots, t_l\}$. Now we claim that $\xi = \frac{T}{2}$ is a lower bound for S_b.

Suppose $d \in Z(b)$ with $\|d\| = \xi$. Now since D is atomic, d must be divisible by some atom a. It must be the case that $\max(a) = A_i$ for some i. We have $\xi < \nu_M(a_i)$ for all $M \in \max(a_i)$. But this is impossible as it implies d divides the atom a_i. Therefore, we conclude that S_b has a lower bound. □

Let D be a Prüfer domain. We let $M_{\mathcal{I}^2} = \{M \in \mathrm{Max}(D) : M = M^2\}$ and $M_{\mathcal{I}} = \{M \in \mathrm{Max}(D) : M \neq M^2\}$

DEFINITION 4.4. We say $M_{\mathcal{I}^2}$ is covered by nonidempotents if

$$\cup_{M \in M_{\mathcal{I}^2}} M \subset \cup_{M \in M_{\mathcal{I}}} M.$$

THEOREM 4.5. *Let D be a one-dimensional Prüfer domain of finite character. If* $M_{\mathcal{I}^2}$ *covered by nonidempotents, then D is a BFD.*

Proof. Take $b \in D$, then $\inf S_b = 1$, since every b is contained in an M with $G_M \cong \mathbb{N}_0$. Thus, D is a BFD. □

THEOREM 4.6. *Let D be a one-dimensional Prüfer domain of finite character with* $M_{\mathcal{I}^2} = \{M\}$. *If M is covered by nonidempotents, then D is an FFD.*

Proof. Suppose M is covered by nonidempotents. Let $b \in D$. Now if $\max(b) \subset M_{\mathcal{I}}$ the number of divisors of b is bounded by $\prod_{M \in \max(b)} (\nu_M(b) + 1)$. Suppose $b \in M$ and b has infinitely many divisors, then there must exist nonassociate $c, d \in Z(b)$ with

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$\nu_P(c) = \nu_P(d)$ for all $P \in \max(b) \setminus \{M\}$. Now without loss of generality we may assume that $\nu_M(c) < \nu_M(d)$. But now $\max(\frac{d}{c}) = \{M\}$ which is a contradiction. □

COROLLARY 4.7. *The atomic Prüfer domain constructed in [11] is an FFD.*

We now present a characterization of FFDs in the class of Prüfer domains of finite character. We let $H = \{(d_1, d_2) \in Z(b) \times Z(b) : d_1 \neq d_2\}$. For $M \in \max(b)$, we define

$$W_M(b) = \inf_H \{|\nu_M(d_1) - \nu_M(d_2)| : \nu_M(d_1) \neq \nu_M(d_2)\}.$$

THEOREM 4.8. *Let D be a one-dimensional Prüfer domain of finite character. D is an FFD if and only if for all $b \in D$ and for all $M \in max(b)$ we have $W_M(b) > 0$.*

Proof. If $W_M(b) = 0$ for some $b \in D$ and some $M \in \max(b)$ there must be infinitely many divisors, hence D is not an FFD. Thus, the forward direction holds.

Now suppose for a given $b \in D$ with $\max(b) = \{M_1, M_2, \ldots, M_k\}$, we have $W_{M_i}(b) > 0$ for all $i = 1, 2, \ldots, k$. Set $W_{M_i}(b) = r_i > 0$ for some $r_i \in \mathbb{R}$. Now there are only $\lceil \frac{\nu_{M_i}(b)}{r_i} \rceil$ possible values for any divisor contained in M_i, since any two distinct values must be at least r_i apart. Thus, the number of divisors for b is bounded above by

$$\prod_{i=1}^{k} \left\lceil \frac{\nu_{M_i}(b)}{r_i} \right\rceil.$$

We conclude that every $b \in D$ has only finitely many divisors, hence D is an FFD. □

We finish the paper by returning to the norm we defined on the set of ideals of a one-dimensional Prüfer domain.

DEFINITION 4.9. Let D be a one-dimensional Prüfer domain, and let I be an ideal of D. We say I is real if $\hat{N}(I)$ contains no surreal entries.

THEOREM 4.10. *Let D be a one-dimensional Prüfer domain, and let I be an ideal of D. If I is invertible, then I is real. Moreover if D is of finite character, then I is invertible if and only if I is real.*

Proof. Suppose I is invertible, then I is finitely generated, say $I = (g_1, g_2, \ldots, g_n)$. Let $b = r_1g_1 + r_2g_2 + \cdots + r_ng_n$. Now for all maximal ideals $M \supseteq I$, we have $\nu_M(b) = \nu_M(r_1g_1 + r_2g_2 + \cdots + r_ng_n) \geq \min\{\nu_M(r_1g_1), \nu_M(r_2g_2), \ldots, \nu_M(r_ng_n)\} \geq \min\{\nu_M(g_1), \nu_M(g_2), \ldots, \nu_M(g_n)\}$. Now it is clear that

$$s_M(I) = \min\{\nu_M(g_1), \nu_M(g_2), \ldots, \nu_M(g_n)\}.$$

Thus, $s_M(I) \in T_M(I)$, hence $\nu_M(I) \in G_M$. We conclude for all maximal ideals M containing I that $\nu(I)$ is real.

Suppose I is real and D is of finite character. Now I is contained in only finitely many maximal ideals say $M_1, M_2, \ldots, M_k$. Since I is real, there exists $b_i \in I$ with $\nu_{M_i}(b_i) = \nu_{M_i}(I)$, for $i = 1, 2, \ldots, k$. We have $N(I) = N((b_1, b_2, \ldots, b_l))$, hence $I = (b_1, b_2, \ldots, b_l)$. Since I is finitely generated, I is invertible. □

If D is a one-dimensional Prüfer domain of finite character, then all ideals factor uniquely into products of primal ideals (see **[8]**). The exact factorization is easily seen using our norm. In particular, $I \subseteq D$ is contained in only finitely many maximal ideals we have

$$I =^{v_{M_1}(I)} M_1 \ldots {}^{v_{M_k}(I)} M_k.$$

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