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A NOTE ON A GENERIC HYPERPLANE SECTION OF AN ALGEBRAIC VARIETY

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1. Introduction. Let V be an irreducible algebraic variety of dimension > 1 defined over a field k in an affine n-space over k, and let H be the generic hyperplane defined by $u_0 + u_1X_1 + \ldots + u_nX_n = 0$, where $u_0, u_1, \ldots, u_n$ are indeterminates over k. It is well known that:

(1) if V is normal over k, then $V \cap H$ is normal over $k(u_0, \ldots, u_n)$ (see [**6**]), and

(2) if P is in the intersection $V \cap H$, then P is absolutely simple on $V \cap H$ over $k(u_0, \ldots, u_n)$ if and only if P is absolutely simple on V over k (see [**2**; **5**]).

In this paper we prove:

(1′) if V is factorial over k, then $V \cap H$ is also factorial over $k(u_0, \ldots, u_n)$ (Theorem 3), and

(2′) if P is in $V \cap H$, then P is normal on $V \cap H$ over $k(u_0, \ldots, u_n)$ if and only if P is normal on V over k (Theorem 2).

The relative case of (2) is a special case of Bertini's theorem [**7**, p. 138]; it can also be proved by the same argument as [**6**, Theorem 1]. In this paper, we give a new proof of the relative case of (2). In addition, we prove that if V is factorial over k at P, then $V \cap H$ is also factorial over $k(u_0, \ldots, u_n)$ at P (Theorem 4). I thank Professor A. Seidenberg for his suggestion to remove the restriction of k being infinite from Lemma 4.

2. Notation and terminology. Let V be an irreducible algebraic variety defined over a field k in an affine n-space A^n over k, i.e. V is a subset of A^n consisting of all zeros of a finite collection of polynomials in the polynomial ring $k[x_1, \ldots, x_n]$ which generate a prime ideal in $k[x_1, \ldots, x_n]$. Let $(\xi) = (\xi_1, \ldots, \xi_n)$ be a generic point of V over k, let $Q = (q_1, \ldots, q_n)$ be a point on V with $\mathfrak{q}$ as the prime ideal in the coordinate ring $k[\xi]$, and let $k[\xi]_\mathfrak{q}$ be the local ring of V at Q in the function field $k(\xi)$. Q is simple on V over k if $k[\xi]_\mathfrak{q}$ is a regular local ring. Q is factorial on V over k if $k[\xi]_\mathfrak{q}$ is factorial (i.e. a unique factorization domain). Q is normal on V over k if $k[\xi]_\mathfrak{q}$ is integrally closed in $k(\xi)$. V is factorial over k if $k[\xi]$ is factorial. The term simple point as defined here is a relative notion over k in contrast to the notion of an absolute simple point over k, which is defined by the classical Jacobian

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criterion. Namely, if $F_1, \ldots, F_s$ is a set of defining polynomials of V over k, P is absolutely simple on V over k if

$$\operatorname{rank}\left(\frac{\partial F_i}{\partial x_j}\right)_P = n - r,$$

where $r = \dim V$. These two concepts are equivalent if k is a perfect field (see [**8**]). Let $u_0, u_1, \ldots, u_n$ be $n+1$ indeterminates over $k(\xi)$. The plane H, given by $u_0 + u_1X_1 + \ldots + u_nX_n = 0$, is called a generic hyperplane. $V \cap H$ is an irreducible variety defined over $k(u) = k(u_0, \ldots, u_n)$ (see [**6**]). Let τ be an indeterminate over $k(\xi)$. If P is a prime ideal in $k[\xi]$, then $\mathfrak{p} \cdot k(\tau)[\xi]$ is a prime ideal in $k(\tau)[\xi]$ (see [**2**]). We denote the prime ideal $\mathfrak{p} \cdot k(\tau)[\xi]$ by $\mathfrak{p}^e$, and for a prime ideal $\mathfrak{q}$ in $k(\tau)[\xi]$, we denote the prime ideal $\mathfrak{q} \cap k[\xi]$ by $\mathfrak{q}^c$. Let A be an integral domain with K as its quotient field and let A' be the integral closure of A in K. If $\mathfrak{C} = \{x \in A \mid xA' \subset A\}$, we call $\mathfrak{C}$ the conductor of A. A is integrally closed if and only if $\mathfrak{C} = (1) \cdot A$ (see [**9**]).

3. Results.

LEMMA 1. (a) *$k[\xi]_{\mathfrak{p}}$ is regular if $k(\tau)[\xi]_{\mathfrak{p}^e}$ is regular, and* (b) *$k(\tau)[\xi]_{\mathfrak{q}}$ is regular if $k[\xi]_{\mathfrak{q}^c}$ is regular.*

Proof. [**7**, p. 132, Lemma 2].

LEMMA 2. *Let $\mathfrak{C}$ and $\mathfrak{C}_\tau$ be the conductors of $k[\xi]$ and $k(\tau)[\xi]$, respectively. Then $\mathfrak{C} \cdot k(\tau)[\xi] = \mathfrak{C}_\tau$ and $\mathfrak{C}_\tau \cap k[\xi] = \mathfrak{C}$.*

Proof. Let $k[\xi]'$ and $k(\tau)[\xi]'$ be the integral closures of $k[\xi]$ and $k(\tau)[\xi]$ in $k(\xi)$ and $k(\tau, \xi)$, respectively. If $\alpha(\tau, \xi) \in k(\tau)[\xi]'$, then there exists $d(\tau) \in k[\tau]$ such that $d(\tau)\alpha(\tau, \xi)$ is integral over the polynomial ring $k(\xi)[\tau]$. Hence $d(\tau)\alpha(\tau, \xi) \in k(\xi)[\tau]$, since $k(\xi)[\tau]$ is a unique factorization domain and therefore is integrally closed. Thus $d(\tau)\alpha(\tau, \xi) = a_0(\xi) + \ldots + a_n(\xi)\tau^n$. Replacing τ by $n+1$ values λ_i from the algebraic closure $\bar{k}$ of k, we see that $a_0(\xi) + a_1(\xi)\lambda_i + \ldots + a_n(\xi)\lambda_i^n$ is integral over $\bar{k}[\xi]$ for each i. Therefore each $a_i(\xi)$ is integral over $\bar{k}[\xi]$ and hence, integral over $k[\xi]$. Now, for $z \in \mathfrak{C}$, $za_i(\xi) \in k[\xi]$, for $i = 0, 1, 2, \ldots, n$, and $z\alpha(\tau, \xi) \cdot d(\tau) \in k[\xi][\tau]$. Thus $\mathfrak{C} \cdot k(\tau)[\xi] \subset \mathfrak{C}_\tau$. On the other hand, let $z \in \mathfrak{C}_\tau$. Then there exists $e(\tau) \in k[\tau]$ such that $ez \in k[\xi][\tau]$. Therefore, $ez = b_0(\xi) + \ldots + b_m(\xi)\tau^m$, where $b_i(\xi) \in k[\xi]$ for $i = 0, 1, 2, \ldots, m$. Let β be any element in $k[\xi]'$ so that $z\beta \in k(\tau)[\xi]$ and $ez\beta \in k[\tau][\xi]$. Thus, $b_i(\xi)\beta \in k[\xi]$ for $i = 0, 1, 2, \ldots, m$, and hence $b_i(\xi) \in \mathfrak{C}$ for $i = 0, 1, 2, \ldots, m$. It follows that $\mathfrak{C}_\tau \subset \mathfrak{C} \cdot k(\tau)[\xi]$. The second equality in the lemma follows immediately from the first one and the fact that $k(\tau)[\xi] \cap k(\xi) = k[\xi]$.

As a consequence we have the following result.

COROLLARY 1. *$k[\xi]$ is integrally closed if and only if $k(\tau)[\xi]$ is integrally closed.*

COROLLARY 2. (a) *$k[\xi]_{\mathfrak{p}}$ is integrally closed if $k(\tau)[\xi]_{\mathfrak{p}^e}$ is integrally closed.*
(b) *$k(\tau)[\xi]_{\mathfrak{q}}$ is integrally closed if $k[\xi]_{\mathfrak{q}^c}$ is integrally closed.*

Proof. (a) By [**9**, p. 269, Lemma], the conductors of $k(\tau)[\xi]_{\mathfrak{p}^e}$ and $k[\xi]_{\mathfrak{p}}$ are $\mathfrak{C}_\tau \cdot k(\tau)[\xi]_{\mathfrak{p}^e}$ and $\mathfrak{C} \cdot k[\xi]_{\mathfrak{p}}$, respectively. As $k(\tau)[\xi]_{\mathfrak{p}^e} \cap k(\xi) = k[\xi]_{\mathfrak{p}}$, it follows that $\mathfrak{C}_\tau \cdot k(\tau)[\xi]_{\mathfrak{p}^e} \cap k[\xi]_{\mathfrak{p}} = \mathfrak{C} \cdot k[\xi]_{\mathfrak{p}}$. Thus, if $k(\tau)[\xi]_{\mathfrak{p}^e}$ is integrally closed, then $\mathfrak{C}_\tau \cdot k(\tau)[\xi]_{\mathfrak{p}^e} = (1) \cdot k(\tau)[\xi]_{\mathfrak{p}^e}$. It follows that $\mathfrak{C} \cdot k[\xi]_{\mathfrak{p}} = (1) \cdot k[\xi]_{\mathfrak{p}}$ and hence $k[\xi]_{\mathfrak{p}}$ is integrally closed.

(b) This is immediate, since $k(\tau)[\xi]_{\mathfrak{q}} \supset k[\xi]_{\mathfrak{q}^c}$ and also $\mathfrak{C} \cdot k[\xi]_{\mathfrak{q}^c} = (1) \cdot k[\xi]_{\mathfrak{q}^c}$ implies that $\mathfrak{C}_\tau \cdot k(\tau)[\xi]_{\mathfrak{q}} = (1) \cdot k(\tau)[\xi]_{\mathfrak{q}}$.

LEMMA 3. *Let $u_1, \ldots, u_n$ be indeterminates over $k(\xi)$, let*

$$\bar{u}_0 = -(u_1\xi_1 + \ldots + u_n\xi_n),$$

and let the conductor of $k(u_1, \ldots, u_n, \bar{u}_0)[\xi]$ be $\mathfrak{C}_0$. Then

$$\mathfrak{C}_0 = \mathfrak{C} \cdot k(u_1, \ldots, u_n, \bar{u}_0)[\xi].$$

Proof. By Lemma 2, $\mathfrak{C} \cdot k(u_1, \ldots, u_n)[\xi]$ is the conductor of $k(u_1, \ldots, u_n)[\xi]$. Also $k(u_1, \ldots, u_n, \bar{u}_0)[\xi] = k(u_1, \ldots, u_n)[\xi]_S$, where $S = k[u_1, \ldots, u_n, \bar{u}_0] - \{0\}$. By [**9**, p. 269, Lemma], $\mathfrak{C}_0 = \mathfrak{C} \cdot k(u_1, \ldots, u_n)[\xi]_S = \mathfrak{C} \cdot k(u_1, \ldots, u_n, \bar{u}_0)[\xi]$.

THEOREM 1. *Let V be an irreducible γ-dimensional variety defined over k with $\gamma \geqq 2$. Let P be a point on the generic hyperplane section $V \cap H$ of V and H over $k(u_0, \ldots, u_n)$. Then P is simple on V over k if and only if P is simple on $V \cap H$ over $k(u_0, \ldots, u_n)$.*

Proof. Let $\mathfrak{p}_u'$ be the prime ideal of P in $k(u)[\xi]$. $k(u)[\xi]_{\mathfrak{p}_u'}$ is the local ring of V at P over $k(u)$. Since $(u_0 + u_1\xi_1 + \ldots + u_n\xi_n) \cdot k(u)[\xi]_{\mathfrak{p}_u'}$ is a prime ideal [**6**, p. 367, Lemma 3], the local ring of $V \cap H$ at P over $k(u)$ is isomorphic to $k(u)[\xi]_{\mathfrak{p}_u'}/(u_0 + u_1\xi_1 + \ldots + u_n\xi_n) \cdot k(u)[\xi]_{\mathfrak{p}_u'}$. If P is simple on V over k, then, by Lemma 1, $k(u)[\xi]_{\mathfrak{p}_u'}$ is a regular local ring. In order to prove that P is simple on $V \cap H$ over $k(u)$, we need only prove that $u_0 + u_1\xi_1 + \ldots + u_n\xi_n \notin \mathfrak{M}_{\mathfrak{p}_u'}{}^2$, where $\mathfrak{M}_{\mathfrak{p}_u'}$ is the maximal ideal of $k(u)[\xi]_{\mathfrak{p}_u'}$. Suppose that $u_0 + u_1\xi_1 + \ldots + u_n\xi_n \in \mathfrak{M}_{\mathfrak{p}_u'}{}^2$. Taking the partial derivative with respect to u_0, we have $1 \in \mathfrak{M}_{\mathfrak{p}_u'}$, a contradiction. Therefore $u_0 + u_1\xi_1 + \ldots + u_n\xi_n \notin \mathfrak{M}_{\mathfrak{p}_u'}{}^2$, and it follows from [**10**, p. 303, Theorem 26] that

$$k(u)[\xi]_{\mathfrak{p}_u'}/(u_0 + u_1\xi_1 + \ldots + u_n\xi_n)k(u)[\xi]_{\mathfrak{p}_u'}$$

is a regular local ring. Hence P is simple on $V \cap H$ over $k(u)$. Conversely, if P is simple on $V \cap H$ over $k(u)$, then

$$k(u)[\xi]_{\mathfrak{p}_u'}/(u_0 + u_1\xi_1 + \ldots + u_n\xi_n)k(u)[\xi]_{\mathfrak{p}_u'}$$

is a regular local ring. But the prime ideal $(u_0 + u_1\xi_1 + \ldots + u_n\xi_n) \cdot k(u)[\xi]_{\mathfrak{p}_u'}$ is of height 1. It follows from [**4**, p. 28, (9.11)] that $k(u)[\xi]_{\mathfrak{p}_u'}$ is a regular local ring. Now $(\mathfrak{p}_u'^c)^e \subset \mathfrak{p}_u'$ is a prime ideal, therefore

$$k(u)[\xi]_{(\mathfrak{p}_u'^c)^e} = (k(u)[\xi]_{\mathfrak{p}_u'})_{(\mathfrak{p}_u'^{ce}) \cdot k(u)[\xi]\mathfrak{p}_u'}$$

is a regular local ring. It follows from Lemma 1 that $k[\xi]_{\mathfrak{p}_u'^c}$ is a regular local ring and hence P is simple on V over K.

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THEOREM 2. *Let V be an irreducible γ-dimensional variety defined over k with $\gamma \geqq 2$. Let P be a point on the generic hyperplane section $V \cap H$ over $k(u_0, \ldots, u_n)$. P is normal on V over k if and only if P is normal on $V \cap H$ over $k(u_0, \ldots, u_n)$.*

Proof. Let P be a normal point on V over k and let (η) be a generic point of $V \cap H$ over $k(u)$. Let $l = u_0 + u_1\xi_1 + \ldots + u_n\xi_n$, and

$$\bar{u}_0 = -(u_1\xi_1 + \ldots + u_n\xi_n),$$

and let (l) be $(l) \cdot k(u)[\xi]$, the principal ideal in $k(u)[\xi]$ generated by l. We have

$$(l) \cap k(u_1, \ldots, u_n)[\xi] = (0),$$
$$k(u_1, \ldots, u_n, \bar{u}_0)[\xi] \cong k(u)[\xi]/(l) \cong k(u)[\eta],$$

as pointed out in [**6**, p. 367, proof of Lemma 3]. Let $\mathfrak{p}_u$, $\mathfrak{p}_u'$ be the prime ideals determined by P in $k(u)[\eta]$ and $k(u)[\xi]$ respectively, and let $\mathfrak{p}_0$ be the isomorphic copy of $\mathfrak{p}_u$ in $k(u_1, \ldots, u_n, \bar{u}_0)[\xi]$. Then $\mathfrak{p}_u' \cap k(u_1, \ldots, u_n)[\xi]$ is the prime ideal $\mathfrak{p}$ determined by P in $k(u_1, \ldots, u_n)[\xi] = \mathfrak{p}_0 \cap k(u_1, \ldots, u_n)[\xi]$.

Let $\mathfrak{C}$ be the conductor of $k(u_1, \ldots, u_n)[\xi]$. Since

$$k(u_1, \ldots, u_n)[\xi]_{\mathfrak{p}} \subset k(u_1, \ldots, u_n, \bar{u}_0)[\xi]_S \subset k(u_1, \ldots, u_n, \bar{u}_0)[\xi]_{\mathfrak{p}_0},$$

where $S = k(u_1, \ldots, u_n)[\xi] - \mathfrak{p}$, we see that

$$\mathfrak{C} \cdot k(u_1, \ldots, u_n)[\xi]_{\mathfrak{p}} \subset \mathfrak{C} \cdot k(u_1, \ldots, u_n, \bar{u}_0)[\xi]_{\mathfrak{p}_0},$$

which according to [**9**, p. 269, Lemma] and Lemma 3, are conductors of $k(u_1, \ldots, u_n)[\xi]_{\mathfrak{p}}$ and $k(u_1, \ldots, u_n, \bar{u}_0)[\xi]_{\mathfrak{p}_0}$, respectively. By Corollary 2 to Lemma 2, $k(u_1, \ldots, u_n)[\xi]_{\mathfrak{p}}$ is integrally closed. Therefore

$$\mathfrak{C} \cdot k(u_1, \ldots, u_n)[\xi]_{\mathfrak{p}} = (1) \cdot k(u_1, \ldots, u_n)[\xi]_{\mathfrak{p}},$$

and we have $\mathfrak{C} \cdot k(u_1, \ldots, u_n, \bar{u}_0)[\xi]_{\mathfrak{p}_0} = (1) \cdot k(u_1, \ldots, u_n, \bar{u}_0)[\xi]_{\mathfrak{p}_0}$. Hence P is normal on $V \cap H$ over $k(u_0, \ldots, u_n)$.

The converse follows immediately from [**4**, p. 134, (36.9)].

LEMMA 4. *$k[\xi]$ is factorial if and only if $k(\tau)[\xi]$ is factorial.*

Proof. If $k[\xi]$ is factorial, then $k[\xi][\tau]$ is factorial and hence $k[\xi][\tau]_S$ is factorial, where $S = k[\tau] - \{0\}$. Thus $k(\tau)[\xi]$ is factorial. Conversely, let $k(\tau)[\xi]$ be factorial and let $f(\xi) \in k[\xi]$ be an irreducible element. To show that $k[\xi]$ is factorial, we only need to show that the principal ideal $(f(\xi))$ is prime in $k[\xi]$. Suppose that $f(\xi) = g_1(\tau, \xi) \cdot h_1(\tau, \xi)$ in $k(\tau)[\xi]$. Multiplying both sides of $f(\xi) = g_1(\tau, \xi) \cdot h_1(\tau, \xi)$ by the denominator of $g_1(\tau, \xi) \cdot h_1(\tau, \xi)$, we may write $v(\tau)f(\xi) = g(\tau, \xi)h(\tau, \xi)$, where $v, f, g, h \in k[\tau, \xi]$ $(= k[\tau][\xi])$. We may also suppose that v is monic. Let $\bar{k}$ be the algebraic closure of k in $k(\xi)$. Then $\bar{k} \subset k[\xi]$. Indeed, we observe that $k(\tau)[\xi] \cap k(\xi) = k[\xi]$, and hence $k[\xi]$ is integrally closed. Now let $a \in \bar{k}$. Then a is in $k(\xi)$ and is integral over $k[\xi]$, hence a is also in $k[\xi]$. Therefore $\bar{k} \subset k[\xi]$. Thus, $k[\xi] = \bar{k}[\xi]$ and

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$k(\tau)[\xi] = \bar{k}(\tau)[\xi]$. Thus we may assume without loss of generality that k is algebraically closed in $k(\xi)$. Let $g_0(\xi)$ and $h_0(\xi)$ be the leading coefficients of $g(\tau, \xi)$ and $h(\tau, \xi)$, respectively. Then

$$v(\tau) = \frac{g(\tau, \xi)}{g_0(\xi)} \cdot \frac{h(\tau, \xi)}{h_0(\xi)}.$$

The coefficients of $g(\tau, \xi)/g_0(\xi)$ and of $h(\tau, \xi)/h_0(\xi)$ are algebraic over k since they are algebraic sums of products of roots of $v(\tau) = 0$; and they are in $k(\xi)$. Hence they are in k. Thus $g(\tau, \xi)/g_0(\xi)$ and $h(\tau, \xi)/h_0(\xi)$ are units in $k(\tau)[\xi]$. We have $f(\xi) = g_0(\xi) \cdot h_0(\xi)$, and so g_0 or h_0 is a unit in $k[\xi]$. Hence g or h is a unit in $k(\tau)[\xi]$. Thus $f(\xi)$ is irreducible in $k(\tau)[\xi]$ and $(f(\xi)) \cdot k(\tau)[\xi]$ is a prime ideal. Hence $(f(\xi)) \cdot k(\tau)[\xi] \cap k[\xi] = (f(\xi)) \cdot k[\xi]$ is a prime ideal.

THEOREM 3. *Let V be an irreducible γ-dimensional variety defined over k with $\gamma \geqq 2$. If V is factorial over k, then $V \cap H$ is factorial over $k(u_0, \ldots, u_n)$.*

Proof. By Lemma 4, $k(u_1, \ldots, u_n)[\xi]$ is factorial. Let (y) be a generic point of $V \cap H$ over $k(u)$. Then $k(u_1, \ldots, u_n, u_0)[y] \cong k(u_1, \ldots, u_n, \bar{u}_0)[\xi]$, where $\bar{u}_0 = -(u_1\xi_1 + \ldots + u_n\xi_n)$. But $k(u_1, \ldots, u_n, \bar{u}_0)[\xi] = k(u_1, \ldots, u_n)[\xi]_S$, where $S = k[u_1, \ldots, u_n, \bar{u}_0] - \{0\}$ and $k(u_1, \ldots, u_n)[\xi]_S$ is factorial. Hence $k(u)[y]$ is factorial.

Let R be a commutative ring with identity, and let $S \subset R$ be a multiplicative system which does not contain 0. Let f be the canonical homomorphism of R into R_S. For an ideal $\mathfrak{A}$ in R, let $\mathfrak{A}_e = f(\mathfrak{A}) \cdot R_S$, and for an ideal $\mathfrak{X}$ in R_S, let $\mathfrak{X}_c = f^{-1}(\mathfrak{X})$. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two ideals in R and let $\mathfrak{X}$ and $\mathfrak{D}$ be two ideals in R_S. With respect to the operation of the quotient of two ideals, we have $(\mathfrak{A}:\mathfrak{B})_e \subset \mathfrak{A}_e:\mathfrak{B}_e$ and $(\mathfrak{X}:\mathfrak{D})_c \subset \mathfrak{X}_c:\mathfrak{D}_c$ [**9**, p. 219]. We give an elementary proof to the following proposition.

PROPOSITION 1. *Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{X}$, and $\mathfrak{D}$ be as above. Then*
(a) $(\mathfrak{A}:\mathfrak{B})_e = \mathfrak{A}_e:\mathfrak{B}_e$, *if* $\mathfrak{A} \supset \operatorname{Ker} f$ *and* $\mathfrak{B}$ *is finitely generated and*
(b) $(\mathfrak{X}:\mathfrak{D})_c = \mathfrak{X}_c:\mathfrak{D}_c$ *if* $\mathfrak{D}$ *is finitely generated.*

Proof. Let $\mathfrak{B} = (b_1, \ldots, b_t) \cdot R$. We have $\mathfrak{B}_e = (f(b_1), \ldots, f(b_t)) \cdot R_S$. Let $x \in \mathfrak{A}_e:\mathfrak{B}_e$. Then $x\mathfrak{B}_e \subset \mathfrak{A}_e$ and $xf(b_i) = f(a_i)/f(s_i)$ for some $a_i \in \mathfrak{A}$, and $s_i \in S$, where $i = 1, 2, \ldots, t$. Therefore

$$f\left(\prod_{i=1}^{t} s_i\right)xf(b_j) \in f(\mathfrak{A}) \qquad \text{for } j = 1, 2, \ldots, t.$$

For each $b \in f(\mathfrak{B})$, $b = \sum_{j=1}^{t} f(r_j)f(b_j)$ for some $r_1, \ldots, r_t \in R$. Now

$$f\left(\prod_{i=1}^{t} s_i\right)xb = \sum_{j=1}^{t} f\left(\prod_{i=1}^{t} s_i\right)xf(r_j)f(b_j),$$

which is in $f(\mathfrak{A})$. Therefore $f(\prod_{i=1}^{t} s_i)x \in f(\mathfrak{A}):f(\mathfrak{B})$. Hence

$$x \in (f(\mathfrak{A}):f(\mathfrak{B})) \cdot R_S.$$

Since $\mathfrak{A} \supset \operatorname{Ker} f$, by [**9**, p. 148, (15)], $f(\mathfrak{A}):f(\mathfrak{B}) = f(\mathfrak{A}:\mathfrak{B})$. Therefore $x \in (\mathfrak{A}:\mathfrak{B})_e$ and $\mathfrak{A}_e:\mathfrak{B}_e = (\mathfrak{A}:\mathfrak{B})_e$. The proof of (b) is similar.

Let k be an infinite field, and let τ be an indeterminate over k. Let $\mathfrak{A}$ be an ideal in the polynomial ring $k(\tau)[x_1, \ldots, x_n]$. Let

$$\bar{\mathfrak{A}} = \{g(a, x_1, \ldots, x_n) \mid g(\tau, x_1, \ldots, x_n) \in k[\tau, x_1, \ldots, x_n] \cap \mathfrak{A}\}$$

be the specialization of $\mathfrak{A}$ over the specialization $\tau \to a \in k$. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two ideals in $k(\tau)[x_1, \ldots, x_n]$ and let $\mathfrak{A}:\mathfrak{B} = \{\gamma \in k(\tau)[x_1, \ldots, x_n] \mid \gamma\mathfrak{B} \subset \mathfrak{A}\}$. It is well known that $\overline{\mathfrak{A}:\mathfrak{B}} = \bar{\mathfrak{A}}:\bar{\mathfrak{B}}$ almost always, i.e. $\overline{\mathfrak{A}:\mathfrak{B}} = \bar{\mathfrak{A}}:\bar{\mathfrak{B}}$ for all but a finite number of elements in k (see [**1**]). Let $Q = (q_1, \ldots, q_n)$ be a point in the affine space A^n over k with $\mathfrak{q}$ as its prime ideal in $k(\tau)[x_1, \ldots, x_n]$, and let $\mathfrak{q}^c = \mathfrak{q} \cap k[x_1, \ldots, x_n]$. Let $\mathfrak{X}$ be an ideal in the local ring $k(\tau)[x_1, \ldots, x_n]_\mathfrak{q}$, and let

$$\tilde{\mathfrak{X}} = \{f(a, x_1, \ldots, x_n) \mid f(\tau, x_1, \ldots, x_n) \in k[\tau, x_1, \ldots, x_n] \cap \mathfrak{X}\} \cdot k[x_1, \ldots, x_n]_{\mathfrak{q}^c}$$

be the specialization of $\mathfrak{X}$ over the specialization $x \to a \in k$. Thus $\tilde{\mathfrak{X}} = (\overline{\mathfrak{X}_c})_e$.

PROPOSITION 2. *Let $\mathfrak{X}$ and $\mathfrak{D}$ be two ideals in $k(\tau)[x_1, \ldots, x_n]_\mathfrak{q}$. Then $\widetilde{\mathfrak{X}:\mathfrak{D}} = \tilde{\mathfrak{X}}:\tilde{\mathfrak{D}}$ almost always.*

Proof. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two ideals in $k(\tau)[x_1, \ldots, x_n]$ such that $\mathfrak{A} \cdot k(\tau)(x_1, \ldots, x_n]_\mathfrak{q} = \mathfrak{X}$ and $\mathfrak{B} \cdot k(\tau)[x_1, \ldots, x_n]_\mathfrak{q} = \mathfrak{D}$. By Proposition 1, we have $\widetilde{\mathfrak{X}:\mathfrak{D}} = \widetilde{\mathfrak{A}_e:\mathfrak{B}_e} = (\widetilde{\mathfrak{A}:\mathfrak{B}})_e$ and $(\widetilde{\mathfrak{A}:\mathfrak{B}})_e = \overline{((\mathfrak{A}:\mathfrak{B})_{ec})_e} = (\overline{(\mathfrak{A}_e:\mathfrak{B}_e)_c})_e = (\overline{\mathfrak{A}_{ec}:\mathfrak{B}_{ec}})_e$. By [**1**, p. 59, *Satz* 3], $(\overline{\mathfrak{A}_{ec}:\mathfrak{B}_{ec}})_e = (\overline{\mathfrak{A}_{ec}}:\overline{\mathfrak{B}_{ec}})_e$ almost always, and by Proposition 1, $(\overline{\mathfrak{A}_{ec}}:\overline{\mathfrak{B}_{ec}})_e = (\overline{\mathfrak{A}_{ec}})_e:(\overline{\mathfrak{B}_{ec}})_e = \tilde{\mathfrak{A}}_e:\tilde{\mathfrak{B}}_e = \tilde{\mathfrak{X}}:\tilde{\mathfrak{D}}$. Thus we have $\widetilde{\mathfrak{X}:\mathfrak{D}} = \tilde{\mathfrak{X}}:\tilde{\mathfrak{D}}$ almost always.

LEMMA 5. *If $k[\xi]_{\mathfrak{q}^c}$ is factorial, then $k(\tau)[\xi]_\mathfrak{q}$ is factorial. Conversely, if $k(\tau)[\xi]_{\mathfrak{p}^e}$ is factorial and k is infinite, then $k[\xi]_\mathfrak{p}$ is factorial.*

Proof. Assume that $k[\xi]_{\mathfrak{q}^c}$ is factorial. Since $k(\tau)[\xi]_\mathfrak{q} = k[\tau][\xi]_{\mathfrak{q} \cap k[\tau][\xi]} = (k[\xi]_{\mathfrak{q}^c}[\tau])_{\mathfrak{q} \cap k[\tau][\xi]}$, we see that $k(\tau)[\xi]_\mathfrak{q}$ is factorial.

For the converse we use the fact that an integral domain R is factorial if and only if for every two elements a and b in R, $(a):(b)$ is a principal ideal in R [**3**, p. 370, Lemma 1]. Let $a(\xi)$ and $b(\xi)$ be any two elements which are non-zero and non-unit in $k[\xi]_\mathfrak{p}$. We proceed to prove that

$$(a(\xi)) \cdot k[\xi]_\mathfrak{p}:(b(\xi)) \cdot k[\xi]_\mathfrak{p}$$

is principal. Indeed, since $k(\tau)[\xi]_{\mathfrak{p}^e}$ is factorial,

$$(a(\xi)) \cdot k(\tau)[\xi]_{\mathfrak{p}^e}:(b(\xi)) \cdot k(\tau)[\xi]_{\mathfrak{p}^e} = (c(\tau, \xi)) \cdot k(\tau)[\xi]_{\mathfrak{p}^e}$$

for some $c(\tau, \xi) \in k[\tau, \xi]$. Let $\mathfrak{T}$ be the prime ideal of V in $k[x_1, \ldots, x_n]$ and let $\mathfrak{P}$ be the prime ideal in $k[x_1, \ldots, x_n]$ containing $\mathfrak{T}$ such that $\mathfrak{P}/\mathfrak{T} = \mathfrak{p}$.

Assume that $a(\xi)$, $b(\xi) \in k[\xi]$. We have, by [**9**, p. 148, (21)], that

$$(a(x_1, \ldots, x_n), \mathfrak{T}) \cdot k(\tau)[x_1, \ldots, x_n]_{\mathfrak{P}^e} : (b(x_1, \ldots, x_n), \mathfrak{T}) \cdot k(\tau)[x_1, \ldots, x_n]_{\mathfrak{P}^e}$$
$$= (c(\tau, x_1, \ldots, x_n), \mathfrak{T}) \cdot k(\tau)[x_1, \ldots, x_n]_{\mathfrak{P}^e}.$$

By Proposition 2, there exists $\alpha \in k$ such that

$$(a(x_1, \ldots, x_n), \mathfrak{T}) \cdot k[x_1, \ldots, x_n]_{\mathfrak{P}} : (b(x_1, \ldots, x_n), \mathfrak{T}) \cdot k[x_1, \ldots, x_n]_{\mathfrak{P}}$$
$$= (c(\alpha, x_1, \ldots, x_n), \mathfrak{T}) \cdot k[x_1, \ldots, x_n]_{\mathfrak{P}}.$$

Passing to the quotient, we have, by [**9**, p. 148, (15)],

$$(a(\xi)) \cdot k[\xi]_{\mathfrak{p}} : (b(\xi)) \cdot k[\xi]_{\mathfrak{p}} = (c(\alpha, \xi)) \cdot k[\xi]_{\mathfrak{p}}.$$

THEOREM 4. *Let V be an irreducible γ-dimensional variety defined over k with $\gamma \geqq 2$. Let P be a point on the generic hyperplane section $V \cap H$ of V and H over $k(u_0, \ldots, u_n)$. If V is factorial at P over k, then $V \cap H$ is factorial at P over $k(u_0, \ldots, u_n)$.*

Proof. Using Lemma 5 and the inclusion relation,

$$k(u_1, \ldots, u_n)[\xi]_{\mathfrak{p}} \subset k(u_1, \ldots, u_n, \bar{u}_0)[\xi]_S \subset k(u_1, \ldots, u_n, \bar{u}_0)[\xi]_{\mathfrak{p}_0},$$

where $S = k(u_1, \ldots, u_n)[\xi] - \mathfrak{p}$ that appeared in the proof of Theorem 2, we see that $k(u_1, \ldots, u_n)[\xi]_{\mathfrak{p}}$ is factorial. Since

$$k(u_1, \ldots, u_n, \bar{u}_0)[\xi]_S = (k(u_1, \ldots, u_n)[\xi]_S)_{k[u_1, \ldots, u_n, \bar{u}_0] - \{0\}}$$
$$= (k(u_1, \ldots, u_n)[\xi]_{\mathfrak{p}})_{k[u_1, \ldots, u_n, \bar{u}_0] - \{0\}},$$

and $(k(u_1, \ldots, u_n)[\xi]_{\mathfrak{p}})_{k[u_1, \ldots, u_n, \bar{u}_0] - \{0\}}$ is factorial, we see that

$$k(u_1, \ldots, u_n, \bar{u}_0)[\xi]_S$$

is factorial. On the other hand, we observe that $S \subset k(u_1, \ldots, u_n, \bar{u}_0)[\xi] - \mathfrak{p}_0$ since $\mathfrak{p} = \mathfrak{p}_0 \cap k(u_1, \ldots, u_n, \bar{u}_0)[\xi]$. Therefore

$$k(u_1, \ldots, u_n, \bar{u}_0)[\xi]_{\mathfrak{p}_0} = (k(u_1, \ldots, u_n, \bar{u}_0)[\xi]_S)_{k(u_1, \ldots, u_n, \bar{u}_0)[\xi] - \mathfrak{p}_0}$$

is factorial.

REFERENCES

1. W. Krull, *Parameterspezialisierung in Polynomringen*, Arch. Math. *1* (1948), 56–64.

2. S. Lang, *Introduction to algebraic geometry* (Interscience, New York, 1964).

3. D. Mumford, *Introduction to modern algebraic geometry* (preprint, Harvard University).

4. M. Nagata, *Local rings* (Interscience, New York, 1962).

5. Y. Nakai, *Note on the intersection of an algebraic variety with the generic hyperplane*, Mem. Coll. Sci. Univ. Kyoto, Ser. A Math. *26* (1951), 185–187.

6. A. Seidenberg, *The hyperplane section of normal varieties*, Trans. Amer. Math. Soc. *50* (1941), 357–386.

7. O. Zariski, *The theorem of Bertini on the variable singular points of a linear system of varieties*, Trans. Amer. Math. Soc. *56* (1944), 130–140.

8. ——— *The concept of a simple point of an abstract algebraic variety*, Trans. Amer. Math. Soc. *62* (1947), 1–52.

9. O. Zariski and P. Samuel, *Commutative algebra*, Vol. I, The University series in Higher Mathematics (Van Nostrand, Princeton, N.J.—Toronto—London—New York, 1958).

10. ——— *Commutative algebra*, Vol. II, The University series in Higher Mathematics (Van Nostrand, Princeton, N.J.—Toronto—London—New York, 1960).

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