Algebraic and geometric definitions of the cross product: the link

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1. *Introduction*

Given two vectors $\vec{u} = (u_1, u_2, u_3)^t$ and $\vec{y} = (v_1, v_2, v_3)^t$ in $\mathbb{R}^3$, the cross product $\vec{u} \times \vec{v}$ is defined as follows (see [1] or [2]):

Algebraic definition: $\vec{u} \times \vec{v} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$.

It is a coordinate-dependent definition, but does not depend on a drawing convention to sketch the basis vectors $\vec{i} = (1, 0, 0)^t, \vec{j}\,(0, 1, 0)^t$ and $\vec{k}\,(0, 0, 1)^t$. To associate a direction to $\vec{u} \times \vec{v}$, we need to link it to a geometric definition.

In an attempt to have a geometric interpretation of the cross product, it is pertinent to read the following coordinate-free definition. According to [2], this definition appeals to the physicists who hate to choose axes and coordinates.

Geometric definition. There exists a unique vector $\vec{w}$ satisfying the following conditions:

G1: $\vec{w}$ is perpendicular to $\vec{u}$ and $\vec{v}$.

G2: The length of $\vec{w}$ is equal to the area $A(\vec{u}, \vec{v})$ of the parallelogram determined by $\vec{u}$ and $\vec{v}$.

G3: (RHR) Using your right hand, curl your fingers from $\vec{u}$ to $\vec{v}$ or, equivalently, put your index finger in direction of $\vec{u}$ and your middle finger in direction $\vec{v}$. Then your thumb points in a direction which we will consider orthogonal to both $\vec{u}$ and $\vec{v}$. We will denote $\vec{n}(\vec{u}, \vec{v}) = \vec{n}_R(\vec{u}, \vec{v})$ the unit vector in the direction indicated by the thumb, and set $\vec{w} = A(\vec{u}, \vec{v})\,\vec{n}(\vec{u}, \vec{v})$.

G3: (LHR) Using your left hand, curl your fingers from $\vec{u}$ to $\vec{v}$ or, equivalently, put your index finger in the direction of $\vec{u}$ and your middle finger in direction $\vec{v}$. Then your thumb points in a direction which we will consider orthogonal to both $\vec{u}$ and $\vec{v}$. We will denote $\vec{n}(\vec{u}, \vec{v}) = \vec{n}_R(\vec{u}, \vec{v})$ the unit vector in the direction indicated by the thumb, and set $\vec{w} = A(\vec{u}, \vec{v})\,\vec{n}(\vec{u}, \vec{v})$.

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This geometric definition leads to two geometric cross products. One for the right hand, say $\vec{w}_R = A(\vec{u}, \vec{v})\vec{n}_R(\vec{u}, \vec{v})$, and one for the left hand, say $\vec{w}_L = A(\vec{u}, \vec{v})\vec{n}_L(\vec{u}, \vec{v})$. Obviously $\vec{w}_R = -\vec{w}_L$. In physics, it doesn't matter whether you use your right or left hand to compute the direction of a vector product, this property is known as 'conservation of parity'.

We will see that the link between the two definitions is based on a convention for drawing axes. Our presentation requires only some elementary knowledge of linear algebra. It contains, extends, and simplifies other relatively simple presentations, for example see [3, 4, 5, 6, 7, 8]. Those interested by more advanced topics on this subject could read [9].

2. *Algebraic Considerations*

The cross product can be expressed as the symbolic determinant

$$\vec{u} \times \vec{v} = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}.$$

And as a simple example we get $\vec{i} \times \vec{j} = \vec{k}$. Using the dot product defined by

$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3,$$

we have

$$\left(\vec{u} \times \vec{v}\right) \cdot \vec{w} = \det\left(\vec{u}, \vec{v}, \vec{w}\right).$$

It follows that $\vec{u} \times \vec{v}$ is orthogonal to $\vec{u}$ and $\vec{v}$, so it satisfies (G1) of the geometric definition.

Now we would like to show that $\vec{u} \times \vec{v}$ satisfies (G2) of the geometric definition. This will be done using a representation of $\vec{v}$ and rotations.

We assume that $\vec{u}$ and $\vec{v}$ are linearly independent and have $\vec{u} \times \vec{v} \neq \vec{0}$. Consider $\theta \in (0, \pi)$ to be such that

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta.$$

Let us set

$$\begin{cases} \vec{i}_0 = \dfrac{\vec{u}}{\|\vec{u}\|}, \\ \vec{j}_0 = \dfrac{\vec{v} - \left(\vec{i}_0 \cdot \vec{v}\right)\vec{i}_0}{\left\|\vec{v} - \left(\vec{i}_0 \cdot \vec{v}\right)\vec{i}_0\right\|} = \dfrac{\vec{v} - \left(\vec{i}_0 \cdot \vec{v}\right)\vec{i}_0}{\|\vec{v}\| \sin\theta}, \end{cases}$$

which form an orthonormal basis for the two-dimensional vector subspace

$$\operatorname{lin}\left\{\vec{u}, \vec{v}\right\} = \left\{r\vec{u} + s\vec{v} \mid r, s \in \mathbb{R}\right\} = \left\{\alpha\vec{i}_0 + \beta\vec{j}_0 \mid \alpha, \beta \in \mathbb{R}\right\} = \operatorname{lin}\left\{\vec{i}_0, \vec{j}_0\right\}.$$

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Then we get the useful representation of $\vec{v}$

$$\vec{v} = \|\vec{v}\| \left(\cos\theta \vec{i}_0 + \sin\theta \vec{j}_0\right),$$

from which it follows that

$$\vec{u} \times \vec{v} = \|\vec{u}\| \|\vec{v}\| \sin\theta \, \vec{i}_0 \times \vec{j}_0. \tag{1}$$

Since $\sin\theta > 0$, we get

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin\theta \, \|\vec{i}_0 \times \vec{j}_0.\| . \tag{2}$$

Now we will use rotations to send $\vec{i}_0$ to $\vec{i}$ and $\vec{j}_0$ to $\vec{j}$. Each rotation we use will leave one coordinate fixed. The fixed coordinate corresponds to a basis vector $\vec{d}$, which can be equal to $\vec{i}$ or $\vec{j}$ or $\vec{k}$. The rotation, denoted by $R_{\vec{d},\omega}$, is given by

$$R_{\vec{d}\omega} = \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega \\ 0 & \sin\omega & \cos\omega \end{pmatrix} & \text{for } \vec{d} = \vec{i}, \\ \begin{pmatrix} \cos\omega & 0 & \sin\omega \\ 0 & 1 & 0 \\ -\sin\omega & 0 & \cos\omega \end{pmatrix} & \text{for } \vec{d} = \vec{j}, \\ \begin{pmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{for } \vec{d} = \vec{k}. \end{cases}$$

We have $\det\left(R_{\vec{d},\omega}\right) = +1$, and $R_{\vec{d},\omega}^{-1} = R_{\vec{d},-\omega} = R_{\vec{d},\omega}^{t}$. Also

$$R_{\vec{d},\omega}\vec{u} \cdot R_{\vec{d},\omega}\vec{v} = \vec{u} \cdot \vec{v},$$

so

$$\left\|R_{\vec{d},\omega}\vec{u}\right\| = \|\vec{u}\|,$$

and

$$\vec{u} \cdot \vec{v} = 0 \qquad \text{if, and only if,} \qquad R_{\vec{d},\omega}\vec{u} \cdot R_{\vec{d},\omega}\vec{v} = 0.$$

We first rotate $\vec{i}_0$ using $R_{\vec{i},\alpha}$ to send $\vec{i}_0$ to $\vec{i}_1$ in $\text{lin}\left\{\vec{i}, \vec{j}\right\}$. The angle α is related to the orthogonal projection of $\vec{i}_0$ onto $\text{lin}\left\{\vec{j}, \vec{k}\right\}$. In fact, we can write

$$\vec{i}_0 = \left(\vec{i} \cdot \vec{i}_0\right)\vec{i} + r\left(\cos\alpha \, \vec{j} + \sin\alpha \, \vec{k}\right)$$

with $r = \left\|\vec{i}_0 - \left(\vec{i} \cdot \vec{i}_0\right)\right\| = \sqrt{1 - \left(\vec{i} \cdot \vec{i}_0\right)^2}$, so $\alpha = -\overline{\alpha}$. We have

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$$\begin{cases} R_{\vec{i},\alpha}\vec{i}_0 = \vec{i}_1 \in \operatorname{lin}\left\{\vec{i}, \vec{j}\right\}, \\ R_{\vec{i},\alpha}\vec{j}_0 = \vec{j}_1. \end{cases}$$

If $\vec{i}_0$ is already in $\operatorname{lin}\left\{\vec{i}, \vec{j}\right\}$ then we set $\alpha = 0$ and we use $R_{\vec{i},0} = I$, which means we do nothing. Then we rotate $\vec{i}_1$ using $R_{\vec{k},\beta}$ to send $\vec{i}_1$ to $\vec{i}_2 = \vec{i}$. The angle β is related to the expression of $\vec{i}_1$ in $\operatorname{lin}\left\{\vec{i}, \vec{j}\right\}$. We can write

$$\vec{i}_1 = \cos\overline{\beta}\,\vec{i} + \sin\overline{\beta}\,\vec{j},$$

so $\beta = -\overline{\beta}$. We have

$$\begin{cases} R_{\vec{k},\beta}\vec{i}_1 = \vec{i}_2, \\ R_{\vec{k},\beta}\vec{j}_1 = \vec{j}_2 \in \operatorname{lin}\left\{\vec{j}, \vec{k}\right\}, \end{cases}$$

because $\vec{j}_2 \cdot \vec{i} = \vec{j}_2 \cdot \vec{i}_2 = \vec{j}_1 \cdot \vec{i}_1 = \vec{j}_0 \cdot \vec{i}_0 = 0$. Finally, we rotate $\vec{j}_2$ using $R_{\vec{i},\gamma}$ to send $\vec{j}_2$ to $\vec{j}_3 = \vec{j}$. The angle γ is related to the expression of $\vec{j}_2$ in $\operatorname{lin}\left\{\vec{j}, \vec{k}\right\}$. We can write

$$\vec{j}_2 = \cos\bar{\gamma}\,\vec{j} + \sin\bar{\gamma}\,\vec{k},$$

so $\gamma = -\bar{\gamma}$. We have

$$\begin{cases} R_{\vec{i},\gamma}\vec{j}_2 = \vec{j}_3 = \vec{j}, \\ R_{\vec{i},\gamma}\vec{i}_2 = \vec{i}_3 = \vec{i}, \end{cases}$$

because $\vec{i}_2 = \vec{i}$.

The successive application of the three rotations, given by

$$R = R_{\vec{i},\gamma} R_{\vec{k},\beta} R_{\vec{i},\alpha},$$

is such that

$$R\,\vec{i}_0 = \vec{i} \qquad \text{and} \qquad R\,\vec{j}_0 = \vec{j}.$$

We also have $RR^t = I$ and $\det(R) = 1$.

The next result, which can be found in [9], will allow us to say that the rotation of the cross product of two vectors is the cross product of the two rotated vectors.

Lemma 1: If R is such that $RR^t = I$ and $\det(R) = 1$, then

$$R\left(\vec{u} \times \vec{v}\right) = R\vec{u} \times R\vec{v}.$$

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Proof: Consider any $\vec{w}$ in $\mathbb{R}^3$, with

$$\begin{aligned}\left(R\vec{u} \times R\vec{v}\right) \cdot R\vec{w} &= \det\left(R\vec{u}, R\vec{v}, R\vec{w}\right)\\ &= \det\left(R\left(\vec{u}, \vec{v}, \vec{w}\right)\right)\\ &= \det\left(R\right)\det\left(\vec{u}, \vec{v}, \vec{w}\right)\\ &= \det\left(\vec{u}, \vec{v}, \vec{w}\right)\\ &= \left(\vec{u} \times \vec{v}\right) \cdot \vec{w},\end{aligned}$$

and

$$\left(R\vec{u} \times R\vec{v}\right) \cdot R\vec{w} = R^t\left(R\vec{u} \times R\vec{v}\right) \cdot \vec{w}.$$

Then

$$R^t\left(R\vec{u} \times R\vec{v}\right) \cdot \vec{w} = \left(\vec{u} \times \vec{v}\right) \cdot \vec{w}$$

for all $\vec{w} \in \mathbb{R}^3$. So we get $R^t\left(R\vec{u} \times R\vec{v}\right) = \left(\vec{u} \times \vec{v}\right)$, and the result follows.

From this result we have

$$R\left(\vec{i}_0 \times \vec{j}_0\right) = R\left(\vec{i}_0\right) \times R\left(\vec{j}_0\right) = \vec{i} \times \vec{j} = \vec{k},$$

so we obtain

$$\left\|\vec{i}_0 \times \vec{j}_0\right\| = \left\|R\left(\vec{i}_0 \times \vec{j}_0\right)\right\| = \left\|R\left(\vec{i}_0\right) \times R\left(\vec{j}_0\right)\right\| = \left\|\vec{i} \times \vec{j}\right\| = \left\|\vec{k}\right\| = 1.$$

Consequently, from (2) we obtain $\left\|\vec{u} \times \vec{v}\right\| = \left\|\vec{u}\right\|\left\|\vec{v}\right\|\sin\theta$, which corresponds to the area of the parallelogram determined by $\vec{u}$ and $\vec{v}$. So $\vec{u} \times \vec{v}$ satisfies (G2).

3. *Convention for drawing axes*

To relate the geometric definition to the algebraic one, we need a way to sketch the vectors $\vec{u}$, $\vec{v}$ and $\vec{u} \times \vec{v}$. So, a drawing convention for the basis $\left\{\vec{i}, \vec{j}, \vec{k}\right\}$ is necessary. We can consider two ways to sketch the axes based on the fact that the right-hand rule, or the left-hand rule, holds for $\vec{i}$, $\vec{j}$ and $\vec{k}$, that is to say that $\vec{i} \times \vec{j} = \vec{k} = \vec{n}\left(\vec{i}, \vec{j}\right)$.

Right-hand drawing convention for $\left\{\vec{i}, \vec{j}, \vec{k}\right\}$

Fix $\vec{i}$ and $\vec{j}$ and determine $\vec{n}\left(\vec{i}, \vec{j}\right)$ from G3 (RHR). Then set $\vec{k} = \vec{n}_R\left(\vec{i}, \vec{j}\right)$.

Left-hand drawing convention for $\left\{\vec{i}, \vec{j}, \vec{k}\right\}$

Fix $\vec{i}$ and $\vec{j}$ and determine $\vec{n}\left(\vec{i}, \vec{j}\right)$ from G3 (LHR). Then set $\vec{k} = \vec{n}_L\left(\vec{i}, \vec{j}\right)$.

Suppose you sketch the system of axes $\left\{\vec{i}, \vec{j}, \vec{k}\right\}$ using the right-hand, respectively the left-hand, drawing convention. Once the coordinate system is fixed, then you use (G3-RHR), respectively (G3-LHR), to determine the direction of the cross product of any two vectors. So, in particular, for $\vec{i}$ and $\vec{j}$, you will get the expected and desired result

$$\vec{n}\left(\vec{i}, \vec{j}\right) = \vec{k} = \vec{i} \times \vec{j}.$$

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On the other hand, suppose you sketch the system of axes $\{\vec{i}, \vec{j}, \vec{k}\}$ using the right-hand, respectively the left-hand, drawing convention. Once the coordinate system is fixed, then you use (G3-LHR), respectively (G3-RHR), to determine the direction of the cross product of any two vectors. So, in particular, for $\vec{i}$ and $\vec{j}$ you will get the undesired result

$$\vec{n}\left(\vec{i}, \vec{j}\right) = -\vec{k} = -\left(\vec{i} \times \vec{j}\right).$$

These observations mean that for the geometric method, to determine the direction of the cross product for any pair of vectors use the hand you used to fix the axes.

4. *Orientation of the cross product*

We will say that the right-hand rule, or the left-hand rule, holds if from both the algebraic and the geometric definitions we get the same vector

$$\vec{u} \times \vec{v} = \vec{w} = A\left(\vec{u}, \vec{v}\right) \vec{n}\left(\vec{u}, \vec{v}\right).$$

Since $\vec{u} \times \vec{v}$ already verifies (G1) and (G2), it remains to consider the direction.

The next result, which is similar to Lemma 1, will allow us to say that the rotation of the geometric direction of two vectors is the geometric direction of the two rotated vectors.

Lemma 2: For any rotation $R_{\vec{d},\omega}$ we have

$$R_{\vec{d},\omega}\vec{n}\left(\vec{u}, \vec{v}\right) = \vec{n}\left(R_{\vec{d},\omega}\vec{u}, R_{\vec{d},\omega}\vec{v}\right).$$

Proof: Since a rotation does not change the length of a vector, nor the angle between two vectors, following [4], and also [6], we can say that, when we apply the rotation simultaneously to the vectors and to the right or left hand, the vectors and the right or left hand continuously still fit. So we get the result.

Now we have the following result concerning the correspondence of the algebraic and the geometric definitions of the cross product.

Theorem 3: The right-hand rule, or the left-hand rule, holds for $\vec{u}$, $\vec{v}$, which means that $\vec{u} \times \vec{v} = \vec{w} = A\left(\vec{u}, \vec{v}\right) \vec{n}\left(\vec{u}, \vec{v}\right)$, if, and only if, it holds for $\vec{i}$, $\vec{j}$ and $\vec{k}$, which means that $\vec{i} \times \vec{j} = \vec{k} = \vec{n}\left(\vec{i}, \vec{j}\right)$.

Proof: From (1), $\vec{u} \times \vec{v}$ and $\vec{i}_0 \times \vec{j}_0$ both point in the same direction. Moreover, if you curl your fingers from $\vec{u}$ to $\vec{v}$ or from $\vec{i}_0$ to $\vec{j}_0$, in both cases your thumb points in the same direction, so $\vec{n}\left(\vec{u}, \vec{v}\right) = \vec{n}\left(\vec{i}_0, \vec{j}_0\right)$. Then, from (1) and (G3), we have

$$\begin{cases} \vec{u} \times \vec{v} = A\left(\vec{u}, \vec{v}\right)\vec{i}_0 \times \vec{j}_0, \\ \quad\ \ \vec{w} = A\left(\vec{u}, \vec{v}\right)\vec{n}\left(\vec{i}_0, \vec{j}_0\right). \end{cases}$$

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Moreover since $\vec{i}_0 \times \vec{j}_0$ and $\vec{n}(\vec{i}_0, \vec{j}_0)$ are both unit vectors perpendicular to $\vec{i}_0$ and $\vec{j}_0$, there exists a constant $\lambda(\vec{i}_0, \vec{j}_0)$ such that $\lambda(\vec{i}_0, \vec{j}_0)$ is equal to +1 or −1, and

$$\vec{i}_0 \times \vec{j}_0 = \lambda(\vec{i}_0, \vec{j}_0)\vec{n}(\vec{i}_0, \vec{j}_0).$$

We will get the result if, and only if, $\vec{n}(\vec{i}_0, \vec{j}_0) = \vec{i}_0 \times \vec{j}_0$ or if, and only if, $\lambda(\vec{i}_0, \vec{j}_0) = 1$.

Going back to the rotations used in the algebraic context, we apply successively each rotation, simultaneously, on the vectors and on the right-hand, or on the left-hand. From Lemma 2, we have

$$\begin{cases} R_{\vec{i},\alpha}\vec{n}(\vec{i}_0, \vec{j}_0) & = & \vec{n}(R_{\vec{i},\alpha}\vec{i}_0, R_{\vec{i},\alpha}\vec{j}_0) & = & \vec{n}(\vec{i}_1, \vec{j}_1) \\ R_{\vec{k},\beta}\vec{n}(\vec{i}_1, \vec{j}_1) & = & \vec{n}(R_{\vec{k},\beta}\vec{i}_1, R_{\vec{k},\beta}\vec{j}_1) & = & \vec{n}(\vec{i}_2, \vec{j}_2) \\ R_{\vec{i},\gamma}\vec{n}(\vec{i}_2, \vec{j}_2) & = & \vec{n}(R_{\vec{i},\gamma}\vec{i}_2, R_{\vec{i},\gamma}\vec{j}_2) & = & \vec{n}(\vec{i}_3, \vec{j}_3) \end{cases}$$

and $\vec{n}(\vec{i}_3, \vec{j}_3) = \vec{n}(\vec{i}, \vec{j})$. So $R\vec{n}(\vec{i}_0, \vec{j}_0) = \vec{n}(R\vec{i}_0, R\vec{j}_0) = \vec{n}(\vec{i}, \vec{j})$.

Now using Lemma 1 and Lemma 2, we have successively

$$\begin{array}{ccc} \text{(algebraic context)} & & \text{(geometric context)} \\ \vec{i}_0 \times \vec{j}_0 & = & \lambda(\vec{i}_0, \vec{j}_0)\vec{n}(\vec{i}_0, \vec{j}_0) \\ R(\vec{i}_0 \times \vec{j}_0) & = & \lambda(\vec{i}_0, \vec{j}_0)R\vec{n}(\vec{i}_0, \vec{j}_0) \\ R\vec{i}_0 \times R\vec{j}_0 & = & \lambda(\vec{i}_0, \vec{j}_0)\vec{n}(R\vec{i}_0, R\vec{j}_0) \\ \vec{i} \times \vec{j} & = & \lambda(\vec{i}_0, \vec{j}_0)\vec{n}(\vec{i}, \vec{j}) \\ \vec{k} & = & \lambda(\vec{i}_0, \vec{j}_0)\vec{n}(\vec{i}, \vec{j}), \end{array}$$

so, $\lambda(\vec{i}_0, \vec{j}_0) = 1$ if, and only if, $\vec{n}(\vec{i}, \vec{j}) = \vec{k} = \vec{i} \times \vec{j}$.

5. *Conclusion*

We have explained the link between the algebraic and the geometric definitions of the cross product. The conditions under which the right-hand rule and the left-hand rule hold are clearly presented.

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The answers to the *Nemo* page from July 2024 on curves were:

1.	Thomas Hardy	The Woodlanders	Chapter 1	
2.	Marianne Moore	Marriage		
3.	Laurence Sterne	Tristram Shandy	Book 3	Chapter 25
4.	PG Wodehouse	Piccadilly Jim	Chapter 23	
5.	TE Lawrence	Seven Pillars of Wisdom	Chapter 3	
6.	James Joyce	Ulysses	Circe	

Congratulations to Martin Lukarevski on tracking all of these down. It is time to revisit equations, and quadratics in particular. Quotations are to be identified by reference to author and work. Solutions are invited to the Editor by 23rd January 2025.

1. For everything else is on the square
   If done by the best quadratics;
   And nothing is low in High Finance
   Or the Higher Mathematics.

2. He tried it by practice and the unitary method, by multiplication, and by rule-of-three-and-three-quarters. He tried it by decimals and by compound interest. He tried it by square root and by cube root. He tried it by addition, simple and otherwise, and he tried it by mixed examples in vulgar fractions. But it was all of no use. Then he tried to do the sum by algebra, by simple and by quadratic equations, by trigonometry, by logarithms, and by conic sections.

Continued on page 483