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A LOWER BOUND FOR $K_X L$ OF QUASI-POLARIZED SURFACES (X, L) WITH NON-NEGATIVE KODAIRA DIMENSION

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ABSTRACT. Let X be a smooth projective surface over the complex number field and let L be a nef-big divisor on X. Here we consider the following conjecture; If the Kodaira dimension $\kappa(X) \geq 0$, then $K_X L \geq 2q(X) - 4$, where $q(X)$ is the irregularity of X. In this paper, we prove that this conjecture is true if (1) the case in which $\kappa(X) = 0$ or 1, (2) the case in which $\kappa(X) = 2$ and $h^0(L) \geq 2$, or (3) the case in which $\kappa(X) = 2$, X is minimal, $h^0(L) = 1$, and L satisfies some conditions.

0. **Introduction.** Let X be a smooth projective manifold over $\mathbb{C}$ with $\dim X \geq 2$, and L a Cartier divisor on X. Then (X, L) is called a *pre-polarized manifold*. In particular, if L is ample (resp. nef-big), then (X, L) is said to be a *polarized* (resp. *quasi-polarized*) manifold. We define the sectional genus $g(L)$ of a pre-polarized manifold (X, L) is defined by the following formula;

$$g(L) = 1 + \frac{1}{2}\Big(K_X + (n-1)L\Big)L^{n-1},$$

where K_X is the canonical divisor of X.

Then there is the following conjecture.

CONJECTURE 0. *Let (X, L) be a quasi-polarized manifold. Then $g(L) \geq q(X)$, where $q(X) = \dim H^1(X, \mathcal{O}_X)$.*

In this paper, we consider the case in which X is a smooth surface. If $\dim X = 2$ and $h^0(L) > 0$, then this conjecture is true. But in general, it is unknown whether this conjecture is true or not. In the papers [Fk1] and [Fk4], the author proved that $L^2 \leq 4$ if L is ample, $g(L) = q(X)$, $h^0(L) > 0$ and $\kappa(X) \geq 0$. By this result, we think that the degree of (X, L) is bounded from above by using $m = g(L) - q(X)$ if $\kappa(X) \geq 0$. By studying some examples of (X, L), we conjectured the following.

CONJECTURE 1. *If (X, L) is a quasi-polarized surface with $\kappa(X) \geq 0$. Then $L^2 \leq 2m + 2$ if $g(L) = q(X) + m$.*

We remark that m is non-negative integer if $h^0(L) > 0$. This conjecture is equivalent to the following conjecture.

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CONJECTURE 1′. *If* (X, L) *is a quasi-polarized surface with* $\kappa(X) \geq 0$.
Then $K_X L \geq 2q(X) - 4$.

This conjecture 1′ is thought to be a generalization of the fact that $\deg K_C = 2g(C) - 2$ if C is a smooth projective curve.

In this paper, we consider the above conjecture. The main results are the following.

MAIN THEOREM 1. *Let* (X, L) *be a quasi-polarized surface with* $\kappa(X) = 0$ *or* 1. *Then* $K_X L \geq 2q(X) - 4$.

If this equality holds and (X, L) *is L-minimal, then* (X, L) *is one of the following;*

(1) $\kappa(X) = 0$ *case. X is an Abelian surface and L is any nef and big divisor.*
(2) $\kappa(X) = 1$ *case.* $X \cong F \times C$ *and* $L \equiv C + (m+1)F$, *where F and C are smooth curves with* $g(C) \geq 2$ *and* $g(F) = 1$, *and* $m = g(L) - q(X)$.

(See Theorem 2.1.)

MAIN THEOREM 2. *Let* (X, L) *be a quasi-polarized surface with* $\kappa(X) = 2$ *and* $h^0(L) \geq 2$. *Then* $K_X L \geq 2q(X) - 2$.

If this equality holds and (X, L) *is L-minimal, then* (X, L) *is the following;* $X \cong F \times C$ *and* $L \equiv C + 2F$, *where F and C are smooth curves with* $g(F) = 2$ *and* $g(C) \geq 2$.

(See Theorem 3.1)

MAIN THEOREM 3. *Let X be a minimal smooth surface of general type and let D be a nef-big effective divisor with* $h^0(D) = 1$ *on X. If D is not the following type* $(\star)$, *then* $K_X D \geq 2q(X) - 4$*:*

$(\star)$ $D = C_1 + \sum_{j \geq 2} r_j C_j$; $C_1^2 > 0$ *and the intersection matrix* $\|(C_j, C_k)\|_{j \geq 2, k \geq 2}$ *of* $\sum_{j \geq 2} r_j C_j$ *is negative semidefinite.*

(See Section 4.)

MAIN THEOREM 4. *Let X be a minimal smooth projective surface with* $\kappa(X) = 2$ *and let D be a nef-big effective divisor on X such that D is the type* $(\star)$. *Then* $D^2 \leq 4m + 4$ *if* $m = g(D) - q(X)$.

We remark that the classification of polarized surfaces (X, L) with $\kappa(X) \geq 1$ and $K_X L \leq 2$ is obtained by [DP]. We work over the complex number field $\mathbb{C}$.

1. **Preliminaries.**

DEFINITION 1.1. Let (X, L) be a quasi-polarized surface.

(1) We call (X_1, L_1) a minimalization of (X, L) if $\varphi : X \longrightarrow X_1$ is a minimal model of X and $L_1 = \varphi_* L$ in the sense of cycle theory. (We remark that L_1 is nef and big (resp. ample) on X_1 if so is L.)

(2) We say that (X, L) is L-minimal if $LE > 0$ for any (-1)-curve E on X. For any quasi-polarized surface (X, L), there exists a birational morphism $\rho : (X, L) \longrightarrow (X_0, L_0)$ such that $L = \rho^* L_0$ and (X_0, L_0) is L_0-minimal. Then we call (X_0, L_0) an L-minimalization of (X, L).

LEMMA 1.2 (DEBARRE). *Let X be a minimal surface of general type with $q(X) \geq 1$. Then $K_X^2 \geq 2p_g(X)$. (Hence $K_X^2 \geq 2q(X)$ for any minimal surface of general type.)*

PROOF. See [D]. ∎

THEOREM 1.3. *Let (X, L) be an L-minimal quasi-polarized surface with $\kappa(X) \geq 0$. If $h^0(L) \geq 2$, then (X, L) satisfies one of the following conditions.*

(1) $g(L) \geq 2q(X) - 1$.

(2) For any linear pencil $\Lambda \subseteq |L|$, the fixed part $Z(\Lambda)$ of Λ is not zero and $\mathrm{Bs}\,\Lambda_M = \phi$, where Λ_M is movable part of Λ. Let $f\colon X \longrightarrow C$ be the fiber space induced by Λ_M. Then $g(L) \geq g(C) + 2g(F) \geq q(X) + g(F)$, $g(C) \geq 2$, $LF = 1$ and $L - aF$ is numerically equivalent to an effective divisor for a general fiber F of f, where $a \geq 2$.

PROOF. See Theorem 3.1 in [Fk3]. ∎

LEMMA 1.4. *Let $f\colon X \longrightarrow C$ be a relatively minimal elliptic fibration with $q(X) = g(C) + 1$. If $LF = 1$ for a nef-big divisor L on X, then $X \cong F \times C$ and $f\colon X \longrightarrow C$ is the natural projection, where F is a general fiber of f.*

PROOF (SEE [FJ3]). By hypothesis f is a quasi-bundle (see Lemma 1.5 and Lemma 1.6 in [S]). Let $\Sigma \subset C$ be the singular locus of f and $U = C - \Sigma$. We fix an elliptic curve $E \cong f^{-1}(x)$ for $x \in U$. Then by [Fj3], we have a map $\varphi\colon \pi_1(U) \longrightarrow \mathrm{Aut}(E, L_E)$. Since the translations of E preserving L_E are of order $d = \deg L_E$ by Abel's Theorem, $\mathrm{Aut}(E, L_E)$ is finite group. Let $G = \mathrm{Im}\,\varphi$. Then by [Fj3], there exists a Galois covering $\pi\colon D \longrightarrow C$ such that $G = \mathrm{Gal}(D/C)$ acts effectively on the polarized pair (E, L_E) and $X \cong (D \times E)/G$, where D is a smooth projective curve. Since $q(X) = g(C) + 1$, we have $g(E/G) = 1$. Hence G acts on E as translations. Therefore any element of G is of order $d = \deg L_E = 1$. So $X \cong D \times E \cong C \times F$, and $f\colon X \longrightarrow C$ is the natural projection by construction. ∎

LEMMA 1.5. *Let X be a smooth algebraic surface, C a smooth curve, $f\colon X \longrightarrow C$ a surjective morphism with connected fibers, and F a general fiber of f. Then $q(X) \leq g(C) + g(F)$. Moreover if this equality holds and $g(F) \geq 2$, then $X \sim_{\mathrm{bir}} F \times C$.*

PROOF. See *e.g.* [Be] p. 345 or [X]. ∎

LEMMA 1.6. *Let X be a minimal smooth surface of general type. Then $K_X^2 \geq 6q(X) - 13$ unless $X \cong C_1 \times C_2$ for some smooth curves C_1 and C_2.*

PROOF. We assume that $X \ncong C_1 \times C_2$ for smooth curves C_1 and C_2. By Théorème 6.3 in [D], we have $K_X^2 \geq 2p_g(X) + 2\big(q(X) - 4\big) + 1$. On the other hand, $p_g(X) \geq 2q(X) - 3$ by [Be]. Hence $K_X^2 \geq 6q(X) - 13$. ∎

PROPOSITION 1.7. *Let X be a minimal smooth surface of general type and let C be an irreducible reduced curve with $C^2 > 0$. Then $K_XC \geq (3/2)q(X) - 3$.*

PROOF. If $q(X) \leq 2$, then this inequality is true. So we assume $q(X) \geq 3$.

If $X \cong C_1 \times C_2$ for some smooth curves C_1 and C_2, then $K_XC \geq 2q(X) - 4 > (3/2)q(X) - 3$. So we may assume $X \ncong C_1 \times C_2$. Let $x \in \mathbb{Q}$ with $x \geq 1$. We put $m_x = g(xC) - q(X)$.

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CLAIM 1.7.1. *If* $2m_x + 2 \geq (2/3)\big(q(X) - 2\big) + 1$, *then* $(xC)^2 \leq 2m_x + 2$.

PROOF. Assume that $(xC)^2 > 2m_x + 2$. Then $(xC)^2 > (2/3)\big(q(X) - 2\big) + 1$. Hence

$$\begin{aligned}(K_X)^2(xC)^2 &> \Big(6\big(q(X)-2\big)-1\Big)\Big(\frac{2}{3}\big(q(X)-2\big)+1\Big)\\ &= 4\big(q(X)-2\big)^2 + 6\big(q(X)-2\big) - \frac{2}{3}\big(q(X)-2\big) - 1\\ &= 4\big(q(X)-2\big)^2 + \frac{16}{3}\big(q(X)-2\big) - 1\end{aligned}$$

by Lemma 1.6.

By Hodge index Theorem, we get $(xCK_X)^2 \geq (xC)^2(K_X)^2 > 4\big(q(X)-2\big)^2$ and we have $xCK_X > 2\big(q(X)-2\big)$. Therefore

$$\begin{aligned}g(xC) &> 1 + \frac{1}{2}\Big(2\big(q(X)-2\big) + 2m_x + 2\Big)\\ &= q(X) + m_x\end{aligned}$$

and this is a contradiction.

This completes the proof of Claim 1.7.1.

We continue the proof of Proposition 1.7.

We have

$$\begin{aligned}q(X) + m_x = g(xC) &= g(C) + (x-1)g(C) + \frac{x-1}{2}(xC^2 - 2)\\ &\geq q(X) + (x-1)q(X) + \frac{x-1}{2}(xC^2 - 2)\end{aligned}$$

since $g(C) \geq q(X)$.

Hence $m_x \geq (x-1)q(X) + \big((x-1)/2\big)(xC^2 - 2)$. Here we put $x = (4/3)$. Then $m_x \geq (1/3)q(X) - (1/9) > (1/3)q(X) - (7/6)$. Therefore by Claim 1.7.1, we have

$$\Big(\frac{4}{3}C\Big)^2 \leq 2m_x + 2.$$

In particular, $(4/3)CK_X \geq 2q(X) - 4$. Therefore $K_XC \geq (3/2)q(X) - 3$. This completes the proof of Proposition 1.7. ∎

LEMMA 1.8. *Let X be a minimal smooth surface of general type. Then there are only finitely many irreducible curves C on X up to numerical equivalence such that K_XC is bounded.*

Moreover there are only finitely many irreducible curves C on X such that K_XC is bounded and $C^2 < 0$.

PROOF. See Proposition 3 in [Bo].

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DEFINITION 1.9 (SEE *e.g.* [BABE], [BEFS], AND [BES]). Let X be a projective variety over $\mathbb{C}$ and let Z be a 0-dimensional subscheme of X. A 0-dimensional subscheme Z_1 of X is called a *subcycle* of Z if $\mathcal{I}_Z \subset \mathcal{I}_{Z_1}$, where $\mathcal{I}_Z$ (resp. $\mathcal{I}_{Z_1}$) is the ideal sheaf which defines Z (resp. Z_1). Let L be a Cartier divisor on X. Let W be a subspace of $H^0(L)$ and k a non-negative integer. Then W is called *k-very ample* if the restriction map $W \longrightarrow H^0(L \otimes \mathcal{O}_Z)$ is surjective for any 0-dimensional subscheme Z with length $\leq k+1$. If $W = H^0(L)$, then L is said to be *k-very ample*. (We remark that L is 0-very ample if and only if L is spanned and L is 1-very ample if and only if L is very ample.)

2. **The case in which** $\kappa(X) = 0$ **or** 1. In this section, we will prove conjecture 1′ for the case in which $\kappa(X) = 0$ or 1.

THEOREM 2.1. *Let* (X, L) *be a quasi-polarized surface with* $\kappa(X) = 0$ *or* 1. *Then* $K_XL \geq 2q(X) - 4$.

If this equality holds and (X, L) *is L-minimal, then* (X, L) *is one of the following;*

(1) $\kappa(X) = 0$ *case. X is an Abelian surface and L is any nef and big divisor.*

(2) $\kappa(X) = 1$ *case.* $X \cong F \times C$ *and* $L \equiv C + (m+1)F$, *where F and C are smooth curves with* $g(C) \geq 2$ *and* $g(F) = 1$, *and* $m = g(L) - q(X)$.

PROOF. (1) The case in which $\kappa(X) = 0$. Then $q(X) \leq 2$ by the classification theory of surfaces. Hence $K_XL \geq 0 \geq 2q(X) - 4$.

If $K_XL = 2q(X) - 4$, then $q(X) = 2$ and $K_XL = 0$. Since (X, L) is L-minimal, we get that X is minimal, in particular, X is an Abelian surface. Conversely, let (X, L) be any quasi-polarized surface which is L-minimal, and let X be an Abelian surface. Then $K_XL = 0 = 2q(X) - 4$.

(2) The case in which $\kappa(X) = 1$. Let $f: X \longrightarrow C$ be an elliptic fibration, $\mu: X \longrightarrow X'$ the relatively minimal model of X, and let $f': X' \longrightarrow C$ be the relatively minimal elliptic fibration such that $f = f' \circ \mu$. Let $L' = \mu_* L$. Then L' is nef and big, and $K_XL \geq K_{X'}L'$.

By the canonical bundle formula for elliptic fibrations, we have

$$K_{X'} \equiv \big(2g(C) - 2 + \chi(\mathcal{O}_{X'})\big)F' + \sum_i (m_i - 1)F_i,$$

where F' is a general fiber of f' and m_iF_i is a multiple fiber of f' for any i.

Hence

$$\begin{aligned} K_{X'}L' &\geq \big(2g(C) - 2 + \chi(\mathcal{O}_{X'})\big) \geq 2g(C) - 2 \\ &= 2\big(g(C) + 1\big) - 4 \\ &\geq 2q(X) - 4. \end{aligned}$$

Therefore $K_XL \geq K_{X'}L' \geq 2q(X) - 4$.

Assume that $K_XL = 2q(X) - 4$.

Since $\kappa(X) = 1$, we get $K_XL > 0$. Hence $q(X) \geq 3$ and $g(C) \geq 2$. By the above argument, we obtain $K_XL = K_{X'}L' = 2q(X) - 4$. Since (X, L) is L-minimal, we obtain that X is minimal. Because $K_XL = 2q(X) - 4$ and $2g(C) - 2 + \chi(\mathcal{O}_{X'}) > 0$, we obtain the following.

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(2-1) f has no multiple fibers.
(2-2) $\chi(\mathcal{O}_X) = 0$.
(2-3) $q(X) = g(C) + 1$.
(2-4) $LF = 1$.

By (2-3), (2-4), and Lemma 1.4, we obtain $X \cong F \times C$ and $f: X \longrightarrow C$ is the natural projection. Because of $\kappa(X) = 1$, we have $g(C) \geq 2$. Then $f^* \circ f_*\big(\mathcal{O}(L)\big) \longrightarrow \mathcal{O}(L - Z)$ is surjective, where Z is a section of f. Let $L|_{F_t} \sim p_t$, where $F_t = f^{-1}(t)$ and $t \in C$. Let (y, t) be a point of $F \times C$ and $\big(y(p_t), t\big)$ the point $p_t \in F \times C$. Then the morphism $h: F \times C \longrightarrow F \times C; h(y, t) = \big(y - y(p_t), t\big)$ is an isomorphism. Hence $L = h^*(\{0\} \times C) + f^*D$. Therefore $L = C + f^*D$ via h, where $D \in \mathrm{Pic}(C)$. But $L^2 = 2m + 2$ for $m = g(L) - q(X)$. Hence $L \equiv C + (m + 1)F$. This completes the proof of Theorem 2.1. ∎

3. The case in which $\kappa(X) = 2$ and $h^0(L) \geq 2$.

THEOREM 3.1. *Let (X, L) be a quasi-polarized surface with $\kappa(X) = 2$ and $h^0(L) \geq 2$. Then $K_X L \geq 2q(X) - 2$.*

If this equality holds and (X, L) is L-minimal, then (X, L) is the following; $X \cong F \times C$ and $L \equiv C + 2F$, where F and C are smooth curves with $g(F) = 2$ and $g(C) \geq 2$.

PROOF. (A) The case in which X is minimal. Then we use Theorem 1.3.

(A-1) The case in which $g(L) \geq 2q(X) - 1$. Then $q(X) + m = g(L) \geq 2q(X) - 1$. So we obtain $m \geq q(X) - 1$.

(A-1-1) The case where $q(X) \geq 1$. Then by Lemma 1.2, we obtain $K_X^2 \geq 2p_g(X) \geq 2q(X)$. If $L^2 \geq 2m$, then

$$\begin{aligned}(K_X L)^2 &\geq K_X^2 L^2 \geq \big(2q(X)\big)(2m)\\ &\geq 4q(X)\big(q(X) - 1\big).\end{aligned}$$

Hence $K_X L > 2\big(q(X) - 1\big)$. But this is impossible because

$$\begin{aligned}q(X) + m = g(L) &> 1 + \frac{1}{2}\big(2q(X) - 2 + 2m\big)\\ &= q(X) + m.\end{aligned}$$

Therefore $L^2 \leq 2m - 1$, that is, $K_X L \geq 2q(X) - 1$.

(A-1-2) The case where $q(X) = 0$. Then $K_X L > 0 > 2q(X) - 2$.

(A-2) The case in which $g(L) < 2q(X) - 1$. Then by Theorem 1.3, there is a fiber space $f: X \longrightarrow C$ such that C is a smooth curve with $g(C) \geq 2$, $LF = 1$, and $L - aF$ is numerically equivalent to an effective divisor, where F is a general fiber of f and $a \geq 2$. So there exists a section C' of f such that C' is an irreducible component of L, and we obtain that $L - aF \equiv C' + D'$, where D' is an effective divisor such that $f(D')$ are points.

Since f is relatively minimal, the relative canonical divisor $K_{X/C} = K_X - f^*K_C$ is nef by Arakelov's Theorem. So we have $K_{X/C}L \geq 2K_{X/C}F$. Hence

$$\begin{aligned} g(L) &= g(C) + \frac{1}{2}(K_{X/C}L) + \frac{1}{2}L^2 \\ &\geq g(C) + K_{X/C}F + \frac{1}{2}L^2 \\ &= g(C) + 2g(F) - 2 + \frac{1}{2}L^2 \\ &= g(C) + g(F) + g(F) - 2 + \frac{1}{2}L^2 \\ &\geq q(X) + \frac{1}{2}L^2 \end{aligned}$$

because $g(F) \geq 2$ and $g(C) + g(F) \geq q(X)$.

Since $q(X) + m = g(L)$, we obtain $L^2 \leq 2m$. Namely $K_XL \geq 2q(X) - 2$.

Next we assume $K_XL = 2q(X) - 2$.

Then $g(L) < 2q(X) - 1$ by the above argument. Moreover the following are satisfied by the above argument of (A-2);

(a) $K_{X/C}C' = 0$, $K_{X/C}D' = 0$.

(b) $a = 2$.

(c) $g(F) = 2$.

(d) $q(X) = g(C) + g(F)$.

Since X is minimal, we obtain $X \cong F \times C$ by (d) and Lemma 1.5. Moreover $f\colon X \longrightarrow C$ is the natural projection. Since D' is contained in fibers of f and $K_{X/C}D' = 0$, we obtain $D' = 0$. Since $K_{X/C} \equiv \big(2g(F) - 2\big)C$ and $K_{X/C}C' = 0$, we have $CC' = 0$. Hence C' is a fiber of $F \times C \longrightarrow F$. Therefore $L \equiv C + 2F$ by (b).

(B) The case in which X is not minimal.

Let $X = X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_{n-1} \longrightarrow X_n$ be the minimalization of X, where $\mu_i\colon X_i \longrightarrow X_{i+1}$ is the blowing down of (-1)-curve E_i. Let $L_i = (\mu_{i-1})_*(L_{i-1})$, $L_0 = L$, and $L_{i-1} = (\mu_{i-1})^*L_i - m_{i-1}E_{i-1}$, where $m_{i-1} \geq 0$. We remark that $h^0(L_i) = h^0\big((\mu_{i-1})^*L_i\big) \geq h^0(L_{i-1})$. Then $L^2 = (L_n)^2 - \sum_{i=0}^{n-1} m_i^2$ and $K_XL = K_{X_n}L_n + \sum_{i=0}^{n-1} m_i$. By the case (A), we have $K_{X_n}L_n \geq 2q(X) - 2$. Hence $K_XL \geq 2q(X) - 2 + \sum_{i=0}^{n-1} m_i \geq 2q(X) - 2$.

Next we consider (X, L) such that $K_XL = 2q(X) - 2$ and (X, L) is L-minimal, Then $\sum_{i=0}^{n-1} m_i = 0$ since $K_XL = 2q(X) - 2$ and so we have $m_i = 0$ for any i. But then X is minimal because (X, L) is L-minimal. This is a contradiction. This completes the proof of Theorem 3.1. ∎

4. The case in which $\kappa(X) = 2$ and $h^0(L) = 1$.

In this section, we consider the case in which $\kappa(X) = 2$ and $h^0(L) = 1$. We put $m = g(L) - q(X)$.

LEMMA 4.1. *If $g(L) \geq 2q(X)$, then $K_XL \geq 2q(X) - 1$.*

PROOF. Then $q(X) + m = g(L) \geq 2q(X)$. Hence $m \geq q(X)$. Assume that $L^2 \geq 2m$. So we obtain $L^2 \geq 2q(X)$. Let $\mu\colon X \longrightarrow X'$ be the minimalization of X and $L' = \mu_*L$.

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Then $K_X L \geq K_{X'}L'$ and $(L')^2 \geq L^2 \geq 2q(X)$. Since $K_{X'}^2 \geq 2q(X)$ by Lemma 1.2, we have $(K_{X'}L')^2 \geq (K_{X'})^2(L')^2 \geq \big(2q(X)\big)^2$ by Hodge index Theorem. So we obtain $K_{X'}L' \geq 2q(X)$. But this is impossible because

$$q(X) + m = g(L) \geq 1 + q(X) + m.$$

Hence $L^2 < 2m$, that is, $K_X L \geq 2q(X) - 1$. This completes the proof of Lemma 4.1. ∎

LEMMA 4.2. *If for any minimal quasi-polarized surfaces (X, L) with $\kappa(X) = 2$ and $h^0(L) \geq 1$ we can prove that $K_X L \geq 2q(X) - 4$, then this inequality holds for any quasi-polarized surface (Y, A) with $\kappa(Y) = 2$ and $h^0(A) \geq 1$.*

PROOF. It is easy. ∎

By Lemma 4.2, it is sufficient to prove Conjecture 1 (or Conjecture 1′) under the following assumption (4-1);

(4-1) X is minimal.

Here we consider Conjecture 1 (or Conjecture 1′) for the following divisors.

DEFINITION 4.3. Let X be a smooth projective surface and let D be an effective divisor on X. Then D is called a *CNNS-divisor* if the following conditions hold:

(1) D is connected.

(2) the intersection matrix $\|(C_i, C_j)\|_{i,j}$ of $D = \sum_i r_i C_i$ is not negative semidefinite.

REMARK 4.4. If L is an effective nef and big divisor, then L is a CNNS-divisor.

Let D be a CNNS-divisor on a minimal smooth projective surface X with $\kappa(X) = 2$, and $D = \sum_i r_i C_i$ its prime decomposition.

We divide three cases:

(α) $\sum_{i\in S} r_i \geq 2$;

(β) $\sum_{i\in S} r_i = 1$;

(γ) $\sum_{i\in S} r_i = 0$,

where $S = \{i \mid C_i^2 > 0\}$.

First we consider the case (α).

THEOREM 4.5. *Let D be a* CNNS-*divisor on a minimal smooth surface X with $\kappa(X) = 2$, and let $D = \sum_i r_i C_i$ be its prime decomposition. If $\sum_{i\in S} r_i \geq 2$, then $K_X D \geq 2q(X) - 1$.*

PROOF. Let $A = \sum_{i\in S} r_i C_i$ and $B = D - A$. Then A is nef and big. We remark that $K_X D \geq K_X A$ since X is minimal with $\kappa(X) = 2$. So it is sufficient to prove that $g(A) \geq 2q(X)$ by Lemma 4.1. By assumption here, there are curves C_1 and C_2 (possibly $C_1 = C_2$) such that $C_1^2 > 0$ and $C_2^2 > 0$ and $A - C_1 - C_2$ is effective. Let $A_{12} = A - C_1 - C_2$. Then

$$g(A) = g(C_1 + C_2) + \frac{1}{2}(K_X + A + C_1 + C_2)A_{12}.$$

Since $K_X + A$ is nef and A is 1-connected, we have $(K_X + A)A_{12} \geq 0$ and $(C_1 + C_2)A_{12} \geq 0$. Hence $g(A) \geq g(C_1 + C_2)$. On the other hand, $g(C_1 + C_2) = g(C_1) + g(C_2) + C_1 C_2 - 1$.

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Because $C_1^2 > 0$ and $C_2^2 > 0$, we obtain $C_1C_2 > 0$. Hence $g(C_1+C_2) \geq g(C_1)+g(C_2) \geq 2q(X)$. Therefore by Lemma 4.1, we obtain $K_X(C_1+C_2) \geq 2q(X)-1$. So we have $K_XD \geq K_X(C_1+C_2) \geq 2q(X)-1$. This completes the proof of Theorem 4.5. ∎

Next we consider the case (γ).

THEOREM 4.6. *Let D be a* CNNS-*divisor on a minimal smooth projective surface X with $\kappa(X)=2$ and let $D=\sum_i r_iC_i$ be its prime decomposition. If $\sum_{i\in S} r_i = 0$ and there exists a curve C_i such that $C_i^2=0$, then $K_XD \geq 2q(X)-4$.*

PROOF. Assume that $C_1^2=0$. We may assume that $q(X)\geq 1$. Since D is a CNNS-divisor, D has at least two irreducible components. Let C_2 be another irreducible component of D such that $C_1\cap C_2 \neq \phi$. Then

$$g(D) = g(C_1+C_2) + \frac{1}{2}(K_X+D+C_1+C_2)D_{12},$$

where $D_{12} = D-(C_1+C_2)$.

We put $l = g(C_1+C_2)-q(X)$ and $m = g(D)-q(X)$. Since $K_XD_{12} \geq 0$, we have $2m-2l \geq (D+C_1+C_2)D_{12}$. Let $X_0 = X$, $C_{1,0}=C_1$, $C_{2,0}=C_2$, and $\mu_i\colon X_i \longrightarrow X_{i-1}$ blowing up at a point of $C_{1,i-1}\cap C_{2,i-1}$, where $C_{1,i}$ (resp. $C_{2,i}$) is the strict transform of $C_{1,i-1}$ (resp. $C_{2,i-1}$), and let E_i be an exceptional divisor such that $\mu_i(E_i)$ is a point. We put $\mu = \mu_1\circ\cdots\circ\mu_n$, where n is the natural number such that $C_{1,n-1}\cap C_{2,n-1}\neq\phi$ and $C_{1,n}\cap C_{2,n}=\phi$. Let $C_{1,i} = \mu_i^*C_{1,i-1}-b_iE_i$, $C_{2,i}=\mu_i^*C_{2,i-1}-d_iE_i$, and $a_i = b_i+d_i$. We remark that $b_i\geq 1$ and $d_i \geq 1$. Let $X'_0 = X_n$, $C'_{1,0}=C_{1,n}$, $C'_{2,0}=C_{2,n}$, $E'_{0,0}=E_n$, and $\mu'_i\colon X'_i \longrightarrow X'_{i-1}$ blowing up at a point $x\in \big(\mathrm{Sing}(C'_{1,i-1})\cup \mathrm{Sing}(C'_{2,i-1})\big)\setminus \big((C'_{1,i-1}\cap E'_{0,i-1})\cup(C'_{2,i-1}\cap E'_{0,i-1})\big)$, where $C'_{1,i}$ (resp. $C'_{2,i}$, $E'_{0,i}$) is the strict transform of $C'_{1,i-1}$ (resp. $C'_{2,i-1}$, $E'_{0,i-1}$), and let E'_i be an exceptional divisor on X'_i such that $\mu'_i(E'_i)$ is a point. Let $C'_{1,i}+C'_{2,i} = (\mu'_i)^*(C'_{1,i-1}+C'_{2,i-1})-a'_iE'_i$. We assume that $\big(\mathrm{Sing}(C'_{1,t})\cup\mathrm{Sing}(C'_{2,t})\big)\setminus\big((C'_{1,t}\cap E'_{0,t})\cup(C'_{2,t}\cap E'_{0,t})\big)=\phi$.

CLAIM 4.7. $g(C'_{1,t}+C'_{2,t}+E'_{0,t}) \geq q(X'_t)$.

PROOF. Let $\alpha(C'_{1,t}+C'_{2,t}+E'_{0,t}) = \dim\mathrm{Ker}\big(H^1(\mathcal{O}_{X'_t}) \longrightarrow H^1(\mathcal{O}_{C'_{1,t}+C'_{2,t}+E'_{0,t}})\big)$. By Lemma 3.1 in [Fk4], it is sufficient to prove $\alpha(C'_{1,t}+C'_{2,t}+E'_{0,t})=0$ since $C'_{1,t}+C'_{2,t}+E'_{0,t}$ is 1-connected. Assume that $\alpha(C'_{1,t}+C'_{2,t}+E'_{0,t})\neq 0$. Since $q(X)\geq 1$, there is a morphism $f\colon X'_t \longrightarrow G$ such that $f(X)$ is not a point and $f(C'_{1,t}+C'_{2,t}+E'_{0,t})$ is a point, where G is an Abelian variety. On the other hand, a $(\mu_1\circ\cdots\circ\mu_n\circ\mu'_1\cdots\circ\mu'_t)$-exceptional divisor is contracted by f because G is an Abelian variety. Therefore $(\mu')^*(C_1+C_2)$ is contracted by f. But $(e_1C_1+C_2)^2>0$ for sufficient large e_1. This is impossible. Hence $\alpha(C'_{1,t}+C'_{2,t}+E'_{0,t})=0$. This completes the proof of Claim 4.7.

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Hence

$$\begin{aligned} g(C_{1,n}+C_{2,n}+E_n) &= g(C'_{1,t}+C'_{2,t}+E'_{0,t})+\sum_{i=1}^{t}\frac{1}{2}a'_i(a'_i-1) \\ &\geq q(X'_t)+\sum_{i=1}^{t}\frac{1}{2}a'_i(a'_i-1) \\ &= q(X)+\sum_{i=1}^{t}\frac{1}{2}a'_i(a'_i-1). \end{aligned}$$

On the other hand,

$$g(C_1+C_2)=g(C_{1,n}+C_{2,n}+E_n)+\sum_{i=1}^{n-1}\frac{1}{2}a_i(a_i-1)+\frac{1}{2}(a_n-1)(a_n-2).$$

Therefore

$$g(C_1+C_2)\geq q(X)+\sum_{i=1}^{n-1}\frac{1}{2}a_i(a_i-1)+\frac{1}{2}(a_n-1)(a_n-2)+\sum_{k=1}^{t}\frac{1}{2}a'_k(a'_k-1).$$

Since $l=g(C_1+C_2)-q(X)$, we obtain

$$2l\geq\sum_{i=1}^{n-1}a_i(a_i-1)+(a_n-1)(a_n-2)+\sum_{k=1}^{t}a'_k(a'_k-1).$$

Let $C_1C_2=x$. Then $x=\sum_{i=1}^{n}b_id_i$ and $(C_1+C_2)^2\leq 2x$ by hypothesis.

CLAIM 4.8.

$$2x-\sum_{i=1}^{n-1}a_i(a_i-1)-(a_n-1)(a_n-2)\leq 2.$$

PROOF.

$$\begin{aligned} &2x-\sum_{i=1}^{n-1}a_i(a_i-1)-(a_n-1)(a_n-2) \\ &\qquad= 2\sum_{i=1}^{n}b_id_i-\sum_{i=1}^{n-1}(b_i+d_i)(b_i+d_i-1)-(b_n+d_n-1)(b_n+d_n-2). \end{aligned}$$

For each $i(\neq n)$,

$$\begin{aligned} 2b_id_i-(b_i+d_i)(b_i+d_i-1) &= -b_i^2-d_i^2+b_i+d_i \\ &= b_i(1-b_i)+d_i(1-d_i)\leq 0, \end{aligned}$$

and for $i=n$,

$$\begin{aligned} 2b_nd_n-(b_n+d_n-1)(b_n+d_n-2) &= -b_n^2-d_n^2+3b_n+3d_n-2 \\ &= b_n(3-b_n)+d_n(3-d_n)-2\leq 2. \end{aligned}$$

Therefore we obtain Claim 4.8.

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By Claim 4.8, we obtain

$$\begin{aligned}D^2 &= (C_1+C_2)^2+(D+C_1+C_2)D_{12}\\ &\le 2x+2m-2l\\ &\le 2x+2m-\sum_{i=1}^{n-1}a_i(a_i-1)-(a_n-1)(a_n-2)-\sum_{k=1}^{t}a'_k(a'_k-1)\\ &\le 2m+2-\sum_{k=1}^{t}a'_k(a'_k-1)\\ &\le 2m+2.\end{aligned}$$

Therefore $K_XD \ge 2q(X)-4$. This completes the proof of Theorem 4.6. ∎

Next we consider the case in which the equality in Theorem 4.6 holds.

THEOREM 4.9. *Let D be a* CNNS-*divisor on a minimal smooth surface X with $\kappa(X)=2$, and let $D=\sum_i r_iC_i$ be its prime decomposition. Assume that $\sum_{i\in S} r_i = 0$, there exists a curve C_i such that $C_i^2=0$, and $K_XD=2q(X)-4$. Then there are two irreducible curves C_1 and C_2 such that $D=C_1+C_2$ with $C_1^2=C_2^2=0$.*

Moreover if C_1 or C_2 is not smooth, then $g(D)-q(X)=1$ or 3, and $\sharp(C_1\cap C_2)=1$.

(1) If $g(D)-q(X)=1$, then C_i is smooth but C_j is not smooth only at $x\in C_1\cap C_2$ and $\mathrm{mult}_x(C_j)=2$ *for $i\ne j$ and $\{i,j\}=\{1,2\}$, where* $\mathrm{mult}_x(C_j)$ *is the multiplicity of C_j at x.*

(2) If $g(D)-q(X)=3$, then C_1 and C_2 are not smooth only at $x\in C_1\cap C_2$ and $\mathrm{mult}_x(C_i)=2$ *for $i=1,2$.*

PROOF. Let $D=C_1+C_2+D_{12}$, where $C_1^2=0$ and C_2 is an irreducible curve such that $C_1C_2>0$. By the proof of Theorem 4.6, we have $K_XD_{12}=0$. If $D_{12}\ne 0$, then $K_XC=0$ for any irreducible curve C of D_{12} because K_X is nef.

CLAIM 4.10. *$C^2=0$ for any irreducible curve C of D.*

PROOF. By hypothesis, there is an irreducible curve B of D such that $B^2=0$. Let B' be any irreducible curve of D such that $B\ne B'$ and $BB'>0$. By the proof of Theorem 4.6 and the assumption that $K_XD=2q(X)-4$, we have $(B')^2=0$. By repeating this argument, this completes the proof because D is connected.

By this Claim, $C^2=0$ for any irreducible curve C of D_{12}. So $C\equiv 0$ by Hodge index Theorem. But this is a contradiction.

Therefore $D_{12}=0$ and so we have $D=C_1+C_2$ with $C_1^2=C_2^2=0$. Next we consider the singularity of C_1 and C_2.

We remark that C_1 (resp. C_2) is smooth on $C_1\setminus\{C_1\cap C_2\}$ (resp. $C_2\setminus\{C_1\cap C_2\}$) since $K_XD=2q(X)-4$ and $\sum_{k=1}^{t}a'_k(a'_k-1)=0$ (here we use the notation in Theorem 4.6).

We assume that $\sharp C_1\cap C_2\ge 2$. Then the number n of blowing up $\mu=\mu_1\circ\cdots\circ\mu_n$ is greater than 1. Since $K_XD=2q(X)-4$, we obtain $b_1=d_1=1$. By interchanging the point of the first blowing up, we obtain that C_1 and C_2 are smooth on $C_1\cap C_2$.

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We assume $\sharp C_1 \cap C_2 = 1$. If the number n of blowing up μ is greater than 1, then $b_1 = d_1 = 1$ by the proof of Theorem 4.6. So C_1 and C_2 are smooth at $x \in C_1 \cap C_2$. Hence we assume that the number of blowing up is one. Then $C_1C_2 = b_1d_1$. By the proof of Theorem 4.6, $b_1(3 - b_1) + d_1(3 - d_1) = 4$. Hence $(b_1, d_1) = (1, 1), (1, 2), (2, 1)$, or $(2, 2)$.

If $(b_1, d_1) = (1, 1)$, then C_1 and C_2 are smooth at x.

If $(b_1, d_1) = (1, 2)$ or $(2, 1)$, then C_i is smooth at x and C_j is not smooth at x for $i \neq j$ and $\{i, j\} = \{1, 2\}$, and $\mathrm{mult}_x(C_j) = 2$, where $\mathrm{mult}_x(C_j)$ is the multiplicity of C_j at x. In this case, $C_1C_2 = 2$ and $g(D) - q(X) = 1$.

If $(b_1, d_1) = (2, 2)$, then C_1 and C_2 are not smooth at x, and $\mathrm{mult}_x(C_i) = 2$ for $i = 1, 2$. In this case, $C_1C_2 = 4$ and $g(D) - q(X) = 3$. This completes the proof of Theorem 4.9. ∎

Next we consider the following case $(*)$:

$(*)$ Let D be a CNNS-divisor on a minimal surface of general type, and let $D = \sum_i r_iC_i$ be its prime decomposition. Then we assume $C_i^2 < 0$ for any i.

THEOREM 4.11. *Let (X,D) be $(*)$. Then $K_XD \geq 2q(X) - 3$.*

Before we prove this theorem, we state some definitions and notations which is used in the proof of Theorem 4.11.

DEFINITION 4.12. Let D be an effective divisor on X. Then the dual graph $G(D)$ of D is defined as follows.

(1) The vertices of $G(D)$ corresponds to irreducible components of D.

(2) For any two vertices v_1 and v_2 of $G(D)$, the number of edges joining v_1 and v_2 equal $\sharp\{B_1 \cap B_2\}$, where B_i is the component of D corresponding to v_i for $i = 1, 2$.

REMARK 4.12.1. Let $G(D)$ be the dual graph of an effective divisor D. We reject one edge e of $G(D)$ and $G = G(D) - \{e\}$. Let v_1 and v_2 be vertices of $G(D)$ which are terminal points of the edge e. Let C_1 and C_2 be the irreducible curve of D corresponding v_1 and v_2 respectively. Then G is the dual graph of the effective divisor which is the strict transform of D by the blowing up at a point x corresponding to e if $i(C_1, C_2; x) = 1$, where $i(C_i, C_j; x)$ is the intersection number of C_i and C_j at x.

NOTATION 4.13. Let (X, D) be $(*)$. We take a birational morphism $\mu'\colon X' \longrightarrow X$ such that $C_i' \cap C_j' \cap C_k' = \phi$ for any distinct C_i', C_j', and C_k', and if $C_i' \cap C_j' \neq \phi$, then $i(C_i', C_j'; x) = 1$ for $x \in C_i' \cap C_j'$, where $D' = (\mu')^*(D) = \sum_i r_i'C_i'$. Let $\mu_i\colon X_i \longrightarrow X_{i-1}$ be one point blowing up such that $\mu' = \mu_1 \circ \cdots \circ \mu_t$, $X_0 = X$ and $X_t = X'$. Let $D_i = \mu_i^*D_{i-1}$ and $D_0 = D$. Let b_i be an integer such that $(\mu_i)^*\big((D_{i-1})_{\mathrm{red}}\big) - b_iE_i = (D_i)_{\mathrm{red}}$, where E_i is a μ_i-exceptional curve.

REMARK 4.14. (a) No two $(\mu_1 \circ \cdots \circ \mu_i)$-exceptional curves on X_i which are not (-1) curve intersect at a point on (-1)-curve on X_i contracted by some μ_j $(j \leq i)$.

(b) The point x which is a center of blowing up $\mu_i\colon X_i \longrightarrow X_{i-1}$ is contained in one of the following types;

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(1) the strict transform of the irreducible components of D;
(2) the intersection of the strict transform of the irreducible components of D and one (-1)-curve on X_i contracted by some μ_j $(j \leq i)$;
(3) the intersection of the strict transform of the irreducible components of D and one $(\mu_1 \circ \cdots \circ \mu_i)$-exceptional curve on X_i which is not (-1)-curve and one (-1)-curve on X_i contracted by some μ_j $(j \leq i)$.

We assume that (X, D) satisfies $(*)$ and we use Notation 4.13 unless specifically stated otherwise.

DEFINITION 4.15. (1) Let $\pi: \tilde{X} \longrightarrow X$ be a birational morphism, and let $\tilde{X}$ and X be smooth surfaces. Let $\pi = \pi_1 \circ \cdots \circ \pi_n$, $X_0 = X$, and $X_n = \tilde{X}$, where $\pi_i: X_i \longrightarrow X_{i-1}$ is one point blowing up. Let E_i be the exceptional divisor of π_i. Let D be an effective divisor on X and we put $D_0 = D$. Let $D_i = \pi^*(D_{i-1})$. Then the multiplicity of E_i in D_i is called the *E_i-multiplicity* of D.

(2) We use Notation 4.13. Let $x_i = \mu_i(E_i)$. If x_i is the type (3) in Remark 4.14(b), then the $(\mu_1 \circ \cdots \circ \mu_i)$-exceptional curve which is not (-1)-curve is said to be an *e-curve*, and x_i is said to be an *e-point*.

We remark that there is at most one e-curve throughout x_i.

REMARK 4.16. We consider Notation 4.13. Let E an e-curve on X_i and let x_i be the e-point associated with E. Then we must be blowing up at x_i by considering Notation 4.13. Let $\tilde{E}$ be a strict transform of E by blowing up $\mu_{i+1}: X_{i+1} \longrightarrow X_i$ at x_i. Then $(\tilde{E})^2 = E^2 - 1 \leq -3$ and $K_{X_{i+1}}\tilde{E} = K_{X_i}E + 1 \geq 1$.

DEFINITION 4.17. Let $\delta: \tilde{X} \longrightarrow X$ be any birational morphism, $\tilde{E}$ a union of δ-exceptional curve, and let D be an effective divisor on X. We put $B = \delta(\tilde{E}) = \{y_1, \ldots, y_s\}$. Then we can describe δ as $\delta = \delta_s \circ \cdots \circ \delta_1$, where δ_i is the map whose image of a union of δ_i-exceptional curves is y_i. For each $y_k \in B$, we define a new graph $G = G(y_k, D)$ which is called the *river* of the birational map δ_k and D.

(STEP 1). Let $E_{0,0}$ be a (-1)-curve obtained by blowing up at y_k. Let $v_{0,0}$ be a vertex of the graph G which corresponds to $E_{0,0}$. We define the weight $u(0, 0; G)$ of $v_{0,0}$ as follows:

$$u(0, 0; G) = \text{the } E_{0,0}\text{-multiplicity of } D.$$

(STEP 2). Let $E_{1,1}, \ldots, E_{1,t}$ be (-1)-curves obtained by blowing up at distinct points $\{x_{1,1}, \ldots, x_{1,t}\}$ on $E_{0,0}$. Let $v_{1,1}, \ldots, v_{1,t}$ be vertices of the graph G which correspond to $E_{1,1}, \ldots, E_{1,t}$ respectively. We join $v_{1,j}$ and $v_{0,0}$ by directed line which goes from $v_{1,j}$ to $v_{0,0}$. For $j = 1, \ldots, t$, we define the weight $u(1, j; G)$ of $v_{1,j}$ as follows:

$$u(1, j; G) = e_{1,j} - u(0, 0; G),$$

where $e_{1,j} =$ the $E_{1,j}$-multiplicity of D.

(STEP 3). In general, let $E_{i,1}, \ldots, E_{i,t_i}$ be disjoint (-1)-curves obtained by blowing up at distinct points $\{x_{i,1}, \ldots, x_{i,t_i}\}$ on $\bigcup_k E_{i-1,k}$. Let $v_{i,1}, \ldots, v_{i,t_i}$ be vertices of the graph

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G which correspond to $E_{i,1}, \ldots, E_{i,t_i}$ respectively. We join $v_{i,j}$ and $v_{i-1,k}$ by directed line which goes from $v_{i,j}$ to $v_{i-1,k}$ if $E_{i,j}$ is contracted in $E_{i-1,k}$. Let $e_{i,j}$ = the $E_{i,j}$-multiplicity of D for $j = 1, \ldots, t_i$. Then we define the weight $u(i,j;G)$ of $v_{i,j}$ as follows:

$$u(i,j;G) = e_{i,j} - \sum_{v_{p,q} \in SP(i,j;G)} u(p,q;G),$$

where $P(i,j;G)$ denotes the path between $v_{0,0}$ and $v_{i,j}$, and $SP(i,j;G) = P(i,j;G) - \{v_{i,j}\}$.

By the above steps, we obtain the graph G for each y_k.

NOTATION 4.18.

$$w(i,j;G) = \begin{cases} \deg(v_{i,j}) - 1, & \text{if } v_{i,j} \neq v_{0,0}, \\ \deg(v_{0,0}). \end{cases}$$

LEMMA 4.19. *Let $\mu: Y \longrightarrow X$ be a birational morphism between smooth surfaces X and Y, and let D be an effective divisor on X. Let $D' = \mu^* D$, and E a union of all μ-exceptional curves.*

Let $B = \mu(E)$ and $M(D')$ = sum of the multiplicity of (-1)-curves on Y in D'. Then

$$M(D') = \sum_{y \in B} \Big[\sum_{v_{i,j} \in G(y)} \Big\{ \sum_{v_{p,q} \in P\left(i,j;G(y)\right)} u\big(p,q;G(y)\big) \Big\} \theta\big(i,j;G(y)\big) \Big] + \sum_{y \in B} \Big\{ \sum_{v_{i,j} \in G(y)} u\big(i,j;G(y)\big) \Big\},$$

where $G(y) = G(y,D)$ and

$$\theta\big(i,j;G(y)\big) = \begin{cases} w\big(i,j;G(y)\big) - 1 & \text{if } w\big(i,j;G(y)\big) \geq 1, \\ 0 & \text{if } w\big(i,j;G(y)\big) = 0. \end{cases}$$

PROOF. We may assume that $B = \{y\}$. Let $G = G(y,D)$. Let $A = \{v_{i,j} \in G \mid \deg(v_{i,j}) = 1, v_{i,j} \neq v_{0,0}\}$ and $\rho = \sharp A - \deg(v_{0,0})$.

If $A = \phi$, then $M(D') = u(0,0;G)$.

So we assume $A \neq \phi$. We prove this lemma by induction on the value of ρ. We remark that by construction the following fact holds;

FACT. *For any $v_{s,t} \in A$, the multiplicity of the (-1)-curve corresponding to $v_{s,t}$ is equal to $\sum_{v_{i,j} \in P(s,t;G)} u(i,j;G)$.*

(1) The case in which $\rho = 0$.

Then $\deg v = 2$ for any $v \notin A$ and $v \neq v_{0,0}$. Hence

$$\begin{aligned} M(D') &= \sum_{v_{i,j} \in G} u(i,j;G) + u(0,0;G)\big(\deg(v_{0,0}) - 1\big) \\ &= \sum_{v_{i,j} \in G} u(i,j;G) + \sum_{v_{i,j} \in G} \Big\{ \sum_{v_{p,q} \in P(i,j;G)} u(p,q;G) \Big\} \theta(i,j;G). \end{aligned}$$

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(2) The case in which $\rho = k > 0$.

We assume that this lemma is true for $\rho \leq k-1$. We take a vertex $v_{s,t} \in A$ such that there is no edge whose terminal points are $v_{0,0}$ and $v_{s,t}$. Let $G^\vee = G - \{v_{s,t}\}$. Let $\mu^-\colon Y \longrightarrow X^-$ be blowing down of (-1)-curves corresponding to $v_{s,t}$ and $\mu = \mu^+ \circ \mu^-$. Let $D^\vee = (\mu^+)^*(D)$. Then we remark that $G^\vee$ is the river of μ^+ and D.

Then by induction hypothesis

$$M(D^\vee) = \sum_{v_{i,j} \in G^\vee} u(i,j;G^\vee) + \sum_{v_{i,j}\in G^\vee} \Big\{ \sum_{v_{p,q} \in P(i,j;G^\vee)} u(p,q;G^\vee) \Big\} \theta(i,j;G^\vee).$$

Next we consider $M(D')$. Let $v_{s-1,l}$ be a vertex such that there is an edge between $v_{s-1,l}$ and $v_{s,t}$.

(2-1) The case in which $w(s-1,l;G) = 1$.

Then $M(D') = M(D^\vee) + u(s,t;G)$. Hence

$$\begin{aligned} M(D') &= \sum_{v_{i,j} \in G^\vee} u(i,j;G^\vee) + u(s,t;G) + \sum_{v_{i,j}\in G^\vee} \Big\{ \sum_{v_{p,q} \in P(i,j;G^\vee)} u(p,q;G^\vee) \Big\} \theta(i,j;G^\vee) \\ &= \sum_{v_{i,j} \in G} u(i,j;G) + \sum_{v_{i,j}\in G} \Big\{ \sum_{v_{p,q} \in P(i,j;G)} u(p,q;G) \Big\} \theta(i,j;G), \end{aligned}$$

because $\theta(s-1,l;G) = \theta(s,t;G) = 0$ and we have $u(i,j;G) = u(i,j;G^\vee)$, $w(i,j;G) = w(i,j;G^\vee)$, and $\theta(i,j;G) = \theta(i,j;G^\vee)$ for $v_{i,j} \neq v_{s,t}$.

(2-2) The case in which $w(s-1,l;G) \geq 2$.

Then

$$M(D') = M(D^\vee) + \sum_{v_{p,q} \in SP(s,t;G)} u(p,q;G) + u(s,t;G).$$

Hence

$$\begin{aligned} M(D') &= \sum_{v_{i,j} \in G^\vee} u(i,j;G^\vee) + u(s,t;G) + \sum_{v_{i,j}\in G^\vee} \Big\{ \sum_{v_{p,q} \in P(i,j;G^\vee)} u(p,q;G^\vee) \Big\} \theta(i,j;G^\vee) \\ &\quad + \sum_{v_{p,q} \in SP(s,t;G)} u(p,q;G) \\ &= \sum_{v_{i,j} \in G} u(i,j;G) + \sum_{v_{i,j}\in G} \Big\{ \sum_{v_{p,q} \in P(i,j;G)} u(p,q;G) \Big\} \theta(i,j;G), \end{aligned}$$

because $\theta(s,t;G) = 0$ and $\theta(s-1,l;G) = \theta(s-1,l;G^\vee) + 1$ and because we have $u(i,j;G) = u(i,j;G^\vee)$, $w(i,j;G) = w(i,j;G^\vee)$, and $\theta(i,j;G) = \theta(i,j;G^\vee)$ for $(i,j) \neq (s,t), (s-1,l)$. This completes the proof of Lemma 4.19. ■

LEMMA 4.20. *Let D be a* CNNS*-divisor on X and we use Notation 4.13. Then*

$$(D'_{\mathrm{red}})^2 \leq 2l - 2 - \sum_{i=1}^{t} b_i(b_i - 1) + \sum_j \big((C'_j)^2 + 2\big),$$

where $l = g(D_{\mathrm{red}}) - q(X)$.

PROOF. First we prove the following Claim.

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CLAIM 4.21.

$$e(D') - o(D') + 1 + \sum_{i=1}^{t} \frac{1}{2} b_i(b_i - 1) \leq l.$$

PROOF. We have $g(D'_{\mathrm{red}}) = g(D_{\mathrm{red}}) - \sum_{i=1}^{t} \frac{1}{2} b_i(b_i - 1)$ by definition. There exists $m = e(D') - o(D') + 1$ edges $e_1, \ldots, e_m$ of $G(D_{\mathrm{red}})$ such that $G - \{e_1, \ldots, e_m\}$ is a tree. Therefore by Remark 4.12.1, there exists a connected effective divisor A on X'' which is obtained by finite number of blowing ups of X' such that $g(D'_{\mathrm{red}}) = g(A) + e(D') - o(D') + 1$. Let $\mu''\colon X'' \longrightarrow X'$ be its birational morphism and A the strict transform of D'_{red} by μ''. Let $\alpha(A) = \dim \operatorname{Ker}\bigl(H^1(\mathcal{O}_{X''}) \longrightarrow H^1(\mathcal{O}_A)\bigr)$. Then we calculate $\alpha(A)$.

If $\alpha(A) \neq 0$, then there exist an Abelian variety T, a surjective morphism $f''\colon X'' \longrightarrow T$ such that $f''(X'')$ is not a point and $f''(A)$ is a point. Then any μ''-exceptional curve is contracted by f'' because T is an Abelian variety. Hence $f''\bigl((\mu'')^* D'_{\mathrm{red}}\bigr)$ is a point. But $(\mu'')^* D'_{\mathrm{red}}$ is not negative semidefinite. Therefore $\alpha(A) = 0$. Since A is reduced and connected, A is 1-connected. Hence $g(A) = h^1(\mathcal{O}_A)$. So we obtain $g(A) = h^1(\mathcal{O}_A) \geq q(X'') = q(X)$.

By the above argument,

$$\begin{aligned} g(D_{\mathrm{red}}) &= g(D'_{\mathrm{red}}) + \sum_{i=1}^{t} \frac{1}{2} b_i(b_i - 1) \\ &= g(A) + e(D') - o(D') + 1 + \sum_{i=1}^{t} \frac{1}{2} b_i(b_i - 1) \\ &\geq q(X) + e(D') - o(D') + 1 + \sum_{i=1}^{t} \frac{1}{2} b_i(b_i - 1). \end{aligned}$$

Therefore

$$e(D') - o(D') + 1 + \sum_{i=1}^{t} \frac{1}{2} b_i(b_i - 1) \leq l.$$

This completes the proof of Claim 4.21.

We continue the proof of Lemma 4.20. By construction, we obtain

$$\begin{aligned} (D'_{\mathrm{red}})^2 &= \sum_j (C'_j)^2 + 2e(D') \\ &= \sum_j (C'_j)^2 + 2\bigl(o(D') + e(D') - o(D')\bigr) \\ &= \sum_j \bigl((C'_j)^2 + 2\bigr) + 2\bigl(e(D') - o(D')\bigr). \end{aligned}$$

By Claim 4.21, we have

$$(D'_{\mathrm{red}})^2 \leq 2l - 2 - \sum_{i=1}^{t} b_i(b_i - 1) + \sum_j \bigl((C'_j)^2 + 2\bigr).$$

This completes the proof of Lemma 4.20. ∎

THEOREM 4.22. *Let X be a minimal smooth projective surface with $\kappa(X) \geq 0$ and D a* CNNS-*divisor on X. Let $D = \sum_j r_j D_j$ be its prime decomposition and $m = g(D) - q(X)$, where $m \in \mathbb{Z}$.*

Then $D^2 \leq 2m - 2 + N(D)$, where

$$N(D) = \sum_{\beta \in \mathbb{Z}} \beta \cdot \sharp\{\text{irreducible curves } C_j \text{ of } D \text{ such that } C_j^2 = -2 + \beta\}.$$

PROOF. We use Notation 4.13 and the notions which is defined above. We may assume that $B = \{y\}$. Let $G = G(y, D)$, $u(i,j) = u(i,j;G)$, $\theta(i,j) = \theta(i,j;G)$, $w(i,j) = w(i,j;G)$, $P(i,j) = P(i,j;G)$, and $SP(i,j) = SP(i,j;G)$. Let $D' = (\mu')^* D$ and $D'_{nr} = D' - D'_{\text{red}}$. Let $D'_{nr} = D'_{ne} + D'_{\text{e}} + D'_{-1}$, where D'_{ne} is the effective divisor which consists of not μ'-exceptional curves, D'_{e} is the effective divisor which consists of curves which are μ'-exceptional curves but not (-1)-curves, and D'_{-1} is the effective divisor which consists of (-1)-curves.

Then

$$K_{X'} D'_{\text{e}} = \sum_{v_{i,j} \in G} \Big\{ \Big(\sum_{v_{p,q} \in P(i,j)} u(p,q) \Big) - 1 \Big\} \theta(i,j) + \sum_{v_{i,j} \in G} \varepsilon(i,j) \big(m(i,j) - 1 \big),$$

where $m(i,j)$ is the multiplicity of e-curve through $x_{i,j}$ in the total transform of D, $x_{i,j}$ is the blowing up point and its (-1)-curve corresponds to $v_{i,j}$, $\varepsilon(i,j) = 1$ if there exists the e-curve through $x_{i,j}$ and $\varepsilon(i,j) = 0$ if there does not exist the e-curve through $x_{i,j}$.

On the other hand,

$$-\sum_{\alpha} (E_\alpha^2 + 2) = \sum_{v_{i,j} \in G - W} \big(w(i,j) - 1 \big) + \sum_{v_{i,j} \in G} \varepsilon(i,j),$$

where E_α is a μ'-exceptional curve on X' and not (-1)-curve, and $W = \{v_{i,j} \in G \mid w(i,j) = 0\}$.

Hence

$$\begin{aligned} K_{X'} D'_{\text{e}} - \sum_{\alpha} (E_\alpha^2 + 2) &= \sum_{v_{i,j} \in G} \Big\{ \Big(\sum_{v_{p,q} \in P(i,j)} u(p,q) \Big) - 1 \Big\} \theta(i,j) \\ &\quad + \sum_{v_{i,j} \in G} \varepsilon(i,j) m(i,j) + \sum_{v_{i,j} \in G - W} \big(w(i,j) - 1 \big). \end{aligned} \tag{4.22.1}$$

Let

$$\beta_{nr} = \text{sum of multiplicity of } \mu'\text{-exceptional } (-1)\text{-curves in } D'_{nr}.$$

Then

$$-\beta_{nr} = K_{X'} D'_{-1}. \tag{4.22.2}$$

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Let $C_{i,j}$ be a strict transform of $C_{i,j-1}$ by μ_j and $C_{i,0} = C_i$. Let $C_{i,j} = \mu_j^*(C_{i,j-1}) - e(i)_j E_j$, where E_j is the (-1)-curve of μ_j. We remark that $e(i)_j \geq 1$ for any i, j.

Then

$$K_{X'}\big((r_i - 1)C_{i,t}\big) \geq \sum_{j=1}^{t}(r_i - 1)e(i)_j$$

because X is minimal.

Hence

$$K_{X'}(D'_{ne}) \geq \sum_i \Big\{\sum_{j=1}^{t}(r_i - 1)e(i)_j\Big\}.$$

On the other hand

$$\sum_i (C_{i,t}^2 + 2) = N(D) - \sum_i \sum_{j=1}^{t} e(i)_j^2$$

because $C_{i,t}^2 = C_i^2 - \sum_{j=1}^{t} e(i)_j^2$.

Hence

$$K_{X'}(D'_{ne}) - \sum_i (C_{i,t}^2 + 2) \geq \sum_i \sum_{j=1}^{t}\big(r_i e(i)_j\big) - N(D) \tag{4.22.3}$$

since $\sum_{j=1}^{t} e(i)_j^2 \geq \sum_{j=1}^{t} e(i)_j$.

By (4.22.1), (4.22.2), and (4.22.3), we obtain

(4.22.4)

$$\begin{aligned} K_{X'}D'_{nr} &- \sum_i (C_{i,t}^2 + 2) - \sum_\alpha (E_\alpha^2 + 2) \\ &\geq -\beta_{nr} + \sum_{v_{i,j}\in G}\Big\{\Big(\sum_{v_{p,q}\in P(i,j)} u(p,q)\Big) - 1\Big\}\theta(i,j) \\ &\quad + \sum_{v_{i,j}\in G}\varepsilon(i,j)m(i,j) + \sum_{v_{i,j}\in G-W}\big(w(i,j) - 1\big) + \sum_i \sum_{j=1}^{t}\big(r_i e(i)_j\big) - N(D). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} q(X) + m = g(D) &= g(D') \\ &= g(D'_{\mathrm{red}}) + \frac{1}{2}(K_{X'} + D' + D'_{\mathrm{red}})D'_{nr} \\ &= g(D_{\mathrm{red}}) - \frac{1}{2}\sum_{i=1}^{t} b_i(b_i - 1) + \frac{1}{2}(K_{X'} + D' + D'_{\mathrm{red}})D'_{nr} \\ &= q(X) + l - \frac{1}{2}\sum_{i=1}^{t} b_i(b_i - 1) + \frac{1}{2}(K_{X'} + D' + D'_{\mathrm{red}})D'_{nr}, \end{aligned}$$

where $l = g(D_{\mathrm{red}}) - q(X)$.

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Hence by (4.22.4), we obtain

$$\begin{aligned}
2m-2l &= (K_{X'}+D'+D'_{\mathrm{red}})D'_{nr} - \sum_{i=1}^{t} b_i(b_i-1)\\
&\geq \sum_i (C_{i,t}^2+2) + \sum_\alpha (E_\alpha^2+2) - \beta_{nr}\\
&\quad + \sum_{v_{i,j}\in G} \Big\{\Big(\sum_{v_{p,q}\in P(i,j)} u(p,q)\Big) - 1\Big\}\theta(i,j)\\
&\quad + \sum_{v_{i,j}\in G} \varepsilon(i,j)m(i,j) + \sum_{v_{i,j}\in G-W} \big(w(i,j)-1\big)\\
&\quad + \sum_i \sum_{j=1}^{t} \big(r_i e(i)_j\big) - N(D) + (D'+D'_{\mathrm{red}})D'_{nr} - \sum_{i=1}^{t} b_i(b_i-1),
\end{aligned}$$

and so we have

$$\begin{aligned}
(D'+D'_{\mathrm{red}})D'_{nr} &\leq -\sum_i (C_{i,t}^2+2) - \sum_\alpha (E_\alpha^2+2) + \beta_{nr}\\
&\quad - \sum_{v_{i,j}\in G} \Big\{\Big(\sum_{v_{p,q}\in P(i,j)} u(p,q)\Big) - 1\Big\}\theta(i,j)\\
&\quad - \sum_{v_{i,j}\in G} \varepsilon(i,j)m(i,j) - \sum_{v_{i,j}\in G-W} \big(w(i,j)-1\big)\\
&\quad - \sum_i \sum_{j=1}^{t} \big(r_i e(i)_j\big) + N(D) + \sum_{i=1}^{t} b_i(b_i-1) + 2m - 2l.
\end{aligned}$$

Therefore by Lemma 4.20, we obtain

$$\begin{aligned}
(D')^2 &= (D'_{\mathrm{red}})^2 + (D'+D'_{\mathrm{red}})D'_{nr}\\
&\leq (2m-2l) + (2l-2) + \sum_{i=1}^{t} b_i(b_i-1) - \sum_{i=1}^{t} b_i(b_i-1)\\
&\quad + \sum_i \big((C_i')^2+2\big) - \sum_i (C_{i,t}^2+2) - \sum_\alpha (E_\alpha^2+2) + \beta_{nr}\\
&\quad - \sum_{v_{i,j}\in G} \Big\{\Big(\sum_{v_{p,q}\in P(i,j)} u(p,q)\Big) - 1\Big\}\theta(i,j)\\
&\quad - \sum_{v_{i,j}\in G} \varepsilon(i,j)m(i,j) - \sum_{v_{i,j}\in G-W} \big(w(i,j)-1\big)\\
&\quad - \sum_i \sum_{j=1}^{t} \big(r_i e(i)_j\big) + N(D)\\
&= (2m-2) + M(D')\\
&\quad - \sum_{v_{i,j}\in G} \Big\{\Big(\sum_{v_{p,q}\in P(i,j)} u(p,q)\Big) - 1\Big\}\theta(i,j)\\
&\quad - \sum_{v_{i,j}\in G} \varepsilon(i,j)m(i,j) - \sum_{v_{i,j}\in G-W} \big(w(i,j)-1\big)\\
&\quad - \sum_i \sum_{j=1}^{t} \big(r_i e(i)_j\big) + N(D),
\end{aligned}$$

where $M(D')$ is the sum of the multiplicity of (-1)-curves in D'.

On the other hand by Lemma 4.19, we have

$$\begin{aligned} M(D') - \sum_{v_{i,j}\in G}\Big\{\Big(\sum_{v_{p,q}\in P(i,j)} u(p,q)\Big) - 1\Big\}\theta(i,j) \\ = M(D') - \sum_{v_{i,j}\in G}\Big\{\sum_{v_{p,q}\in P(i,j)} u(p,q)\Big\}\theta(i,j) + \sum_{v_{i,j}\in G-W}\big(w(i,j)-1\big) \\ = \sum_{v_{p,q}\in G} u(p,q) + \sum_{v_{i,j}\in G-W}\big(w(i,j)-1\big). \end{aligned}$$

Therefore

$$\begin{aligned} (D')^2 &\le 2m-2+\sum_{v_{p,q}\in G} u(p,q) + \sum_{v_{i,j}\in G-W}\big(w(i,j)-1\big) \\ &\quad - \sum_{v_{i,j}\in G}\varepsilon(i,j)m(i,j) - \sum_{v_{i,j}\in G-W}\big(w(i,j)-1\big) \\ &\quad - \sum_i\sum_{j=1}^{t}\big(r_i e(i)_j\big) + N(D) \\ &= 2m-2+N(D) \end{aligned}$$

because we have

$$\sum_{v_{i,j}\in G}\varepsilon(i,j)m(i,j) + \sum_i\sum_{j=1}^{t}\big(r_i e(i)_j\big) = \sum_{v_{p,q}\in G} u(p,q)$$

by considering the definition of $u(p,q)$. This completes the proof of Theorem 4.22. ∎

Theorem 4.11 is obtained by Theorem 4.22.

PROOF OF THEOREM 4.11. It is sufficient to prove $D^2 \le 2m+1$ if $g(D)-q(X)=m$. We consider the following decomposition $(**)$ of D:

$(**)$ $D = D_1 + D_2$, and D_1 and D_2 have no common component, where D_1 and D_2 are non zero effective connected divisors.

CLAIM 4.23. *If* $\big((D_1)_{\mathrm{red}}\big)^2 \le 0$ *and* $\big((D_2)_{\mathrm{red}}\big)^2 \le 0$, *then* $N(D) \le 4$. *If* $\big((D_1)_{\mathrm{red}}\big)^2 < 0$ *or* $\big((D_2)_{\mathrm{red}}\big)^2 < 0$, *then* $N(D) \le 3$.

PROOF. Let $(D_i)_{\mathrm{red}} = \sum_j B_{i,j}$. Then $\sum_j (B_{i,j})^2 = N(D_i) - 2o(D_i)$ and $\sum_{j\ne k} B_{i,j}B_{i,k} \ge e(D_i)$. Hence $\big((D_i)_{\mathrm{red}}\big)^2 \ge 2e(D_i) - 2o(D_i) + N(D_i)$ for $i=1,2$. By hypothesis, we have $0 \ge 2e(D_i) - 2o(D_i) + N(D_i)$ for $i=1,2$. Since the dual graph $G(D_i)$ of D_i is connected, we have $e(D_i) - o(D_i) + 1 \ge 0$. Hence $2e(D_i) - 2o(D_i) \ge -2$ and so we have $N(D_i) \le 2$.

On the other hand, $N(D) = N(D_1) + N(D_2)$ since $D = D_1 + D_2$. Therefore $N(D) \le 4$.

The last part of Claim 4.23 can be proved by the above argument. This completes the proof of Claim 4.23.

Let $S(D)$ be a set of an effective connected reduced divisor $\tilde{D}$ contained in D such that $\tilde{D}$ has a minimum component which satisfies the property that the intersection matrix of $\tilde{D}$ is not negative semidefinite.

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Then $S(D) \neq \phi$ by hypothesis. Let $\bar{D} = \sum_{i\in J} C_i \in S(D)$ and let r_i be the multiplicity of C_i in D. Let $D_\alpha = \sum_{i\in J} r_i C_i$ and $D_\beta = D - D_\alpha$. We remark that possibly $D_\beta = 0$. Then D_α has at least two components since $C_i^2 < 0$ for any i. Let $D_\alpha = D_{\alpha,1} + D_{\alpha,2}$ be the decomposition as $(**)$.

CLAIM 4.24. *We can take this decomposition which satisfies* $(D_{\alpha,1})^2 < 0$.

PROOF. We consider the dual graph $G(D_\alpha)$ of D_α. Then $G(D_\alpha)$ is connected. In Graph Theory, there is the following standard Theorem;

THEOREM 4.25. *Let G be a connected graph which is not one point. Then there are at least two points which are not cutpoints. (Here a vertex v of a graph is called a cutpoint if removal of v increases the number of components.)*

PROOF. See Theorem 3.4 in [H].

We continue the proof of Claim 4.24. By Theorem 4.25, it is sufficient to take $(D_{\alpha,1})_{\mathrm{red}}$ as an irreducible curve corresponding to a vertex of $G(D_\alpha)$ which is not a cutpoint. This completes the proof of Claim 4.24.

We continue the proof of Theorem 4.11.

We have $\big((D_{\alpha,1})_{\mathrm{red}}\big)^2 < 0$ and $\big((D_{\alpha,2})_{\mathrm{red}}\big)^2 \leq 0$ by the choice of D_α. Therefore $N(D_\alpha) \leq 3$ by Claim 4.23.

On the other hand, we have

$$q(X) + m = g(D) = g(D_\alpha) + \frac{1}{2}(K_X + D + D_\alpha)D_\beta.$$

Let $g(D_\alpha) = q(X) + m_\alpha$. Then by Theorem 4.22, $D_\alpha^2 \leq 2m_\alpha - 2 + N(D_\alpha) \leq 2m_\alpha + 1$ since D_α is a CNNS-divisor.

On the other hand, $(K_X + D + D_\alpha)D_\beta = 2(m - m_\alpha)$ and $K_X D_\beta \geq 0$. Hence $(D + D_\alpha)D_\beta \leq 2(m - m_\alpha)$. Therefore

$$\begin{aligned} D^2 &= D_\alpha^2 + (D + D_\alpha)D_\beta \\ &\leq 2m_\alpha + 1 + 2m - 2m_\alpha \\ &= 2m + 1. \end{aligned}$$

This completes the proof of Theorem 4.11. ∎

REMARK 4.26. Let $D = \sum_i r_i C_i$ be an effective divisor on a minimal smooth surface of general type with $C_i^2 < 0$ for any i. If the intersection matrix $\|(C_i \cdot C_j)\|$ is not negative semidefinite, then $K_X D \geq 2q(X) - 3$.

Indeed, let $D_1, \ldots, D_t$ be the connected component of D. Then for some D_k, the intersection matrix of the components of D is not negative semidefinite. By Theorem 4.11, we have $K_X D_k \geq 2q(X) - 3$. Since K_X is nef, we obtain $K_X D \geq 2q(X) - 3$.

COROLLARY 4.27. *Let X be a minimal smooth surface of general type and let D be a nef-big effective divisor with $h^0(D) = 1$ on X. If D is not the following type ($\star$), then $K_X D \geq 2q(X) - 4$;*

($\star$) $D = C_1 + \sum_{j\geq 2} r_j C_j$; $C_1^2 > 0$ and the intersection matrix $\|(C_j, C_k)\|_{j\geq 2, k\geq 2}$ of $\sum_{j\geq 2} r_j C_j$ is negative semidefinite.

PROOF. By Theorem 4.5, Theorem 4.6, Theorem 4.11, and Remark 4.26, we obtain Corollary 4.27. ∎

5. The case in which $\kappa(X) = 2$ and L is an irreducible reduced curve.

NOTATION 5.1. Let X be a smooth projective surface over the complex number field $\mathbb{C}$ and let C be a curve on X with $C^2 > 0$. Let $N(k; C)$ be the set of a 0-dimensional subscheme $\tilde{Z}$ with length $\tilde{Z} = k+1$ and $\operatorname{Supp}\tilde{Z} \subset C$ such that the restriction map $\Gamma\big(\mathcal{O}(K_X+C)\big) \longrightarrow \Gamma\big(\mathcal{O}(K_X+C)\otimes\mathcal{O}_{\tilde{Z}}\big)$ is not surjective. Let $S(\tilde{Z}; C)$ be the set of a subcycle Z of $\tilde{Z} \in N(k; C)$ with length $Z \leq$ length $\tilde{Z}$ such that $\Gamma\big(\mathcal{O}(K_X+C)\big) \longrightarrow \Gamma\big(\mathcal{O}(K_X+C)\otimes\mathcal{O}_Z\big)$ is not surjective but for any subcycle Z' of Z with length $Z' <$ length Z, $\Gamma\big(\mathcal{O}(K_X+C)\big) \longrightarrow \Gamma\big(\mathcal{O}(K_X+C)\otimes\mathcal{O}_{Z'}\big)$ is surjective.

First we prove the following Theorem.

THEOREM 5.2. *Let X be a minimal smooth projective surface with $\kappa(X) = 2$, and let C be an irreducible reduced curve on X with $C^2 > 0$. We put $g(C) = q(X) + m$. We assume that $K_X + C$ is not k-very ample for some integer $k \geq (1/2)(m-1)$, and also assume that*

$$\sharp \bigcup_{\tilde{Z}\in N(k:C)} \Big(\bigcup_{Z\in S(\tilde{Z};C)} \operatorname{Supp} Z \Big) = \infty.$$

Then $C^2 \leq 4(k+1)$.

PROOF. We remark that C is nef and big. Assume that $C^2 > 4(k+1)$. Then we remark that $C^2 \geq 2m+3$ by hypothesis.

If $q(X) \leq 2$, then $K_X C \geq 0 \geq 2q(X) - 4$ and so we have $C^2 \leq 2m+2$ and this is a contradiction. Hence we have $q(X) \geq 3$.

Then by Corollary 2.3 in [BeS], for any $Z \in \bigcup_{\tilde{Z}\in N(k;C)} S(\tilde{Z}; C)$ there exists an effective divisor D_Z on X such that $\operatorname{Supp}(Z) \subset D_Z$ and $C - 2D_Z$ is a $\mathbb{Q}$-effective divisor. Let $A = \{D_Z \mid Z \in \bigcup_{\tilde{Z}\in N(k;C)} S(\tilde{Z}; C)$ and D_Z as above$\}$.

CLAIM 5.3. *Let D be an effective divisor on X and let $D = \sum_i r_i C_i$ be its prime decomposition. If there exists an irreducible component C_i with $C_i^2 > 0$, and $C - 2D$ is $\mathbb{Q}$-effective, then $C^2 \leq 2m$ if $g(C) = q(X) + m$.*

PROOF. By Proposition 1.7, we have

$$\begin{aligned} K_X D &\geq K_X C_i \\ &\geq \frac{3}{2}q(X) - 3 \\ &= q(X) + \frac{1}{2}q(X) - 3. \end{aligned}$$

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Since $q(X) \geq 3$, we obtain that $K_XD \geq q(X) - (3/2)$. Hence $K_XD \geq q(X) - 1$ because K_XD is an integer. Because K_X is nef and $C - 2D$ is $\mathbb{Q}$-effective, we obtain

$$\begin{aligned} g(C) &= 1 + \frac{1}{2}(K_X + C)C \\ &\geq 1 + \frac{1}{2}(K_X)(2D) + \frac{1}{2}C^2 \\ &= 1 + K_XD + \frac{1}{2}C^2 \\ &\geq q(X) + \frac{1}{2}C^2. \end{aligned}$$

Therefore $C^2 \leq 2m$. This completes the proof of Claim 5.3.

We continue the proof of Theorem 5.2.

By Claim 5.3, any $D_Z \in A$ satisfies $C_i^2 \leq 0$ for any irreducible component C_i of D_Z. So $C \not\subset D_Z$ for any $D_Z \in A$ since $C^2 > 0$. Hence by hypothesis, we obtain

$$\dim \bigcup_{D_Z \in A} \Big(\bigcup_{C_{Z,i} \in V(D_Z)} \operatorname{Supp} C_{Z,i} \Big) = 2,$$

where $V(D_Z) =$ the set of irreducible components of D_Z.

Let

$$\bigcup_{D_Z \in A} V(D_Z) = B_1 \cup B_2,$$

where B_1 is the set of irreducible curves C_1 with $C_1^2 < 0$ and B_2 is the set of irreducible curves C_2 with $C_2^2 = 0$.

(1) The case in which $\sharp B_1 = \infty$.

If $C_1 \in B_1$ with $K_XC_1 \geq q(X) - 1$, then $K_XD_Z \geq q(X) - 1$ and so we have $C^2 \leq 2m$ by the same argument as Claim 5.3. So we have $K_XC_1 \leq q(X) - 2$ for any $C_1 \in B_1$. Then the number of such a curve C_1 is at most finite by Lemma 1.8. But this is a contradiction by hypothesis.

(2) The case in which $\sharp B_2 = \infty$.

If $C_2 \in B_2$ with $K_XC_2 \geq q(X) - 1$, then we have $C^2 \leq 2m$ by the same argument as above. So we have $K_XC_2 \leq q(X) - 2$ for any $C_2 \in B_2$. Then there is a subset $B_3 \subset B_2$ such that $\sharp B_3 = \infty$ and $C_s \equiv C_t$ for any distinct $C_s, C_t \in B_3$ by Lemma 1.8. We take a $C_k \in B_3$. Let $\alpha(C_k) = \dim \operatorname{Ker}\big(H^1(\mathcal{O}_X) \longrightarrow H^1(\mathcal{O}_{C_k})\big)$.

(2-1) The case in which $\alpha(C_k) \neq 0$.

Then by Lemma 1.3 in [Fk4], there exist an Abelian variety G and a morphism $f\colon X \longrightarrow G$ such that $f(X)$ is not a point and $f(C_k)$ is a point. Since $C_k^2 = 0$, we obtain $f(X)$ is a curve. By taking Stein factorization, if necessary, there is a smooth curve B, a surjective morphism $h\colon X \longrightarrow B$ with connected fibers, and a finite morphism $\delta\colon B \longrightarrow f(X)$ such that $f = \delta \circ h$. On the other hand, for any $C_n \in B_3$ and $C_n \neq C_k$, we have $C_nC_k = C_k^2 = 0$. Hence any element C_n of B_3 is contained in a fiber of h and $C_n^2 = 0$. Therefore for a general fiber F_h of h, we may assume $F_h \in B_3$. On the other hand, we have $C - 2D_Z \leq C - 2F_h$. So we obtain that $C - 2F_h$ is a $\mathbb{Q}$-effective divisor.

Hence we have

$$\begin{aligned} g(C) &= g(B) + \frac{1}{2}(K_{X/B} + C)C + (CF_h - 1)\big(g(B) - 1\big) \\ &\geq g(B) + \frac{1}{2}(K_{X/B})(2F_h) + \frac{1}{2}C^2 \\ &= g(B) + 2g(F_h) - 2 + \frac{1}{2}C^2 \\ &= g(B) + g(F_h) + \frac{1}{2}C^2 + g(F_h) - 2 \\ &\geq q(X) + \frac{1}{2}C^2 \end{aligned}$$

because $K_{X/B}$ is nef, $g(B) \geq 1$ and $g(F_h) \geq 2$.

Hence $C^2 \leq 2m$. But this is a contradiction because we assume that $C^2 \geq 2m + 3$.

(2-2) The case in which $\alpha(C_k) = 0$.

Then $q(X) \leq h^1(\mathcal{O}_{C_k}) = g(C_k)$. On the other hand, since K_X is nef, $C_k^2 = 0$, $C - 2C_k \geq C - 2D_Z$, and $C - 2D_Z$ is $\mathbb{Q}$-effective, we obtain

$$\begin{aligned} g(C) &= 1 + \frac{1}{2}(K_X + C)C \\ &\geq 1 + \frac{1}{2}(K_X)(2C_k) + \frac{1}{2}C^2 \\ &= 1 + K_X C_k + \frac{1}{2}C^2 \\ &= 1 + 2g(C_k) - 2 + \frac{1}{2}C^2 \\ &\geq 2q(X) - 1 + \frac{1}{2}C^2. \end{aligned}$$

Hence

$$\begin{aligned} C^2 &\leq 2m + 2\big(1 - q(X)\big) \\ &\leq 2m - 4 \end{aligned}$$

since $q(X) \geq 3$.

But this is a contradiction by hypothesis. Therefore $C^2 \leq 4(k+1)$. This completes the proof of Theorem 5.2. ∎

COROLLARY 5.4. *Let X be a minimal smooth projective surface with $\kappa(X) = 2$ and let C be an irreducible reduced curve with $C^2 > 0$. Then $C^2 \leq 4m+4$ if $m = g(C) - q(X)$.*

PROOF. We use Notation 5.1. By Theorem 5.2, it is sufficient to prove that $K_X + C$ is not m-very ample and

$$\sharp \bigcup_{\tilde{Z} \in N(m;C)} \Big(\bigcup_{Z \in S(\tilde{Z};C)} \operatorname{Supp} Z \Big) = \infty.$$

Let $W = \operatorname{Im}\big(H^0(K_X + C) \longrightarrow H^0(\omega_C)\big)$, where ω_C is a dualizing sheaf of C. We remark that ω_C is a Cartier divisor. Let α be the map $H^0(K_X + C) \longrightarrow W$. Then $\dim W = h^0(K_X + C) -$

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$h^0(K_X) = m$ by Riemann-Roch Theorem and Kawamata-Viehweg Vanishing Theorem. Let $P_1, \ldots, P_{m+1}$ be any $m+1$ distinct points on $C \setminus \operatorname{Sing} C$, where $\operatorname{Sing} C$ denotes the singular locus of C. Let Z be a 0-dimensional subscheme such that

(1) $\mathcal{I}_Z \mathcal{O}_{X,y} = \mathcal{O}_{X,y}$ if $y \notin \{P_1, \ldots, P_{m+1}\}$;

(2) $\mathcal{I}_Z \mathcal{O}_{X,y} = (x_i, y_i)$ if $y = P_i$,

where $\mathcal{I}_Z$ is the ideal sheaf of Z and (x_i, y_i) is a local coordinate of X at P_i such that C is defined by (x_i) at P_i. Let β be the restriction map $W \longrightarrow H^0\big((K_X + C) \otimes \mathcal{O}_Z\big)$. If $K_X + C$ is m-very ample at Z, then the restriction $\gamma\colon H^0(K_X+C) \longrightarrow H^0\big((K_X+C)\otimes \mathcal{O}_Z\big)$ is surjective. But we have $\dim W = m$ and $\dim H^0\big((K_X + C) \otimes \mathcal{O}_Z\big) = m + 1$. This is a contradiction since $\gamma = \beta \circ \alpha$. Hence $K_X + C$ is not m-very ample for any 0-dimensional subscheme with length $m+1$ which consists of distinct $m+1$ points of $C \setminus \operatorname{Sing}(C)$. This implies

$$\sharp \bigcup_{\tilde{Z} \in N(m;C)} \Big(\bigcup_{Z \in S(\tilde{Z};C)} \operatorname{Supp} Z \Big) = \infty.$$

This completes the proof of Corollary 5.4. ∎

By Corollary 4.27, in order to solve Conjecture 1 (or Conjecture 1′), it is sufficient to consider the case in which D is the following type $(\star)$:

$(\star)$ $D = C_1 + \sum_{j\geq 2} r_j C_j$; $C_1^2 > 0$ and the intersection matrix $\|C_j, C_k\|_{j\geq 2, k\geq 2}$ of $\sum_{j\geq 2} r_j C_j$ is negative semidefinite.

COROLLARY 5.5. *Let X be a minimal smooth projective surface with $\kappa(X) = 2$ and let D be a nef-big effective divisor on X such that D is the type $(\star)$. Then $D^2 \leq 4m + 4$ if $m = g(D) - q(X)$.*

PROOF. First we obtain

$$g(C_1) = q(X) + m - \frac{1}{2}(K_X + D + C_1)\Big(\sum_{j\geq 2} r_j C_j\Big).$$

By Corollary 5.4, we have

$$\begin{aligned} C_1^2 &\leq 4m + 4 - 2(K_X + D + C_1)\Big(\sum_{j\geq 2} r_j C_j\Big) \\ &\leq 4m + 4 - 2(D + C_1)\Big(\sum_{j\geq 2} r_j C_j\Big). \end{aligned}$$

Hence

$$\begin{aligned} D^2 &= C_1^2 + (D + C_1)\Big(\sum_{j\geq 2} r_j C_j\Big) \\ &\leq 4m + 4 - (D + C_1)\Big(\sum_{j\geq 2} r_j C_j\Big). \end{aligned}$$

On the other hand $D + C_1$ is nef. Hence $(D + C_1)(\sum_{j\geq 2} r_j C_j) \geq 0$ and so we obtain $D^2 \leq 4m + 4$. This completes the proof of Corollary 5.5. ∎

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6. **Higher dimensional case and conjecture.** In this section we consider the case in which $n = \dim X \geq 3$ and $\kappa(X) \geq 0$.

THEOREM 6.1. *Let (X, L) be a quasi-polarized manifold with $\dim X = n \geq 3$ and $\kappa(X) = 0$ or 1. Then $K_X L^{n-1} \geq 2\big(q(X) - n\big)$.*

PROOF. (1) The case in which $\kappa(X) = 0$.

Then $q(X) \leq n$ by [Ka1]. Hence $K_X L^{n-1} \geq 0 \geq 2\big(q(X) - n\big)$.

(2) The case in which $\kappa(X) = 1$.

By Iitaka Theory ([Ii]), there exist a smooth projective variety X_1, a birational morphism $\mu_1 : X_1 \longrightarrow X$, a smooth curve C, and a fiber space $f_1 : X_1 \longrightarrow C$ such that $\kappa(F_1) = 0$, where F_1 is a general fiber of f_1. Let $L_1 = \mu_1^* L$.

(2-1) The case in which $g(C) \geq 1$.

By Lemma 1.3.1 and Remark 1.3.2 in [Fk2] and the semipositivity of $(f_1)_*(mK_{X_1/C})$ for $m \in \mathbb{N}$ ([Fj1], [Ka2]), we have $K_{X_1/C} L_1^{n-1} \geq 0$. Therefore

$$\begin{aligned} K_X L^{n-1} &= K_{X_1} L_1^{n-1} \\ &= K_{X_1/C} L_1^{n-1} + \big(2g(C) - 2\big) L_1^{n-1} F_1 \\ &\geq 2g(C) - 2. \end{aligned}$$

On the other hand, $q(X) \leq g(C) + (n - 1)$ since $q(F_1) \leq n - 1$ by [Ka1]. Hence

$$\begin{aligned} K_X L^{n-1} &\geq 2\big(g(C) - 1\big) \\ &\geq 2\big(q(X) - n\big). \end{aligned}$$

(2-2) The case in which $g(C) = 0$.

Then $q(X) \leq n - 1$ since $q(F_1) \leq n - 1$. Therefore $K_X L^{n-1} \geq 0 > 2\big(q(X) - n\big)$.

This completes the proof of Theorem 6.1. ∎

By considering the above theorem, we propose the following conjecture which is a generalization of Conjecture 1′.

CONJECTURE 6.2. *Let (X, L) be a quasi-polarized manifold with $n = \dim X \geq 3$ and $\kappa(X) \geq 0$. Then $K_X L^{n-1} \geq 2\big(q(X) - n\big)$.*

By Theorem 6.1, this conjecture is true if $\kappa(X) = 0$ or 1. We will study Conjecture 6.2 in a future paper.

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