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Part 7. Queueing theory and Markov processes

ON EXPONENTIAL LIMIT LAWS FOR HITTING TIMES OF RARE SETS FOR HARRIS CHAINS AND PROCESSES

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ON EXPONENTIAL LIMIT LAWS FOR HITTING TIMES OF RARE SETS FOR HARRIS CHAINS AND PROCESSES

BY PETER W. GLYNN

Abstract

This paper provides a simple proof for the fact that the hitting time to an infrequently visited subset for a one-dependent regenerative process converges weakly to an exponential distribution. Special cases are positive recurrent Harris chains and Harris processes. The paper further extends this class of limit theorems to 'rewards' that are cumulated to the hitting time of such a rare set.

Keywords: Hitting time; regenerative process; Harris recurrent Markov chain; Harris recurrent Markov process

2010 Mathematics Subject Classification: Primary 60F05; 60G70; 60J05; 60J25; 60K05

1. Introduction

This paper is concerned with a topic that touches upon two of Søren Asmussen's research interests, namely extreme values (see [1] and [4]) and Harris recurrent Markov chains and processes (see, for example, [2] and [3]). Our main goal here is to present a simple proof of a basic fact in this theory, namely that the distribution of the hitting time to an infrequently visited set (i.e. a 'rare set') is approximately exponential. For Harris chains, this result can be found in [11]. (In later papers, Baccelli and McDonald [6] and Cogburn [8] considered the special case where the Harris chain possesses a classical regenerative structure, so that the embedded cycles are independent and identically distributed (i.i.d.).) The continuous-time result (for processes) derived in the present paper appears to be new.

Our argument is based on exploiting the regenerative structure that is present in such Markovian systems. As has been noted in [10] for Harris chains and in [14] for Harris processes, such models necessarily contain one-dependent regenerative cycles. As might then be expected, the central element of our proof is establishing an exponential limit law for such one-dependent processes; see Section 2. An important implication of our approach is that we can easily extend the asymptotic exponentiality limit theorem to the distribution of the reward cumulated to the hitting time of a rare set; see Section 3. Although not fully developed in this paper, our methodology can also be exploited to extend this 'hitting time for rare sets' limit theory to null recurrent one-dependent regenerative processes; see [5] for the corresponding results in the (classical) regenerative context. While our regenerative setting is closely related to that analyzed in [9], their results use different scalings and limit theorems than those obtained here.

2. Asymptotic exponentiality for one-dependent regenerative processes

As pointed out in the introduction, both Harris recurrent Markov chains and Harris recurrent Markov processes can be viewed as special cases of one-dependent regenerative processes.

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Definition 1. An S-valued stochastic process $X = (X(t)\colon t \geq 0)$ is said to be *one-dependent regenerative* if there exist random times $0 \leq T(0) < T(1) < \cdots$ such that

(i) $(W_n\colon n \geq 1)$ is identically distributed;

(ii) $(W_n\colon n \geq 0)$ is a one-dependent sequence of random elements,

where the *cycles* $(W_n\colon n \geq 0)$ are defined via

$$W_n(t) = \begin{cases} X(T(n-1)+t), & 0 \leq t < \tau_n, \\ \Delta, & t \geq \tau_n, \end{cases}$$

$\tau_n = T(n) - T(n-1)$, $T(-1) = 0$, and $\Delta \notin S$. The process X is said to be *nondelayed* if $T(0) = 0$ almost surely (a.s.) and *delayed* otherwise.

We note that if $(X_n\colon n \geq 0)$ is a discrete-time sequence (as for a Harris chain), $(X_n\colon n \geq 0)$ can be embedded in continuous time by setting $X(t) = X_{\lfloor t \rfloor}$ for $t \geq 0$.

Throughout this paper, we assume that X is a right-continuous process with left limits (i.e. a 'càdlàg' process) taking values in a Polish space S; this serves to simplify certain measurability issues that can otherwise arise. Our basic theorem concerns a sequence $(A_n\colon n \geq 1)$ of subsets of S. Let $T_n = \inf\{t \geq 0\colon X(t) \in A_n\}$ be the hitting time of A_n; we assume that the A_ns have been chosen so that the T_ns are well-defined random variables (e.g. A_n is an open subset of S). Then, $N_n = \inf\{l \geq 0\colon T(l) > T_n\}$ is the index of the cycle containing the first hitting time of A_n. (If X were a classical regenerative process with i.i.d. cycles, N_n would be geometrically distributed, and Theorem 1 below is then essentially trivial.)

Theorem 1. *If X is a one-dependent regenerative process for which* $\mathrm{P}(T_n < T(1)) \to 0$ *as $n \to \infty$ and* $\mathrm{P}(T(0) < T_n < T(1)) > 0$ *for each $n \geq 1$, then*

$$\frac{N_n}{\mathrm{E}\, N_n} \Rightarrow \mathcal{E}\mathrm{xp}(1),$$

where $\mathcal{E}\mathrm{xp}(1)$ is an exponential random variable (RV) with mean 1, and '$\Rightarrow$' denotes weak convergence.

Remark 1. In the i.i.d. setting, $\mathrm{E}\, N_n \sim 1/p_n$ as $n \to \infty$, where $p_n = \mathrm{P}(W_1 \text{ hits } A_n)$. In the one-dependent setting, p_n is not typically the correct renormalization, in the sense that $p_n N_n$ need not converge to $\mathcal{E}\mathrm{xp}(1)$ as $n \to \infty$. Consider, for example, an i.i.d. real-valued sequence $(Y_j\colon j \geq 0)$ and let $W_j = (Y_{j-1}, Y_j)$ for $j \geq 1$; the W_js are one-dependent. Given a subset $B_n \subseteq \mathbb{R}$, set $A_n = (B_n \times \mathbb{R}) \cup (\mathbb{R} \times B_n)$. Then, $p_n = \mathrm{P}(W_1 \in A_n) = \mathrm{P}(\text{either } Y_0 \text{ or } Y_1 \text{ lie in } B_n) = 2\,\mathrm{P}(Y_1 \in B_n) - \mathrm{P}(Y_1 \in B_n)^2$. But $\mathrm{P}(N_n = k) = \mathrm{P}(Y_1 \in B_n) \times \mathrm{P}(Y_1 \in B_n)^{k-1}$ so $\mathrm{P}(Y_1 \in B_n) N_n \Rightarrow \mathcal{E}\mathrm{xp}(1)$ as $n \to \infty$, and, hence, $p_n N_n \Rightarrow 2\mathcal{E}\mathrm{xp}(1)$ as $n \to \infty$. The fact that there appears to be no (simple) normalization factor that can be used in the one-dependent setting complicates the proof relative to the i.i.d. context.

Remark 2. It is easily seen that the proof of Theorem 1 can be extended to include m-dependent stationary sequences.

Proof of Theorem 1. Since the extension to the delayed case is trivial, we assume here that X is nondelayed. Let $I_j^n = I(W_j \text{ visits } A_n)$.

Step 1. Argue that $1/p_n \le \mathrm{E}\, N_n + 1 \le 2/p_n$. To see this, note that

$$\begin{aligned}
\mathrm{E} \sum_{j=1}^{N_n+1} I_j^n &= \mathrm{E}\, I_1^n + \mathrm{E} \sum_{j=2}^{\infty} I_j^n I(N_n \ge j-1) \\
&= \mathrm{E}\, I_1^n + \mathrm{E} \sum_{j=2}^{\infty} I_j^n I(I_1^n = 0, \dots, I_{j-2}^n = 0) \\
&= p_n + p_n \,\mathrm{E} \sum_{j=2}^{\infty} I(I_1^n = 0, \dots, I_{j-2}^n = 0) \\
&= p_n \,\mathrm{E}(N_n + 1),
\end{aligned}$$

where the one-dependence was used for the third equality. On the other hand,

$$1 \le \sum_{j=1}^{N_n+1} I_j^n \le 2,$$

yielding the result.

Step 2. Prove that $(p_n N_n \colon n \ge 1)$ is a uniformly integrable sequence of RVs. This follows as a consequence of the fact that $0 \le p_n N_n \le 2 \min(\tilde{N}_n^1, \tilde{N}_n^2)$, where

$$\tilde{N}_n^1 = \inf\{j \ge 1 \colon I_{2j}^n = 1\} \quad \text{and} \quad \tilde{N}_n^2 = \inf\{j \ge 1 \colon I_{2j-1}^n = 1\}.$$

But the one-dependence implies that $\tilde{N}_n^1$ and $\tilde{N}_n^2$ are both geometric RVs, so $(p_n \tilde{N}_n^1)$ is uniformly integrable, as is $(p_n \tilde{N}_n^2)$ and $(p_n \min(\tilde{N}_n^1, \tilde{N}_n^2))$.

Step 3. It follows from steps 1 and 2 that $(N_n/\mathrm{E}\, N_n \colon n \ge 1)$ is a uniformly integrable sequence of RVs. In particular, $(N_n/\mathrm{E}\, N_n \colon n \ge 1)$ is tight and any (weak) limit point must have unit mean. We can therefore complete the proof of the theorem by establishing that any weak limit of $(N_n/\mathrm{E}\, N_n \colon n \ge 1)$ must be memoryless (so that any weak limit is exponentially distributed with unit mean, i.e. $\mathcal{E}\mathrm{xp}(1)$).

Let $(\overline{F}_k(\cdot) := \mathrm{P}(N_{n_k}/\mathrm{E}\, N_{n_k} > \cdot) \colon k \ge 1)$ be a weakly convergent subsequence associated with $(N_n/\mathrm{E}\, N_n \colon n \ge 1)$, so that there exists a limiting complementary distribution function $\overline{F}(\cdot)$ for which $\overline{F}_k(x) \to \overline{F}(x)$ at all continuity points x of $\overline{F}$. Set $a_k = \mathrm{E}\, N_{n_k}$ and $\tilde{I}_j^k = I_j^{n_k}$. Choose x and $y > 0$ as continuity points of $\overline{F}$. Then, for each $\varepsilon > 0$,

$$\begin{aligned}
\overline{F}_k(x+y) &\le \overline{F}_k(x+y-\varepsilon) \\
&= \mathrm{P}(\tilde{I}_1^k = 0, \dots, \tilde{I}_{\lceil a_k(x+y-\varepsilon)\rceil}^k = 0) \\
&\le \mathrm{P}(\tilde{I}_1^k = 0, \dots, \tilde{I}_{\lceil a_k x\rceil}^k = 0,\ \tilde{I}_{\lceil a_k x\rceil+2}^k = 0, \dots, \tilde{I}_{\lceil a_k(x+y-\varepsilon)\rceil}^k = 0) \\
&= \mathrm{P}(\tilde{I}_1^k = 0, \dots, \tilde{I}_{\lceil a_k x\rceil}^k = 0)\, \mathrm{P}(\tilde{I}_{\lceil a_k x\rceil+2}^k = 0, \dots, \tilde{I}_{\lceil a_k(x+y-\varepsilon)\rceil}^k = 0) \\
&\le \overline{F}_k(x)\overline{F}_k(y-2\varepsilon),
\end{aligned}$$

where we have used the one-dependence for the second equality above.

If we now choose $(\varepsilon_n \colon n \ge 1)$ so that $\varepsilon_n \downarrow 0$ and $y - 2\varepsilon_n$ is a continuity point of $\overline{F}$ for $n \ge 1$, it follows that, for each $n \ge 1$,

$$\varlimsup_{k\to\infty} \overline{F}_k(x+y) \le \overline{F}(x)\overline{F}(y-2\varepsilon_n).$$

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Sending $n \to \infty$, we conclude that $\overline{\lim}_{k\to\infty} \overline{F}_k(x+y) \leq \overline{F}(x)\overline{F}(y)$. On the other hand, for $\varepsilon > 0$,

$$\begin{aligned}
&\overline{F}_k(x+y)\\
&\quad\geq \overline{F}_k(x+y+\varepsilon)\\
&\quad= \mathrm{P}(\tilde{I}_1^k = 0, \dots, \tilde{I}^k_{\lceil a_k(x+y+\varepsilon)\rceil} = 0)\\
&\quad= \mathrm{P}(\tilde{I}_1^k = 0, \dots, \tilde{I}^k_{\lceil a_k x\rceil} = 0,\ \tilde{I}^k_{\lceil a_k x\rceil+2} = 0, \dots, \tilde{I}^k_{\lceil a_k(x+y+\varepsilon)\rceil} = 0)\\
&\qquad - \mathrm{P}(\tilde{I}_1^k = 0, \dots, \tilde{I}^k_{\lceil a_k x\rceil} = 0,\ \tilde{I}^k_{\lceil a_k x\rceil+1} = 1,\ \tilde{I}^k_{\lceil a_k x\rceil+2} = 0, \dots, \tilde{I}^k_{\lceil a_k(x+y+\varepsilon)\rceil} = 0)\\
&\quad\geq \mathrm{P}(\tilde{I}_1^k = 0, \dots, \tilde{I}^k_{\lceil a_k x\rceil} = 0,\ \tilde{I}^k_{\lceil a_k x\rceil+2} = 0, \dots, \tilde{I}^k_{\lceil a_k(x+y+\varepsilon)\rceil} = 0) - p_{n_k}\\
&\quad\geq \overline{F}_k(x)\overline{F}_k(y+2\varepsilon) - p_{n_k}.
\end{aligned}$$

Again, by choosing ε_n so that $y + 2\varepsilon_n$ is a continuity point, we conclude that

$$\underline{\lim}_{k\to\infty} \overline{F}_k(x+y) \geq \overline{F}(x)\overline{F}(y+2\varepsilon_n).$$

Sending $n \to \infty$, the right continuity of $\overline{F}$ guarantees that $\underline{\lim}_{k\to\infty} \overline{F}_k(x+y) \geq \overline{F}(x)\overline{F}(y)$, and, hence, $\overline{F}_k(x+y) \to \overline{F}(x)\overline{F}(y)$ as $k \to \infty$ whenever x and y are continuity points of $\overline{F}$. It follows that $\overline{F}(x+z) = \overline{F}(x)\overline{F}(z)$ for almost every z, so that the right continuity of $\overline{F}$ ensures that $\overline{F}(x+z) = \overline{F}(x)\overline{F}(z)$ for $z \geq 0$. Interchanging the roles of x and z establishes that $\overline{F}(x+z) = \overline{F}(x)\overline{F}(z)$ for $x, z \geq 0$, proving the memorylessness of $\overline{F}$.

The asymptotic exponentiality of T_n follows as an easy consequence of Theorem 1. Note that $T(N_n - 1) \leq T_n \leq T(N_n)$, so that

$$\frac{N_n}{\mathrm{E}\,N_n}\frac{T(N_n-1)}{N_n} \leq \frac{T_n}{\mathrm{E}\,N_n} < \frac{N_n}{\mathrm{E}\,N_n}\frac{T(N_n)}{N_n}. \tag{1}$$

If X is a *positive recurrent* one-dependent regenerative process (so that $\mathrm{E}\,\tau_1 < \infty$), the strong law for i.i.d. sequences yields

$$\frac{T(2n)}{2n} = \frac{1}{2}\frac{\tau_1 + \tau_3 + \cdots + \tau_{2n-1}}{n} + \frac{1}{2}\frac{\tau_2 + \tau_4 + \cdots + \tau_{2n}}{n} \to \mathrm{E}\,\tau_1 \quad \text{a.s.}$$

as $n \to \infty$, so that $T(N_n)/N_n \Rightarrow \mathrm{E}\,\tau_1$ as $n \to \infty$, proving the following corollary.

Corollary 1. *Suppose that X is a positive recurrent one-dependent regenerative process satisfying the conditions of Theorem 1. Then,*

$$\frac{T_n}{\mathrm{E}\,N_n} \Rightarrow \mathrm{E}\,\tau_1 \mathscr{E}\mathrm{xp}(1)$$

as $n \to \infty$.

An argument similar to that used in step 1 of the above proof shows that

$$\mathrm{E}\,T(N_n+1) = \mathrm{E}\,\tau_0 + \mathrm{E}\,\tau_1\,\mathrm{E}(N_n+1). \tag{2}$$

As observed in the discussion proceeding Corollary 1, $T(N_n+1)/\mathrm{E}\,N_n \Rightarrow \mathrm{E}\,\tau_1 \mathscr{E}\mathrm{xp}(1)$ as $n \to \infty$. Relation (2) therefore implies that $(T(N_n+1)/\mathrm{E}\,N_n : n \geq 1)$ is uniformly integrable if $\mathrm{E}(\tau_0+\tau_1) < \infty$ (since uniform integrability is equivalent, in the presence of nonnegativity, to being able to interchange limits and expectations), so that $(T_n/\mathrm{E}\,N_n : n \geq 1)$ is also uniformly integrable; see (1). This proves our next result.

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Corollary 2. *Suppose that X satisfies the same conditions as those for Corollary 1 and, in addition,* $\mathrm{E}\,\tau_0 < \infty$. *Then,*

$$\frac{T_n}{\mathrm{E}\,T_n} \Rightarrow \mathrm{E}\,\tau_1 \mathcal{E}\mathrm{xp}(1)$$

as $n \to \infty$.

This is the desired distributional approximation for T_n. Thus, if A_n is a rarely visited set, we may use the approximation

$$T_n \overset{\mathrm{D}}{\approx} \mathrm{E}\,T_n \mathcal{E}\mathrm{xp}(1)$$

(where '$\overset{\mathrm{D}}{\approx}$' has the nonrigorous meaning 'has approximately the same distribution as'), so that the distribution of T_n is (approximately) determined once we have computed $\mathrm{E}\,T_n$. Since $\mathrm{E}\,T_n$ can be more easily calculated (say, by solving a finite linear system of equations as in the finite-state Markov chain setting) than the distribution of T_n, this approximation greatly simplifies the computation of the distribution of T_n.

3. Extensions

Our principal extension, to be considered in this section, is to obtain exponential approximations to 'rewards' cumulated to the hitting time of a rare set. In particular, given a (suitably measurable) 'reward' function $f: S \to \mathbb{R}_+$, let

$$R_n = \int_0^{T_n} f(X(s))\,\mathrm{d}s.$$

As in (1), the nonnegativity of f allows us to bound $R_n / \mathrm{E}\,N_n$ via

$$\frac{N_n}{\mathrm{E}\,N_n} \frac{\sum_{i=0}^{N_n-1} \tilde{Y}_i}{N_n} \le \frac{R_n}{\mathrm{E}\,N_n} \le \frac{N_n}{\mathrm{E}\,N_n} \frac{\sum_{i=0}^{N_n} \tilde{Y}_i}{N_n},$$

where

$$\tilde{Y}_i = \int_{T(i-1)}^{T(i)} f(X(s))\,\mathrm{d}s.$$

Identical arguments as for Corollaries 1 and 2 then yield the following result.

Proposition 1. (a) *Suppose that X satisfies the conditions of Theorem 1,*

$$\int_0^{T(0)} f(X(s))\,\mathrm{d}s < \infty,$$

and $\mathrm{E}\,\tilde{Y}_1 < \infty$. *Then,*

$$\frac{R_n}{\mathrm{E}\,N_n} \Rightarrow \mathrm{E}\,\tilde{Y}_1 \mathcal{E}\mathrm{xp}(1)$$

as $n \to \infty$.

(b) *Suppose that* $\mathrm{E}(\tilde{Y}_0 + \tilde{Y}_1) < \infty$. *Then,*

$$\frac{R_n}{\mathrm{E}\,R_n} \Rightarrow \mathrm{E}\,\tilde{Y}_1 \mathcal{E}\mathrm{xp}(1)$$

as $n \to \infty$.

Remark 3. By applying Proposition 1 to the positive and negative parts of a function f of mixed sign, our result can easily be extended to this setting.

Remark 4. Note that Proposition 1 can be applied even in the null recurrent setting where $\mathrm{E}\,\tau_1 = \infty$, provided that $\mathrm{E}\,\tilde{Y}_1 < \infty$. When X is a Harris chain or process, the hypothesis $\mathrm{E}\,\tilde{Y}_1 < \infty$ is equivalent to requiring that $\int_S f(x)\eta(\mathrm{d}x) < \infty$, where η is the unique (up to a multiplicative constant) σ-finite invariant measure of the chain/process; see, for example, [13, pp. 234–235] for a proof in the discrete-time setting (the continuous-time proof being analogous).

A different limit distribution arises in the special case where $\mathrm{E}\,\tilde{Y}_1 = 0$ and $\sigma^2 := \mathrm{E}\,\tilde{Y}_1^2 < \infty$.

Proposition 2. *Suppose that X is a one-dependent regenerative process satisfying the conditions of Theorem 1,* $\mathrm{E}\,\tilde{Y}_1 = 0$,

$$\int_0^{T(0)} |f(X(s))|\,\mathrm{d}s < \infty \quad a.s.,$$

and

$$\mathrm{E}\int_{T(0)}^{T(1)} |f(X(s)) + 1|^2\,\mathrm{d}s < \infty. \tag{3}$$

Then,

$$\frac{R_n}{\sqrt{\mathrm{E}\,N_n}} \Rightarrow \sigma B(\mathcal{E}\mathrm{xp}(1))$$

as $n \to \infty$, where $B = (B(t) \colon t \geq 0)$ is a standard Brownian motion independent of $\mathcal{E}\mathrm{xp}(1)$.

Proof. For $\varepsilon > 0$, set

$$\chi_\varepsilon(t) = \varepsilon \int_0^{t/\varepsilon^2} f(X(s))\,\mathrm{d}s, \qquad \chi'_\varepsilon(t) = \varepsilon \sum_{i=0}^{N(t/\varepsilon^2)} \tilde{Y}_i,$$

where $N(t) = \max\{n \geq -1 \colon T(n) \leq t\}$ is the index of the last cycle to complete prior to time t. Set $a_n = \mathrm{E}\,N_n$ and $b_n = a_n^{-1/2}$, and observe that $R_n/\sqrt{\mathrm{E}\,N_n} = \chi_{b_n}(T_n/a_n)$. In the presence of (3), it is easy to see that $\chi_{b_n}(T_n/a_n) - \chi'_{b_n}(T_n/a_n) \Rightarrow 0$ as $n \to \infty$. But

$$\chi'_{b_n}\left(\frac{T_n}{a_n}\right) = b_n \sum_{i=0}^{N(T_n)} \tilde{Y}_i = b_n \sum_{i=0}^{N_n - 1} \tilde{Y}_i = \chi''_{b_n}\left(\frac{N_n - 1}{a_n}\right),$$

where $\chi''_\varepsilon(t) = \varepsilon \sum_{i=0}^{\lfloor t/\varepsilon^2 \rfloor} \tilde{Y}_i$. In view of the fact that the composition operation is a continuous functional on $D[0,\infty) \times \mathbb{R}_+$ at pairs (x,t) for which $x(\cdot)$ is continuous, the proof is therefore complete if we can prove that

$$\left(\chi''_{b_n}, \frac{N_n}{a_n}\right) \Rightarrow (\sigma B, \mathcal{E}\mathrm{xp}(1)) \quad \text{in } D[0,\infty) \times \mathbb{R}_+$$

as $n \to \infty$. Donsker's theorem and Theorem 1 establish that $(\chi''_{b_n} : n \geq 1)$ and $(N_n/a_n \colon n \geq 1)$ are both tight sequences, from which it follows that $(\chi''_{b_n}, N_n/a_n \colon n \geq 1)$ is a tight sequence; see, for example, [7, p. 41]. We need only to verify convergence of the finite-dimensional distributions of $(\chi''_{b_n}, N_n/a_n \colon n \geq 1)$ to finish the proof. We now separate the

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time indices into c_n blocks $\mathcal{S}_1, \dots, \mathcal{S}_{c_n}$ of (essentially) equal size, specifically setting $\mathcal{S}_j = \{\lfloor (j-1)a_n t/c_n + 1 \rfloor, \dots, \lfloor j a_n t/c_n - 1 \rfloor\}$ (so that the single time index $\lfloor j a_n t/c_n \rfloor$ separates $\mathcal{S}_j$ and $\mathcal{S}_{j+1}$, and, consequently, the one-dependence implies that $\{\tilde{Y}_i : i \in \mathcal{S}_j\}$ is independent of $\{\tilde{Y}_i : i \in \mathcal{S}_{j+1}\}$). If $d_n := a_n/c_n$, we will choose the c_ns so that they diverge to ∞ with the block size d_n also diverging to ∞. Then,

$$\chi''_{b_n}(t) - \frac{1}{\sqrt{c_n}} \sum_{i=1}^{c_n} \sum_{j \in \mathcal{S}_i} \frac{\tilde{Y}_j}{\sqrt{d_n}} = \sum_{i=1}^{c_n} \frac{\tilde{Y}_{\lfloor i a_n t/c_n \rfloor}}{\sqrt{a_n}}.$$

Since $(\tilde{Y}_{\lfloor j a_n t/c_n \rfloor} : j \geq 1)$ is an i.i.d. sequence with mean 0 and finite variance, the fact that $\sqrt{a_n/c_n} \to \infty$ guarantees that

$$\sum_{i=1}^{c_n} \frac{\tilde{Y}_{\lfloor i a_n t/c_n \rfloor}}{\sqrt{a_n}} \xrightarrow{\text{P}} 0$$

as $n \to \infty$. Set $\tilde{Z}_i^n = \sum_{j \in \mathcal{S}_i} \tilde{Y}_j / \sqrt{d_n}$ for $1 \leq i \leq c_n$. For arbitrary subsets $\Gamma_1, \dots, \Gamma_{c_n}$, we can now follow the same argument as in step 3 of Theorem 1 to show that

$$\text{P}\left(\tilde{Z}_i^n \in \Gamma_1,\ 1 \leq i \leq c_n,\ \frac{N_n}{a_n} > t \right) \leq \prod_{j=1}^{c_n} \text{P}(\tilde{Z}_i^n \in \Gamma_i,\ I_j^n = 0,\ j \in \mathcal{S}_i)$$

and

$$\text{P}\left(\tilde{Z}_i^n \in \Gamma_1,\ 1 \leq i \leq c_n,\ \frac{N_n}{a_n} > t \right) \geq \prod_{j=1}^{c_n} \text{P}(\tilde{Z}_i^n \in \Gamma_i,\ I_j^n = 0,\ j \in \mathcal{S}_i) - c_n p_n.$$

Because $c_n/a_n \to 0$ as $n \to \infty$,

$$\begin{aligned}
&\text{P}\left(\sum_{i=1}^{c_n} \frac{\tilde{Y}_{\lfloor i a_n t/c_n \rfloor}}{\sqrt{a_n}} \leq x,\ \frac{N_n}{a_n} > t \right) \\
&\quad = \text{P}\left(\sum_{i=1}^{c_n} \frac{\tilde{Z}_i^n}{c_n} \leq x \,\middle|\, I_j^n = 0,\ j \in \mathcal{S}_i,\ 1 \leq i \leq c_n \right) \prod_{i=1}^{n} \text{P}(I_j^n = 0,\ j \in \mathcal{S}_i) + o(1) \\
&\quad = \text{P}\left(\sum_{i=1}^{c_n} \frac{\tilde{Z}_i^n}{c_n} \leq x \,\middle|\, I_j^n = 0,\ j \in \mathcal{S}_i,\ 1 \leq i \leq c_n \right) \text{P}\left(\frac{N_n}{a_n} > t \right) + o(1).
\end{aligned}$$

Conditional on $\{I_j^n = 0,\ j \in \mathcal{S}_i,\ 1 \leq i \leq c_n\}$, the $\tilde{Z}_i^n$s are i.i.d. RVs with common distribution $\text{P}(\tilde{Z}_i^n \in \cdot \mid I_j^n = 0,\ j \in \mathcal{S}_1)$. By letting $d_n \to \infty$ sufficiently slowly, we can now verify the conditions of the Lindeberg–Feller central limit theorem to establish the necessary central limit theorem for $\sum_{i=1}^{c_n} \tilde{Z}_i^n / \sqrt{c_n}$. Since Theorem 1 guarantees the convergence of $\text{P}(N_n/a_n > t)$, we have proved that

$$\text{P}\left(\chi''_{b_n} \leq x,\ \frac{N_n}{a_n} > t \right) \to \text{P}(\sigma B(t) \leq x)\, \text{P}(\mathcal{E}\text{xp}(1) > t)$$

as $n \to \infty$. A similar argument verifies convergence of all the finite-dimensional distributions, completing the proof of the proposition.

Remark 5. The RV $B(\mathcal{E}\mathrm{xp}(1))$ has a Laplace distribution (i.e. a 'double exponential' distribution) with mean 0 and variance 1; see [12].

Remark 6. Note that (3) ensures that the process X is a positive recurrent one-dependent regenerative process. Hence, in the Harris setting, X must be a positive recurrent Harris chain or Harris process. The hypothesis $\mathrm{E}\,\tilde{Y}_1 = 0$ is then equivalent to requiring that $\int_S f(x)\eta(\mathrm{d}x) = 0$, where η is the unique stationary distribution of X.

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